A TEXT BOOK OF HIGHER SECONDARY

Elective Mathematics

[IN ENGLISH]

For Class XI

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N A G B O O K H O U S E

CURRICULUM AND SYLLABUSES FOR THE HIGHER SECONDARY COURSE

CLASS—XI

Algebra :

Theory of Quadratic Equations and Expressions, Permutation and Combination ; Bionomial Theorem for positive integral index.

Elementary idea of an infinite series in connection with infinite geometric series ; The use of the expansion of $(1+x)^n$ where n is fractional or negative (proof for the establishment of this expansion is not required but the restriction on the value of x should be known.

Trigonometry :

Graphs of simple trigonometric functions.

Trigonometric equations and general values ; Inverse Circular Functions.

Relation between sides and angles of a triangle ; In-radius, circum-radius and area of triangle ; Practical solution of a triangle with the help of logarithms ; Simple problems of heights and distances.

Co-ordinate Geometry :

Circle, Chords, Tangents, Normals and elementary properties connected with them : Parabola, Ellipse, Hyperbola referred to their principal axes ; Analytical treatment of these curves in respect of (1) the focus and the directrix properties, (2) tangents and normals and elementary properties connected with them, (3) centre and diameter. (Note—Discussion should always be restricted to rectangular Cartesian co-ordinates).

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TRIGONOMETRY

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APPENDIX

Some Important Formulas and Results

ALGEBRA

Theory of Quadratic Equations :

1. The roots of the equation $ax^2+bx+c=0$, where a, b, c are real, are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

From these roots we have :—

(*i*) If $b^2-4ac=0$, the roots are *real* and *equal*. The roots will be irrational, if either a or b be irrational.

(*ii*) If b^2-4ac is a perfect square, the roots are *rational* and *unequal*.

(*iii*) If b^2-ac is positive, but not a perfect square, the roots are *real*, *irrational* and *unequal*.

(*iv*) If b^2-4ac is negative, the roots are *imaginary* and *unequal*.

2. The equations $ax^2+bx+c=0$ and $a_1x^2+b_1x+c_1=0$ will have a common root if $(ca_1-c_1a)^2=(bc_1-b_1c)(ab_1-a_1b)$, and the common root is *either* $\dfrac{ca_1-c_1a}{ab_1-a_1b}$ *or* $\dfrac{bc_1-b_1c}{ca_1-a_1c}$.

Permulation and Combination :

3. (*a*) (*i*) The number of permutations of n different things taken r things at a time$={}^nP_r=n(n-1)(n-2)\cdots(n-r+1)$.

(*ii*) but when n things are taken at a time, the number $={}^nP_n=n(n-1)(n-2)\cdots3.2.1.$

(*iii*) ${}^nP_n={}^nP_{n-1}$ (*iv*) ${}^nP_r=\dfrac{\lfloor n}{\lfloor n-r}$,

(*v*) $\lfloor 0=1$; (*vi*) $\dfrac{1}{\lfloor -n}=0.$

(*b*) The number (x) of permutations of n things taken all together, when the things are not all different :

$$x=\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r\ldots\ldots}$$

(*c*) ${}^nP_r={}^{n-1}P_r+r.{}^{n-1}P_{r-1}.$

4. (*a*) (*i*) The number of combinations of n dissimilar things taken r at a time $(r \leqslant n)$:

$$^nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r}.$$

(*ii*) $^nC_1 = n$, (*iii*) $^nC_n = 1$ (*iv*) $^nP_r = \lfloor r \times {^nC_r}$ (*v*) $^nC_0 = 1$.

(*b*) The number of combinations of n things taken r at a time in which p things always (*i*) occur (*ii*) do not occur.

(*i*) The number $= {^{n-p}C_{r-p}}$, (*ii*) the number $= {^{n-p}C_r}$

(*c*) (*i*) $^nC_r = {^nC_{n-r}}$ (*ii*) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(*d*) Number of combinations of n different things taken any number at a time $= 2^n - 1$;
also $= {^nC_1} + {^nC_2} + {^nC_3} + \cdots\cdots + {^nC_n}$.

5. The gratest value of $^nC_r = {^nC_{\frac{n}{2}}}$ (when n is even)

and $= {^nC_{\frac{n+1}{2}}} = {^nC_{\frac{n-1}{2}}}$ (when n is odd).

Binomial theorem :

(*a*) (*i*) $(a+x)^n = a^n + {^nC_1}a^{n-1}.x + {^nC_2}a^{n-2}x^2 + \cdots\cdots$

$$+ {^nC_r}a^{n-r}x^r + \ldots\ldots + x^n$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2 + \cdots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1}{\lfloor r} a^{n-r}x^r + \ldots + x^n.$$

(*ii*) $(a-x)^n = a^n + {^nC_1}a^{n-1}(-x) + {^nC_2}a^{n-2}(-x)^2 + \ldots + (-x)^n$

$$= a^n - {^nC_1}a^{n-1}x + {^nC_2}a^{n-2}x^2 - \ldots + (-1)^n x^n.$$

(*b*) **The general term :**

(*i*) The general term in the expansion of $(a+x)^n = {^nC_r}a^{n-r}x^r$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1}{\lfloor r} a^{n-r}x^r \;;$$

(*ii*) The general term in the expansion of $(a-x)^n$

$$= {^nC_r}a^{n-r}(-x)^r = (-1)^r \, {^nC_r}a^{n-r}x^r$$

(c) *Middle term or terms :*

The middle term in the expansion of $(a+x)^n$:

(i) if n is even, the middle term $= {}^nC_{\frac{1}{2}n}\, a^{\frac{1}{2}n} x^{\frac{1}{2}n} = \dfrac{\lfloor n}{\lfloor \frac{1}{2}n \lfloor \frac{1}{2}n} a^{\frac{1}{2}n} x^{\frac{1}{2}n}$

(ii) if n is odd, the two middle terms

$= {}^nC_{\frac{1}{2}(n-1)} a^{\frac{1}{2}(n+1)} x^{\frac{1}{2}(n-1)}$ and ${}^nC_{\frac{1}{2}(n+1)} a^{\frac{1}{2}(n-1)} x^{\frac{1}{2}(n+1)}$

$= \dfrac{\lfloor n}{\lfloor \frac{1}{2}(n-) \lfloor \frac{1}{2}(n+1)} a^{\frac{1}{2}(n+1)} x^{\frac{1}{2}(n-1)}$ and

$\dfrac{\lfloor n}{\lfloor \frac{1}{2}(n+1) \lfloor \frac{1}{2}(n-1)} a^{\frac{1}{2}(n-1)} x^{\frac{1}{2}(n+1)}$

(d) The sum of the coefficients in the expansion of

$(1+x)^n = 2^n$.

(e) Sum of the infinite series $a+ar+ar^2+ar^3+\ldots$ to ∞

$= \dfrac{a}{1-r}$ where $-1<r<1$.

(f) when $-1<x<1$ and n is *fractional* or *negative*

we have $(1+x)^n = 1+nx+\dfrac{n(n-1)}{\lfloor 2}x^2+\dfrac{n(n-1)(n-2)}{\lfloor 3}x^3+\cdots+$

$\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r+\cdots$ to ∞.

(g) The general term t_{r+1} of $(1+x)^n$

$= \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r$, when n is fractional or negative.

(i) t_{r+1} in the expansion of $(1-x)^n$

$= (-1)^r . \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r$

(ii) t_{r+1} ,, ,, $(1+x)^{-n}$

$= (-1)^r . \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}x^r$

(iii) t_{r+1} ,, ,, $(1-x)^{-n}$

$= \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}x^r$.

CHAPTER I

ALGEBRA

THEORY OF QUADRATIC EQUATIONS

You already know how to solve Quadratic Equations. Now you will be acquainted with some theories regarding Quadratic Equations and Expressions.

You know that the equation, in which the highest power of the unknown quantity is 2, is called a quadratic equation.

Let us examine the following two equations :

(1) Solve : $p.\frac{(x-q)(x-r)}{(p-q)(p-r)}+q.\frac{(x-r)(x-p)}{(q-r)(q-p)}=x.$

(2) Solve : $a.\frac{(x-b)(x-c)}{(a-b)(a-c)}+b.\frac{(x-c)(x-a)}{(b-c)(b-a)}$

$$+c.\frac{(x-a)(x-b)}{(c-a)(c-b)}=x.$$

Here you notice that the first equation is satisfied by the values $x=p$ and $x=q$, but not by the value $x=r$. So, the first equation has two roots.

Again, the second equation is satisfied, if $x=a$, or, $x=b$ or $x=c$. So, apparently this equation seems to have three roots. But on careful scrutiny you will find that the second equation is satisfied by any value of x (such as 1, 2,...p, q,...etc.)

Hence it is an *identity* and not an equation, i.e., if a quadratic equation has more than two roots, then it is not an equation but is an identity.

You should know that generally the number of roots of an equation is equal to the index of the highest power of its unknown quantity. So, a quadratic equation has two roots, the equation of the third degree has 3 roots, and so on.

I. **Theorem :** *A quadratic equation cannot have more than two roots.*

Any quadratic equation that you have solved can be reduced to the form $ax^2+bx+c=0$.

If possible, let this equation have three roots α, β, γ.

So the equation must be satisfied by these three roots.

$$\therefore \quad a\alpha^2+b\alpha+c=0 \text{......(i)}$$

$$a\beta^2+b\beta+c=0 \text{......(ii)}$$

$$a\gamma^2+b\gamma+c=0 \text{......(iii)}$$

Subtracting (ii) from (i) we have

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0.$$

$$\text{or,} \quad (\alpha-\beta)\{a(\alpha+\beta)+b\}=0.$$

$\because$ α and β are different roots (i. e., they are unequal),

$$\therefore \quad \alpha-\beta \text{ cannot be zero.}$$

$$\therefore \quad a(\alpha+\beta)+b=0 \text{......(iv).}$$

Similarly, subtracting (iii) from (i) we have

$$a(\alpha+\gamma)+b=0 \text{......(v).}$$

Again subtracting (v) from (iv) we have $a(\beta-\gamma)=0$.

$\therefore$ Either $a=0$, or, $\beta-\gamma=0$, but both are impossible, as both are contrary to the hypothesis. If $\beta-\gamma=0$, we have $\beta=\gamma$, but by hypothesis they are different. Again, if $a=0$, then from (iv) and (v) we get $b=0$ and substituting $a=0$, $b=0$ in (i) we have $c=0$. Then the given equation becomes $0.x^2+0.x+0=0$, and it is satisfied by any value of x. Hence it is not an equation, but an identity.

Hence it is found that proceeding from the supposed hypothesis, we arrive at conclusions contrary to it. So the original supposition is wrong. Hence a quadratic equation cannot have more than two roots.

2. *Relation between the roots and the coefficients of a quadratic equation.*

Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$.

By solving this equation we have $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Then $\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

Now, the sum of the roots $=\alpha+\beta$

$$=\frac{1}{2a}(-b+\sqrt{b^2-4ac})+\frac{1}{2a}(-b-\sqrt{b^2-4ac})$$

$$=\frac{1}{2a}(-2b)=-\frac{b}{a} \qquad \text{......(A)}$$

The product of the roots $=\alpha\beta$

$$=\frac{1}{2a}(-b+\sqrt{b^2-4ac})\times\frac{1}{2a}(-b-\sqrt{b^2-4ac})$$

$$=\frac{1}{4a^2}\{(-b)^2-(\sqrt{b^2-4ac})^2\}=\frac{4ac}{4a^2}=\frac{c}{a} \qquad \text{......(B)}$$

Corollary : (i) If the equation of the form $ax^2+bx+c=0$ be reduced to the form $x^2+\frac{b}{a}.x+\frac{c}{a}=0$ by dividing it by the coefficient of x^2 (here by a), then we have :

(1) **The sum of the roots = the coefficient of x with its sign changed.**

(2) **The product of the roots = the absolute term** (i.e., the term independent of x)

Example. Take the equation $4x^2-3x-1=0$. Dividing it by 4 (the coefficient of x^2) we have $x^2-\frac{3}{4}x-\frac{1}{4}=0$.

Now, changing the sign of the coefficient of x we have $\frac{3}{4}$ and the absolute term here is $-\frac{1}{4}$.

$\therefore$ The sum of the roots $=\frac{3}{4}$, and the product of the roots $=-\frac{1}{4}$.

Corollary (ii) : *Formation of a quadratic equation from the given roots.*

Taking the sum of the given roots and changing the sign we obtain the coefficient of x and the product of the roots will be the absolute term of the equation.

Thus, if α and β be the given roots, then the coefficient of x will be $-(\alpha+\beta)$ and the term independent of x will be $\alpha\beta$.

$\therefore$ The equation required is $x^2-(\alpha+\beta)x+\alpha\beta=0$,

i.e., x^2 – (sum of the roots) x + the product of the roots $=0$.

Example. Let $-\frac{2}{3}$ and $\frac{1}{2}$ be the roots. To form the equation.

Here, the sum of the roots $=-\frac{2}{3}+\frac{1}{2}=-\frac{1}{6}$.
and the product of the roots $=-\frac{2}{3}\times\frac{1}{2}=-\frac{1}{3}$.
$\therefore$ the required equation is $x^2+\frac{1}{6}x-\frac{1}{3}=0$, or, $6x^2+x-2=0$.

3. Conjugate roots.

You know that of the two surds $p+\sqrt{q}$ and $p-\sqrt{q}$, one is called the *conjugate surd* of the other and that each of the complex quantities $p+iq$ and $p-iq$ is called the *conjugate complex* of the other.

Now we shall see that if one root of a quadratic equation be a surd or a complex quantity, then the other root will be its conjugate, i.e., a conjugate surd or a conjugate complex quantity. Such roots are called *conjugate roots.*

(*a*) **Theorem :** *In a quadratic equation with rational coefficients, the irrational roots always occur in conjugate pairs.*

Let a, b and c in the equation $ax^2+bx+c=0$ be rational and let one root of the equation be $\alpha+\sqrt{\beta}$.

To prove that the other root will be $\alpha-\sqrt{\beta}$ (which is conjugate of the root $\alpha+\sqrt{\beta}$).

$\because$ $\alpha+\sqrt{\beta}$ is a root of the equation, $\therefore$ it satisfies the equation.

$\therefore$ we have $a(\alpha+\sqrt{\beta})^2+b(\alpha+\sqrt{\beta})+c=0$,

or, $(a\alpha^2+b\alpha+a\beta+c)+\sqrt{\beta}(2a\alpha+b)=0$; but the sum of a rational quantity and an irrational quantity cannot be zero, unless each of them separately is zero.

$\therefore$ Here $a\alpha^2+b\alpha+a\beta+c=0$......(i) and $2a\alpha+b=0$......(ii)

Now, putting the value $\alpha-\sqrt{\beta}$ for x in ax^2+bx+c, we have

$$\begin{aligned} ax^2+bx+c &= a(\alpha-\sqrt{\beta})^2+b(\alpha-\sqrt{\beta})+c \\ &= (a\alpha^2+b\alpha+a\beta+c)-\sqrt{\beta}(2a\alpha+b) \\ &= 0-\sqrt{\beta}\times 0 \text{ [from (i) and (ii)]} \\ &= 0. \end{aligned}$$

$\therefore$ The root $\alpha-\sqrt{\beta}$ satisfies the given equation.

Hence $\alpha-\sqrt{\beta}$ is the other root of the equation.

(*b*) **Theorem** : *In a quadratic equation with real coefficients, imaginary roots always occur in conjugate pairs.*

Let a, b, c in the equation $ax^2+bx+c=0$ be real and let $\alpha+i\beta$ be one of its roots. To prove that its other root will be $\alpha-i\beta$ (which is conjugate of $\alpha+i\beta$).

$\because$ $\alpha+i\beta$ is a root of the equation, $\therefore$ it satisfies the equation

$\therefore$ we have $a(\alpha+i\beta)^2+b(\alpha+i\beta)+c=0$,

or, $(a\alpha^2+b\alpha-a\beta^2+c)+i(2a\alpha\beta+b\beta)=0$ $[\because i^2=-1]$;

Here we find that the sum of one real quantity and one imaginary quantity is zero, which is impossible unless each separately is zero.

$\therefore$ $a\alpha^2+b\alpha-a\beta^2+c=0$...(i) and $2a\alpha\beta+b\beta=0$...(ii)

Now, putting $\alpha-i\beta$ for x in ax^2+bx+c, we have

$$\begin{aligned} ax^2+bx+c &= a(\alpha-i\beta)^2+b(\alpha-i\beta)+c \\ &= (a\alpha^2+b\alpha-a\beta^2+c)-i\,(2a\alpha\beta+b\beta) \\ &= 0-i\times 0 \text{ [from (i) and (ii)]} \\ &= 0. \end{aligned}$$

$\therefore$ The root $\alpha-i\beta$ satisfies the given equation.

Hence, if one root of the equation is $\alpha+i\beta$, the other root is $\alpha-i\beta$.

[**N. B.** (i) It is evident from Art. 3(*a*) that a quadratic equation with rational coefficients cannot have one root rational and the other irrational.

(ii) Also from Art. 3(*b*) it is evident that a quadratic equation with real coefficients cannot have one root real and the other imaginary.]

4. Reciprocal roots.

When the product of the two roots of a quadratic equation is 1, one root is called the reciprocal of the other and the equation is said to have reciprocal roots.

(*a*) *Condition for reciprocal roots of a quadratic equation.*

Suppose the equation $ax^2+bx+c=0$ has two reciprocal roots.

Let the roots be α and $\frac{1}{\alpha}$.

Now, from Art. 2 (ii) we have $\alpha\times\frac{1}{\alpha}=\frac{c}{a}$,

$$\therefore \quad \frac{c}{a}=1, \quad \text{or,} \quad a=c.$$

Hence, a quadratic equation will have two reciprocal roots, if its absolute term be the same as the coefficient of x^2.

Example. In the equation $2x^2+5x+2=0$, the coefficient of x^2 is 2 and the absolute term (i.e., the term independent of x) is also 2. Hence its roots will be reciprocal to each other. Solving the equation we have $x=-2$ and $-\frac{1}{2}$ which are reciprocal.

[**N. B.** Also the two roots of one quadratic equation may be reciprocals of the roots of another quadratic equation.

Let us take the equations $6x^2-5x+1=0$ and $x^2-5x+6=0$. The roots of the first equation are $\frac{1}{2}$ and $\frac{1}{3}$, while the roots of the other are 2 and 3.]

Examples (1. A)

Ex. 1. If α and β be the rooots of $x^2+px+q=0$, find the value of $\alpha^3+\beta^3$ and $\frac{1}{\alpha}+\frac{1}{\beta}$ in terms of p and q.

$\because$ α and β are the roots of the equation $x^2+px+q=0$,

$\therefore$ $\alpha+\beta=-p$ and $\alpha\beta=q$.

Now, $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=(-p)^3-3q(-p)=-p^3+3pq\ ;$$

and $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-p}{q}=-\frac{p}{q}$.

Ex. 2. If α and β be the roots of $ax^2+bx+c=0$, find an expression for $\alpha^2+\beta^2$ in terms of a, b, c. [A. U, '19]

$\because$ α and β are the roots of $ax^2+bx+c=0$,

i.e , of $x^2+\frac{b}{a}x+\frac{c}{a}=0$,

$\therefore$ we have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Now, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2-2ca}{a^2}$.

Ex. 3. If α and β be the roots of $ax^2+bx+c=0$, find the value of $\frac{1}{(a\alpha+b)^3}+\frac{1}{(a\beta+b)^3}$. [C. U. 1943]

$\because$ α and β are the roots of $ax^2+bx+c=0$,

$\therefore$ $\alpha+\beta=-\frac{b}{a}$, or, $a\alpha+a\beta=-b$ [multiplying by a]

$\therefore$ we have $a\alpha+b=-a\beta$, $a\beta+b=-a\alpha$, and $\alpha\beta=\frac{c}{a}$.

Now, $\frac{1}{(a\alpha+b)^3}+\frac{1}{(a\beta+b)^3}=\frac{1}{(-a\beta)^3}+\frac{1}{(-a\alpha)^3}$

$$=\frac{1}{-a^3\beta^3}+\frac{1}{-a^3\alpha^3}$$

$$=\frac{\alpha^3+\beta^3}{-a^3\alpha^3\beta^3}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{-a^3(\alpha\beta)^3}=\frac{\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{-a^3\left(\frac{c}{a}\right)^3}$$

$$=\frac{-\frac{b^3}{a^3}+\frac{3bc}{a^2}}{-c^3}=\frac{-b^3+3abc}{-a^3c^3}=\frac{b^3-3abc}{a^3c^3}.$$

√Ex. 4. If α, β and α', β' be the roots of $x^2-px+q=0$ and $x^2-p'x+q'=0$ respectively, find the value of $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$. [C. U, '13]

$\because$ α and β are the roots of $x^2-px+q=0$,

$\therefore$ $\alpha+\beta=p$ and $\alpha\beta=q$.

Again, $\because$ α' and β' are the roots of $x^2-p'x+q'=0$,

$\therefore$ $\alpha'+\beta'=p'$ and $\alpha'\beta'=q'$.

Now, $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$

$=2(\alpha^2+\beta^2)+2(\alpha'^2+\beta'^2)-2\alpha'(\alpha+\beta)-2\beta'(\alpha+\beta)$

$=2\{(\alpha+\beta)^2-2\alpha\beta+(\alpha'+\beta')^2-2\alpha'\beta'-(\alpha+\beta)(\alpha'+\beta')\}$

$=2(p^2-2q+p'^2-2q'-pp')$.

[Formation of equations]

Ex. 5. If α, β be the roots of $ax^2+bx+c=0$, form an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [C. U. 1877 ; P. U.]

$\because$ α and β are the roots of $ax^2+bx+c=0$,

$\therefore$ $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Here, the sum of the given roots of the required equation

$$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\frac{c}{a}}=\frac{b^2-2ca}{ca};$$

and the product of the roots $=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1.$

$\therefore$ the required equation is $x^2-\frac{b^2-2ca}{ca}x+1=0,$

i.e., $cax^2-(b^2-2ca)x+ca=0.$

Ex. 6. Form an equation whose roots are the squares of the roots of the equation $x^2+x+1=0.$ [A. U. '20]

Let α and β be the roots of $x^2+x+1=0.$

Then the roots of the required equation are α^2 and β^2.

From the given equation we have $\alpha+\beta=-1$ and $\alpha\beta=1.$

Then the sum of the roots of the required equation

$$=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-1)^2-2\times1=-1,$$

and the product of the roots $=\alpha^2\beta^2=(1)^2=1.$

$\therefore$ the required equation is $x^2+x+1=0.$

Ex. 7. Form the quadratic equation whose roots are the reciprocals of the roots of $x^2-7x+12=0.$

Suppose the roots of $x^2-7x+12=0$ are α and $\beta.$

Then the roots of the required equation are $\frac{1}{\alpha}$ and $\frac{1}{\beta}.$

From the given equation $\alpha+\beta=7$ and $\alpha\beta=12.$

$\therefore$ the sum of the roots of the required equation

$$=\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{7}{12};$$

and their product $=\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{1}{12}.$

$\therefore$ the required equation is $x^2-\frac{7}{12}x+\frac{1}{12}=0,$

or, $12x^2-7x+1=0.$

Ex. 8. Form the equation whose roots α and β satisfy the condition $\alpha^2+\beta^2=113$ and $\alpha\beta=28.$

Here, $(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta=113+28\times2=169,$

$\therefore$ $\alpha+\beta=\pm13.$

$\therefore$ The required equation is $x^2\mp13x+28=0.$

Ex. 9. Find a quadratic equation whose roots are the squares of the sum and difference of the roots of $ax^2+bx+c=0$.

[Raj. '45]

Let α and β be the roots $ax^2+bx+c=0$.

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

By the given condition the roots of the required equation are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.

Now, the sum of these roots $=(\alpha+\beta)^2+(\alpha-\beta)^2$

$$=2(\alpha^2+\beta^2)=2\{(a+\beta)^2-2\alpha\beta\}=2\left\{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\right\}$$

$$=2\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)=\frac{2(b^2-2ac)}{a^2};$$

and the product of the roots $=(\alpha+\beta)^2(\alpha-\beta)^2$

$$=(\alpha+\beta)^2\{(\alpha+\beta)^2-4\alpha\beta\}=\frac{b^2}{a^2}\times\frac{b^2-4ca}{a^2}=\frac{b^2(b^2-4ac)}{a^4}.$$

$\therefore$ The required equation is

$$x^2-\frac{2(b^2-2ca)}{a^2}x+\frac{b^2(b^2-4ca)}{a^4}=0,$$

or, $a^4x^2-2a^2(b^2-2ca)x+b^2(b^2-4ca)=0$.

Ex. 10. If a is not equal to b, but $a^2=5a-3$ and $b^2=5b-3$, find the equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$. [C. U. '50]

Here $a\neq b$, but $a^2=5a-3$ and $b^2=5b-3$, from which it is evident that a and b are the roots of the equation $x^2=5x-3$ or $x^2-5x+3=0$.

$$\therefore \quad a+b=5 \text{ and } ab=3.$$

Again, the roots of the required equation are $\frac{a}{b}$ and $\frac{b}{a}$.

Now, their sum $=\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ab}=\frac{(a+b)^2-2ab}{ab}=\frac{25-6}{3}=\frac{19}{3}$,

and their product $=\frac{a}{b}\times\frac{b}{a}=1$.

$\therefore$ The required equation is $x^2-\frac{19}{3}x+1=0$,

or, $3x^2-19x+3=0$.

Ex. 11. Form a quadratic equation with rational coefficients, one of whose roots is $3-\sqrt{2}$.

By the conditions, the coefficients of the required equation are rational and one of its roots is the irrational quantity $3-\sqrt{2}$. Hence the other root must be its conjugate $3+\sqrt{2}$ [vide Art 3(*a*).]

Now, the sum of the roots $=3-\sqrt{2}+3+\sqrt{2}=6$,
and their product $=(3-\sqrt{2})(3+\sqrt{2})=9-2=7$.

$\therefore$ The required equation is $x^2-6x+7=0$.

Ex. 12. If α, β be the roots of $ax^2+bx+c=0$ and γ, δ be the roots of $lx^2+mx+n=0$, then find the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$. [U. P. '39, '49]

$\because$ α and β are the roots of $ax^2+bx+c=0$,

$\therefore$ $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

Again, $\because$ γ and δ are the roots of $lx^2+mx+n=0$,

$\therefore$ $\gamma+\delta=-\dfrac{m}{l}$ and $\gamma\delta=\dfrac{n}{l}$.

Now, the sum of the roots of the required equation

$$=\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma=\alpha(\gamma+\delta)+\beta(\gamma+\delta)$$

$$=(\alpha+\beta)(\gamma+\delta)=-\frac{b}{a}\times-\frac{m}{l}=\frac{bm}{al};$$

and the product of the roots $=(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)$

$$=\alpha^2\gamma\delta+\beta^2\gamma\delta+\alpha\beta\gamma^2+\alpha\beta\delta^2=(\alpha^2+\beta^2)\gamma\delta+(\gamma^2+\delta^2)\alpha\beta$$

$$=\{(\alpha+\beta)^2-2\alpha\beta\}\gamma\delta+\{(\gamma+\delta)^2-2\gamma\delta\}\alpha\beta$$

$$=\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)\frac{n}{l}+\left(\frac{m^2}{l^2}-\frac{2n}{l}\right)\frac{c}{a}=\frac{n(b^2-2ac)}{a^2l}+\frac{c(m^2-2ln)}{al^2}$$

$$=\frac{ln(b^2-2ac)+ac(m^2-2ln)}{a^2l^2}=\frac{b^2ln+m^2ac-4acln}{a^2l^2}.$$

$\therefore$ the reqd. equation is $x^2-\dfrac{bm}{al}x+\dfrac{b^2ln+m^2ac-4acln}{a^2l^2}=0$,

or, $a^2l^2x^2-ablmx+b^2ln+m^2ac-4acln=0$.

[**Miscellaneous**]

Ex. 13. Find the value of $x^4+4x^3+6x^2+4x+9$, when $x=\sqrt{-2}-1$.

$\because$ $x=\sqrt{-2}-1$, $\therefore$ $x+1=\sqrt{-2}$,

$\therefore$ $x^2+2x+1=-2$ (squaring), or, $x^2+2x+3=0$.

Now the given exp. $=x^2(x^2+2x+3)+2x(x^2+2x+3)$

$-(x^2+2x+3)+12=x^2\times 0+2x\times 0-0+12=12.$

Ex. 14. If the roots of $ax^2+bx+c=0$ bear to one another the ratio of 2 : 3, show that $6b^2=25ac$. [C. U. '49]

Here the roots of $ax^2+bx+c=0$ are in the ratio 2 : 3.

Let the roots be 2α and 3α.

Then their sum $2\alpha+3\alpha=-\frac{b}{a}$, or, $5\alpha=-\frac{b}{a}$......(1) and their product $2\alpha\times 3\alpha=\frac{c}{a}$, or, $6\alpha^2=\frac{c}{a}$..... (2). From (1) we have $25\alpha^2=\frac{b^2}{a^2}$......(3). Dividing (3) by (2) we have $\frac{25\alpha^2}{6\alpha^2}=\frac{\frac{b^2}{a^2}}{\frac{c}{a}}$,

or, $\frac{25}{6}=\frac{b^2}{ac}$, $\therefore$ $6b^2=25ac$.

Ex. 15. If r be the ratio of the roots of the equation $ax^2+bx+c=0$, show that $\frac{(r+1)^2}{r}=\frac{b^2}{ac}$. [C. U. '34]

Let α be one root of the given equation. Then the other root is $r\alpha$ ($\because$ the ratio of the roots is r),

$\therefore$ From the equation we have

$\alpha+r\alpha=-\frac{b}{a}$, or, $\alpha(1+r)=-\frac{b}{a}$......(1)

and $\alpha\times\alpha r=\frac{c}{a}$, or, $\alpha^2 r=\frac{c}{a}$......(2).

Now, dividing the square of (1) by (2) we have

$$\frac{\alpha^2(1+r)^2}{\alpha^2 r}=\frac{\dfrac{b^2}{a^2}}{\dfrac{c}{a}}, \text{ or, } \frac{(1+r)^2}{r}=\frac{b^2}{ac}.$$

Ex. 16. If the ratio of the roots of the equation $lx^2+nx+n=0$ be $p:q$, then prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

[W. B. H. S. '63 ; U. P. '56 ; Raj. '54]

Let α and β be the roots of $lx^2+nx+n=0$.

Then $\alpha+\beta=-\frac{n}{l}$ and $\alpha\beta=\frac{n}{l}$. From the given condition $\frac{\alpha}{\beta}=\frac{p}{q}$.

Now, $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{n}{l}}=\frac{\alpha+\beta}{\sqrt{\alpha\beta}}+\sqrt{\frac{n}{l}}$

$$=\frac{-\frac{n}{l}}{\sqrt{\frac{n}{l}}}+\sqrt{\frac{n}{l}}=-\sqrt{\frac{n}{l}}+\sqrt{\frac{n}{l}}=0.$$

Ex. 17. What relation must exist between the constants of the equation $x^2+px+q=0$, so that the sum of the roots may be equal to three times their difference. [C. U. '47]

Let α and β be the roots of $x^2+px+q=0$, and $\alpha>\beta$.

Now, by the given condition, $\alpha+\beta=3(\alpha-\beta)$,

or, $\alpha=2\beta$......(i).

Again, $\because$ α and β are the roots of $x^2+px+q=0$,

$\therefore$ $\alpha+\beta=-p$, or, $3\beta=-p$......(ii) [from (i)],

and $\alpha\beta=q$, or, $2\beta^2=q$......(iii) [from (i)].

Now, squaring (ii) and dividing the result by (iii),

we have $\frac{9\beta^2}{2\beta^2}=\frac{p^2}{q}$, or, $\frac{9}{2}=\frac{p^2}{q}$, or, $2p^2=9q$.

$\therefore$ The required relation is $2p^2=9q$.

Ex. 18. If the roots of $x^2+2px+q=0$ and $x^2+2qx+p=0$ differ by a constant, show that $p+q+1=0$, unless $p=q$.

[C. U. '44]

Let α and β be the roots of $x^2+2px+q=0$ and γ and δ be the roots of $x^2+2qx+p=0$.

$\therefore\quad \alpha+\beta=-2p$ and $\alpha\beta=q$, and $\gamma+\delta=-2q$ and $\gamma\delta=p$.

From the given condition, the roots of the two equations differ by a constant.

Then $\alpha-\gamma=k$ (k being a constant) and $\beta-\delta=k$ (constant).

$\therefore\quad \alpha-\gamma=\beta-\delta$, or, $\quad \alpha-\beta=\gamma-\delta$.

$\therefore\quad (\alpha-\beta)^2=(\gamma-\delta)^2$, or, $\quad (\alpha+\beta)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta$,

or, $\quad 4p^2-4q=4q^2-4p$, or, $\quad (p^2-q^2)+(p-q)=0$,

or, $\quad (p-q)(p+q+1)=0$.

$\therefore\quad$ either $p-q=0$, or, $\quad p+q+1=0$.

Now, if p and q are not equal, *i.e.*, if $p-q$ is not zero, then $p+q+1=0$.

But if $p=q$, then the two equations become the same equation, so $p\neq q$.

$\therefore\quad p+q+1=0$.

Ex. 19. If the roots of $ax^2+2bx+c=0$ be α, β and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+\delta$ and $\beta+\delta$, show that $\dfrac{b^2-ac}{B^2-AC}=\left(\dfrac{a}{A}\right)^2$.

[C. U. '12]

$\because\quad \alpha, \beta$ are the roots of the first equation,

$\therefore\quad \alpha+\beta=-\dfrac{2b}{a}$ and $\alpha\beta=\dfrac{c}{a}$......(*i*)

Again, $\because\ \alpha+\delta$ and $\beta+\delta$ are the roots of $Ax^2+2Bx+C=0$,

$\therefore\quad \alpha+\beta+2\delta=-\dfrac{2B}{A}$ and $(\alpha+\delta)(\beta+\delta)=\dfrac{C}{A}$......(*ii*).

Now, since $\alpha-\beta=(\alpha+\delta)-(\beta+\delta)$.

$\therefore\quad (\alpha-\beta)^2=\{(\alpha+\delta)-(\beta+\delta)\}^2$,

or, $\quad (\alpha+\beta)^2-4\alpha\beta=\{(\alpha+\delta)+(\beta+\delta)\}^2-4(\alpha+\delta)(\beta+\delta)$,

or, $\quad \left(-\dfrac{2b}{a}\right)^2-4\left(\dfrac{c}{a}\right)=\left(-\dfrac{2B}{A}\right)^2-4\left(\dfrac{C}{A}\right)$ [from (i) & (ii)]

or, $\dfrac{4b^2}{a^2}-\dfrac{4c}{a}=\dfrac{4B^2}{A^2}-\dfrac{4C}{A}$, or, $\dfrac{b^2-ca}{a^2}=\dfrac{B^2-CA}{A^2}$,

$$\therefore \quad \frac{b^2-ca}{B^2-CA}=\frac{a^2}{A^2}=\left(\frac{a}{A}\right)^2.$$

Ex. 20. If the roots of $ax^2+bx+c=0$ are the reciprocals of those of $a'x^2+b'x+c'=0$, show that $a:b:c::c':b':a'$.

Let α and β be the roots of $a'x^2+b'x+c'=0$,

then $\alpha+\beta=-\dfrac{b'}{a'}$, and $\alpha\beta=\dfrac{c'}{a'}$.

$\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ are the reciprocals of α and β respectively.

$$\therefore \quad \frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{\;\frac{-b'}{a'}\;}{\frac{c'}{a'}}=-\frac{b'}{c'}, \text{ and } \frac{1}{\alpha}\times\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{1}{\frac{c'}{a'}}=\frac{a'}{c'}.$$

Hence the equation, whose roots are the reciprocals of the roots of $a'x^2+b'x+c'=0$, is $x^2+\dfrac{b'}{c'}x+\dfrac{a'}{c'}=0$,

i.e., $c'x^2+b'x+a'=0$; but by the problem, $ax^2+bx+c=0$ is the equation.

$\therefore$ the two equations are identical,

$\therefore$ we have $\dfrac{a}{c'}=\dfrac{b}{b'}=\dfrac{c}{a'}$, *i.e.*, $a:b:c::c':b':a'$.

Ex. 21. Find the condition that the roots of the quadratic equation $ax^2+bx+c=0$ should be one positive and the other negative.

Let α and $-\beta$ be the roots of $ax^2+bx+c=0$.

First, when α is numerically greater than β, then the sum of the two roots (*i.e.*, $\alpha-\beta$) is positive and their product (*i.e.*, $\alpha\times-\beta$) is negative.

So, $-\dfrac{b}{a}$ will be positive and $\dfrac{c}{a}$ will be negative.

But $-\frac{b}{a}$ will be positive, when b and a have opposite signs, and $\frac{c}{a}$ will be negative, when c and a have opposite signs.

Hence, we find that if the root that is numerically greater than the other be positive, then b and c of the given equation are of the same sign and the sign of a is opposite to that of b and c.

Secondly, when a is numerically less than b, the sum of the two roots (*i.e.*, $\alpha-\beta$) is negative and their product is also negative.

$\therefore$ both $-\frac{b}{a}$ and $\frac{c}{a}$ are negative. But $-\frac{b}{a}$ is negative, when b and a are of the same sign, while $\frac{c}{a}$ is negative when c and a are of opposite signs. Hence, we find that if the root, that is numerically greater, is negative, then a and b of the equation will be of the same sign and c will be of opposite sign.

Ex. 22. Form the two quadratic equations of which the roots are (*a*) $2+\sqrt{7}$ and $2-\sqrt{7}$ and (*b*) $\sqrt{7}+2$ and $\sqrt{7}-2$ respectively. Distinguish between the coefficients of the two equations and try to explain the difference. [C. U. '48]

(*a*) Here the sum of the roots $=2+\sqrt{7}+2-\sqrt{7}=4$,

and their product $=(2+\sqrt{7})(2-\sqrt{7})=4-7=-3$.

$\therefore$ the required equation is $x^2-4x-3=0$.

(*b*) Here the sum of the roots $=\sqrt{7}+2+\sqrt{7}-2=2\sqrt{7}$,

and their product $=(\sqrt{7}+2)(\sqrt{7}-2)=7-4=3$.

$\therefore$ the required equation is $x^2-2\sqrt{7}\,x+3=0$.

Again, (i) since of the two roots of the first equation, one $(2+\sqrt{7})$ is positive and the other $(2-\sqrt{7})$ is negative, x and the term independent of x are of the same sign (here, negative), while the coefficient of x^2 is of opposite sign (*i.e.*, positive).

As both the roots of the second equation are positive, x^2 and the absolute term have the same sign (here, positive), while the coefficient of x is of opposite sign (*i.e.*, negative).

(ii) As both the sum and the product of the roots of the first equation are rational, all its terms have rational coefficients.

But as the sum of the roots of the second equation is irrational, the coefficient of x in it is irrational.

Exercise 1(A)

1. Prove that the following are identities and not equations :—

(a). $(x-2)(x-4)-2(x-2)(x-5)+(x-4)^2=4.$

(b). $\dfrac{(x-a)(x-b)}{(c-a)(c-b)}+\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}=1.$

(c). $a^2\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\dfrac{(x-a)(x-b)}{(c-a)(c-b)}=x^2.$ [C. U. 1884]

2. If α, β be the roots of the equation $ax^2+bx+c=0$, find the values of the following in terms of a, b and c.

(i) $\alpha-\beta$ (ii) $\alpha^2\beta+\alpha\beta^2$ (iii) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ (iii) $\alpha^3+\beta^3$

(v) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$ (vi) $\left(\dfrac{\alpha}{\beta}-\dfrac{\beta}{\alpha}\right)^2$ (vii) $\alpha^4\beta^{-2}+\beta^4\alpha^{-2}$

(viii) $\dfrac{1}{a\alpha-b}+\dfrac{1}{a\beta-b}$ (ix) $\dfrac{1}{a\alpha+b}+\dfrac{1}{a\beta+b}.$

3. If α, β be the roots of the equation $x^2+px+q=0$, find in terms of p and q the values of the following :—

(i) $\alpha^{-3}+\beta^{-3}$ [C. U. '46] (ii) $\alpha\beta^{-2}+\beta\alpha^{-2}$

(iii) $\alpha^3\beta^{-1}+\beta^3\alpha^{-1}$ (iv) $\alpha^4+\alpha^2\beta^2+\beta^4$

(v) $\alpha^2(\alpha^2\beta^{-1}-\beta)+\beta^2(\beta^2\alpha^{-1}-\alpha)$ [C. U. '41]

(vi) $\alpha^4\beta^7+\beta^4\alpha^7$ (vii) $(\alpha+p)^{-4}+(\beta+p)^{-4}.$

4. Form the equations whose roots are

(i) 3, 4 ; (ii) 5, -7 ; (iii) $1+\sqrt{3}$, $1-\sqrt{3}$;

(iv) $a+b$, $a-b$; (v) a^2+a+1, a^2-a+1.

5. (*a*) Form the equation whose roots are the squares of the roots of $x^2+3x+2=0$.

(*b*) Form the equations whose roots are the reciprocals of the roots of

(i) $x^2+3x+4=0$; (ii) $x^2-x+1=0$. [A. U. '25]

(*c*) Form the equation whose roots are the A. M. and G. M. of the roots of $x^2+bx+c=0$.

6. If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are 2α and 2β.

7. If α, β are the roots of $ax^2+bx+c=0$, find the equation whose roots are—

(i) α^2 and β^2 [C. U. 1933] (ii) $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$ [C. U. 1336]

(iii) $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$ [C. U. 1924]

(iv) $\alpha+\beta^{-1}$ and $\beta+\alpha^{-1}$ [C. U. 1939]

(v) $\dfrac{1}{a\alpha+b}$ and $\dfrac{1}{a\beta+b}$.

8. Form the equation whose roots α and β satisfy the relation $\alpha\beta=768$ and $\alpha^2+\beta^2=1600$. [A. U. '18]

9. If one root of the equation $x^2-px+q=0$ be twice the other, show that $2p^2=9q$. [C. U. '37]

10. If one root of the equation $ax^2+bx+c=0$ be four times the other, show that $4b^2=25ac$. [C. U. 1940]

11. If the roots of the equation $ax^2+bx+c=0$ bear to one another the ratio of 3 : 4, prove that $12b^2=49ac$. [C. U. 1945]

12. Find the value of

(i) $x^3-7x^2+13x-10$, when $x=3+\sqrt{2}$

(ii) $x^3-10x^2+35x-30$, when $x=4+\sqrt{-3}$

(iii) $x^4+7x^3+21x^2+29x+19$, when $x=\sqrt{-3}-2$

(iv) $4x^3-24x^2+49x-40$, when $x=\frac{1}{2}(3+2\sqrt{-1})$.

13. Form the quadratic equation with rational coefficients one of whose roots is $4-\sqrt{5}$.

14. If the roots of $lx^2+mx+m=0$ be in the ratio $p:q$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{m}{l}}=0$. [C. U. '48]

15. If α and β be the roots of $x^2+px+q=0$, show that $\frac{\alpha}{\beta}$ is a root of the equation $qx^2-(p^2-2q)x+q=0$. [C. U. '31]

16. If p and q be the roots of the equation $3x^2+6x+2=0$, show that the equation whose roots are $\frac{-p^2}{q}$ and $\frac{-q^2}{p}$ will be $3x^2-18x+2=0$. [C. U. '54]

17. If α, β be the roots of $ax^2+bx+c=0$, show that the equation whose roots are $\frac{1}{\alpha+\beta}$ and $\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$ is $bcx^2+(b^2+ac)x+ab=0$. [P. U. '37 ; H. S. '65]

18. If α and β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are

(i) $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$. [P. U. '44] (ii) $\frac{1}{\alpha}+\beta^{-1}$ and $\alpha\beta$.

(iii) $2\alpha-\beta$ and $2\beta-\alpha$. [C. U. '28]

19. If α and β are the roots of $3x^2-6x+4=0$, find the value of $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)+2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+3\alpha\beta$. [C. U. '43]

20. If α, β be the roots of the equation $21x^2-x-2=0$, form the equation whose roots are $\alpha\beta$ and $\alpha^2+\beta^2$.

21. If p, q denote the roots of $2x^2-5x+2=0$, find the equation whose roots are $p+q$ and $\frac{1}{2}pq$. [C. U. 1919]

22. If α and β be the roots of $x^2+x+1=0$, form the equation whose roots are α^2 and β^2. Explain why you get the same equation as the given one. [A. U. '24]

23 If α, β be the roots of the equation $x^2+x+1=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. Account for the identity of the equation thus obtained with the original equation. [C. U. '22]

24. Form the quadratic equation whose roots are the reciprocals of the roots of $ax^2+bx+c=0$. [C. U. 1944]

25. Prove that the roots of $2x^2+3x+4=0$ are the reciprocals of the roots of the equation $4x^2+3x+2=0$. [U. P. '33]

26. Find the equation whose roots are twice the reciprocals of the roots of the equation $ax^2+2bx+4c=0$. [U. P. '42]

27. If the sum of the roots of $ax^2+bx+c=0$ be equal to the sum of their squares, then prove that $2ac=ab+b^2$. [Ajmir, '50]

28. Find the relation between a, b and c, if one root of the equation $ax^2+bx+c=0$ be greater than three times the other root by unity.

29. If the ratio of the roots of $ax^2+bx+c=0$, be $m:n$, then prove that $\sqrt{\frac{m}{n}}+\sqrt{\frac{n}{m}}=\sqrt{\frac{b^2}{ac}}$. [U. P. 1958]

30. If the ratio of the roots of the equation $x^2+px+q=0$ be equal to the ratio of the roots of $x^2+lx+m=0$, prove that $p^2m=l^2q$. [Raj. '55]

31. If α and β be the roots of $x^2-(1+k^2)x+\frac{1}{2}(1+k^2+k^4)=0$, show that $\alpha^2+\beta^2=k^2$. [C. U. 1909]

32. Find the relation which exists between a, b, c when one root of $ax^2-bx+c=0$ is five times the other. [C. U. '55]

33. If one root of the equation $x^2+px+q=0$ be the square of the other, show that $p^3-q(3p-1)+q^2=0$. [C. U. '43]

34. If the roots of $x^2+Px+q=0$ be α, β and the roots of $x^2+px+Q=0$ be γ, δ, find the roots of $x^2+px+q=0$ in terms of α, β, γ, δ. [C. U. '14]

35. Express the roots of the equation $q^2x^2-(p^2-2q)x+1=0$ in terms of those of $x^2+px+q=0$. [C. U. '32]

36. Prove that the arithmetic mean of the roots of $x^2-2ax+b^2=0$ is the geometric mean of the roots of $x^2-2bx+a^2=0$ and vice-versa. [C. U. '52]

37. In a quadratic equation of the form $x^2+px+q=0$, the absolute term is misprinted 40 for 24 and the roots are therefore obtained as 4 and 10. Find the roots of the correct equation.

38. Find the condition that the roots of the quadratic equation $ax^2+bx+c=0$ should be

(*i*) equal in magnitude but opposite in sign.

(*ii*) both negative. (*iii*) both positive,

39. For what condition will the equation $x^2+px+q=0$ have

(*i*) one root zero ? (*ii*) both the roots zero ?

40. Find the equation

(*i*) whose roots are each less by 3 than those of $x^2-14x+48=0$;

(*ii*) whose roots are each greater than those of $x^2-10x+21=0$ by unity.

41. If α, β and γ, δ are the roots of $x^2-2px+q^2=0$ and $x^2-2rx+s^2=0$ and if $\alpha\delta=\beta\gamma$, prove that $p^2s^2=q^2r^2$.

42. If the difference of the roots of $x^2-px+q=0$ be unity, show that $p^2+4q^2=(1+2q)^2$. [A. U. '17]

43. If the difference of the roots of the equation $x^2-px+q=0$ be the same as that of the equation $x^2-qx+p=0$, show that $p+q+4=0$, unless $p=q$. [C U. '41]

44. If the roots of $ax^2+bx+c=0$, be the square roots of the roots of the equation $px^2+qx+r=0$, show that $pb^2+qa^2=2apc$ and $pc^2=a^2r$.

45. If α, β be the roots of the equation $ax^2+bx+c=0$ and α_1, β_1 be the roots of $a_1x^2+b_1x+c_1=0$, form the equation having the roots $\alpha\alpha_1+\beta\beta_1$ and $\alpha\beta_1+\alpha_1\beta$.

46. If one root of the equation $ax^2+bx+c=0$ be square of the other, then prove that $b^3+ac^2+a^2c=3abc$. [U. P. '53]

47. If the ratio of the roots of $ax^2+bx+c=0$ be equal to that of the roots of $a_1x^2+b_1x+c_1=0$, then prove that $b^2 : b_1^2 :: ac : a_1c_1$.

48. If the roots of $a_1x^2+b_1x+c_1=0$ differ from those of $a_2x^2+b_2x+c_2=0$ by a constant, show that

$$\frac{b_1^2-4a_1c_1}{a_1^2}=\frac{b_2^2-4a_2c_2}{a_2^2}.$$

49. The sum of the roots of a quadratic equation is 2 and the sum of their cubes is 27. Find the equation. [C. U. '56]

50. If α and β be the roots of $ax^2+bx+c=0$, show that

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

5. Nature of the roots of a quadratic equation.

(A) In the equation $ax^2+bx+c=0$, we suppose that a, b and c are real. Solving it we have $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Here the two roots are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

The character of the roots depends upon the value of b^2-4ac.

(i) **If $\mathbf{b^2-4ac=0}$**, *i.e.*, if $b^2=4ac$, each of the roots reduces to $-\dfrac{b}{2a}$. So in this case both the roots are **real** and **equal.** The roots will be irrational, if either a or b be irrational.

Corollary : From the equation $ax^2+bx+c=0$, we have $\left(x+\dfrac{b}{2a}\right)^2=\dfrac{b^2-4ac}{4a^2}=0$, when $b^2-4ac=0$.

Thus, ax^2+bx+c of the equation $ax^2+bx+c=0$ will be a perfect square, when $b^2-4ac=0$.

(ii) **If $\mathbf{b^2-4ac}$ is positive,** *i.e.*, if $b^2>4ac$ but not a perfect square, $\sqrt{b^2-4ac}$ is irrational and the roots are **real, irrational** and **unequal.**

(iii) **If $\mathbf{b^2-4ac}$ is positive** and *a perfect square*, the roots are **real, rational** and **unequal.** But if either a or b be irrational, the roots will be irrational, even if b^2-4ac be a perfect square.

(iv) If **b^2-4ac is negative,** *i. e.*, if $b^2<4ac$, $\sqrt{b^2-4ac}$ is imaginary and so the roots are **imaginary** and **unequal.**

(v) If **$b=0$**, the roots reduce to $\dfrac{0\pm\sqrt{0-4ac}}{2a}$ or $\pm\sqrt{\dfrac{-c}{a}}$.

Hence the roots are equal but opposite in sign. If a and c are of opposite signs, the roots will be real, but if a and c are of the same sign, the roots will be imaginary.

N. B. As the character of the roots is discriminated (or determined) with the help of b^2-4ac, the quantity b^2-4ac is called the **discriminant.**

The above results may be stated as :

(i) If the *discriminant* be zero, the two roots are real and equal (not different).

(ii) If the discriminant is positive, the roots are real and unequal (different) ;

(iii) If the discriminant is a perfect square, the roots are real, rational and unequal ;

(iv) If the discriminant is negative, the roots are imaginary and unequal.

Corollary : The two roots of a quadratic equation will be real, when its discriminant is positive or zero. This can be expressed in any of the following ways :

(i) The roots are real, when the discriminat $\geqslant 0$.

(ii) The roots are real, when the discriminant $\nless 0$.

(B) The condition that two quadratic equations may have a common root.

Suppose α to be a common root of the equations $ax^2+bx+c=0$ and $a_1x^2+b_1x+c_1=0$. Then α satisfies both the equations.

$\therefore \quad a\alpha^2+b\alpha+c=0$...(i)

and $a_1\alpha^2+b_1\alpha+c_1=0$...(ii)

$\therefore$ By the rule of cross multiplication, we have from (i) and (ii),

$$\frac{\alpha^2}{bc_1-b_1c}=\frac{\alpha}{ca_1-c_1a}=\frac{1}{ab_1-a_1b}\ldots\text{(iii)}$$

$$\therefore\quad\left(\frac{\alpha}{ca_1-c_1a}\right)^2=\frac{\alpha^2}{bc_1-b_1c}\times\frac{1}{ab_1-a_1b},$$

or, $\boldsymbol{(ca_1-c_1a)^2=(bc_1-b_1c)(ab_1-a_1b)}$, which is the required condition.

From (iii) we have

$$\frac{\alpha}{ca_1-c_1a}=\frac{1}{ab_1-a_1b}\text{ and }\frac{\alpha}{bc_1-b_1c}=\frac{1}{ca_1-a_1c}.$$

$\therefore$ The value of the common root is

either $\dfrac{ca_1-c_1a}{ab_1-a_1b}$ or $\dfrac{bc_1-b_1c}{ca_1-a_1c}$. ...(iv)

(C) Function.

$3x^2-4x+9$ is an expression and its value depends on the value of x. It is therefore called a **function of** x. Here the value of x may vary (*i.e.*, may be different),so x is called a **variable** or a variable quantity (or independent variable).

Thus any quantity that involves a variable quantity x is called a function of x and its value depends on that of x.

Such functions are denoted by the symbols $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$ etc.

Suppose $f(x)\equiv 2x^2-3x+5$. Here, if we have to find the value of $f(a)$ or $f(2)$ we have to substitute the definite values a or 2 for x, and the value of the function $f(x)$ thus obtained will be the value of $f(a)$ or $f(2)$.

Just as we obtain the remainder ma^2+na+l on dividing mx^2+nx+l by $x-a$, so also it may be said that when $f(x)=mx^2+nx+l$ is divided by $x-a$, the remainder will be $f(a)$.

When $y=x^2$, then y is said to be a function of x, i.e., $y=f(x)$. It should be noted that only one definite value of y is obtained for any one value of x. Thus if $x=2$ then $y=4$, or if $x=-5$ then $y=25$.

Again, if $y=x^2$, then $x=\pm\sqrt{y}$. But here x is not a function of y $[x\neq f(y)]$, because for any value of y, excepting 0, two values of x are obtained instead of one definite value. $x=\pm\sqrt{y}$ may, however, be separately expressed as two functions, viz., $x=+\sqrt{y}$ and $x=-\sqrt{y}$.

The expression involving two variable quantities is said to be a function of two variables. Thus, $2x^2+3xy+5y^2$ is a function of (x, y), i.e., $f(x, y)$.

Here, for a separate definite value of each of x and y, the value of the expression is definitely obtained.

$f(2, 3)$ signifies the value of $f(x, y)$ obtained by putting 2 for x and 3 for y.

Example. If $f(x, y)=3x^2+2y^2$, find $f(2, 3)$ and $f(3, 2)$.

Here, $f(2, 3)=3.2^2+2.3^2=30$,

and $f(3, 2)=3.3^2+2.2^2=35$.

It may be generally said that in the case of two variables $f(a, b)\neq f(b, a)$; but if the function be a symmetrical expression in x and y, then only $f(a, b)$ may be equal to $f(b, a)$.

Similarly a function may consist of more than two variables.

Note the following examples.

Example 1. If $f(x)\equiv x^2-4$, find the value of $f(-2)$ and $f(2)$. What do the results indicate ?

Here, $f(-2)=(-2)^2-4=0$ and $f(2)=(2)^2-4=0$.

These results indicate that here the expression is exactly divisible by both $x-2$ and $x+2$, i. e., each of $x-2$ and $x+2$ is a factor of the expression.

Example 2. If $f(x)\equiv x^5-61x+p$, find the value of p for which $f(x)$ will be exactly divisible by $x+1$.

Here $f(x)$ will be divisible by $x+1$, if $f(-1)=0$.

Now $f(-1)=(-1)^5-61(-1)+p=60+p$.

$\therefore$ if $f(-1)=0$, then we have $60+p=0$, $\therefore$ $p=-60$.

Examples (1B)

Ex. 1. Discuss the nature of the roots of the equation $4x^2=9$.

The given equation is $4x^2=9$, or, $4x^2-9=0$.

Here, $b=0$, and a and c are of opposite signs. Hence the two roots of the equation are real, equal and opposite in sign.

Ex. 2. Examine the nature of the roots of the equations (i) $2x^2-6x+3=0$ and (ii) $4x^2-12x+9=0$.

(i) Here, the discriminant $(b^2-4ac)=(-6)^2-4\times2\times3=12$.

$\therefore$ The discriminant is positive but not a perfect square.

$\therefore$ The two roots are real, irrational and unequal.

(ii) Here the discriminant $=(-12)^2-4\times4\times9$

$$=144-144=0.$$

$\therefore$ The discriminant is zero and both a and $b(a=4,\ b=-12)$ are rational.

$\therefore$ The two roots are real, rational and equal.

Ex. 3. Discuss the nature of the roots of (i) $x^2-6x+2=0$ and (ii) $x^2-2\sqrt{7}x-2=0$. [G. U. '48]

(i) Here, the discriminant $=(-6)^2-4\times1\times2=36-8=28$.

$\therefore$ the discriminant is positive but not a perfct square.

$\therefore$ the two roots are real, irrational and unequal.

As the product $\frac{c}{a}$ (i.e., 2 here) of the roots is positive, both of them may be either positive or negative, but as their sum (i.e. $-\frac{b}{a}$ or 6 here) is positive so both the roots are positive.

(ii) Here, the discriminant $=(-2\sqrt{7})^2-4\times1\times-2$

$$=28+8=36=(6)^2.$$

$\therefore$ the discriminat is a perfect square, but b is irrational $(b=-2\sqrt{7})$.

$\therefore$ the roots are real, irrational and unequal. One of them is positive and the other is negative, as their product is negative (-2).

Ex. 4. Show that the roots of the equation $3x^2+4x-7=0$ are rational.

Here, the discriminant $=(4)^2-4\times3\times-7=16+84$
$=100$, which is a perfect square.

$\because$ the discriminant is a perfect square and each of the coefficients of x^2 and x is rational,

$\therefore$ the two roots are rational.

Ex. 5. Show that the equation $2x^2+3x+5=0$ cannot have any real value of x.

Here, the discriminant $=(3)^2-4\times2\times5=-31$ (negative)

Since the discriminant is negative, the two roots must be imaginary. $\therefore$ the equation can have no real value of x.

Ex. 6. Show that the roots of the equation $99x^2+100x=101$ are real. [A. U. '21]

From the given equation we have $99x^2+100x-101=0$.

Here, the discriminant $=(100)^2-4\times99\times-101$
$=10000+39996=49996$ (positive).

$\because$ the discriminant is positive (i.e. >0),

$\therefore$ the roots are real.

Ex. 7. Find p, if $x^2+10x+p=0$ has equal roots.

Here, the discriminant $=(10)^2-4p=100-4p$.

The roots will be equal, if the discriminant is zero.

$\therefore$ $100-4p=0$, or, $4p=100$, $\therefore$ $p=25$, which is the required value.

Ex. 8. Find m, if $x^2-2(5+2m)x+3(7+10m)=0$ have (i) equal roots and (ii) reciprocal roots. [C.U. '36]

(i) Here, the discriminant

$$
\begin{aligned}
&=\{-2(5+2m)\}^2-4\times1\times3(7+10m)\\
&=4(25+20m+4m^2)-4(21+30m)\\
&=4(4m^2+20m+25-30m-21)\\
&=4(4m^2-10m+4)=8(2m^2-5m+2)\\
&=8(m-2)(2m-1).
\end{aligned}
$$

The roots of the given equation will be equal, if the discriminant is zero.

$\therefore$ We have $8(m-2)(2m-1)=0$, or, $(m-2)(2m-1)=0$,

$\therefore$ $m=2$ or $\frac{1}{2}$.

Hence $m=2$ or $\frac{1}{2}$, if the roots are equal.

(ii) The equation has reciprocal roots when the coefficient of x^2 and the absolute term are equal.

Here the coefficient of $x^2=1$ and the absolute term$=3(7+10m)$.

$\therefore$ we have $3(7+10m)=1$, or, $30m=-20$, $\therefore m=-\frac{2}{3}$.

$\therefore$ $m=-\frac{2}{3}$, if the roots are reciprocal.

Ex. 9. Are the roots of $x^2-2\sqrt{3}x-13=0$ free from surds though the part of the roots arising from $\sqrt{b^2-4ac}$ is not a surd? If not, why not? [Cf. C.U. '46]

Here, the discriminant$=(-2\sqrt{3})^2-4\times1\times-13=12+52=64$
$=(8)^2$ which is a perfect square.

But b is irrational, it being equal to $-2\sqrt{3}$.

Hence the roots are not free from surds, and they are irrational.

Ex. 10. Solve (*a*) $x^2-2\sqrt{17}x-8=0$ and (*b*) $x^2-10x+8=0$. Distinguish between the roots of the two equations and try to account for the difference. [C. U. '47]

(*a*) Here $x=\dfrac{2\sqrt{17}\pm\sqrt{(-2\sqrt{17})^2-4\times1\times-8}}{2}$

$$=\frac{2\sqrt{17}\pm\sqrt{100}}{2}=\frac{2\sqrt{17}\pm10}{2}=\sqrt{17}\pm5.$$

(*b*) Here $x=\dfrac{10\pm\sqrt{10^2-4\times1\times8}}{2}=\dfrac{10\pm\sqrt{68}}{2}=\dfrac{10\pm2\sqrt{17}}{2}$

$$=5\pm\sqrt{17}.$$

Now, (i) the roots of each of the two equations are real, irrational and unequal.

The discriminant of equation $(a)=10^2$.

Here the roots are irrational, even though the discriminant is a perfect square, for the reason that here the coefficient of x (i.e. $-2\sqrt{17}$) is irrational.

In equation (b), $\sqrt{b^2-4ac}=\sqrt{68}$, which is irrational. Hence the roots are irrational.

(ii) Of the first equation, one root $(\sqrt{17}+5)$ is positive and the other root $(\sqrt{17}-5)$ is negative, while both the roots $(5+\sqrt{17})$ and $(5-\sqrt{17})$ of the second equation are positive.

In the first case the sum of the roots is positive (as the numerically greater root is positive here), and the product of the roots is negative, *i.e.*, $-\dfrac{b}{a}$ is positive and $\dfrac{c}{a}$ is negative. This is possible when b and c have the same sign and a is opposite to them in sign.

Here, the coefficient of x^2 is positive, and the coefficient of x and the absolute term are both negative.

In the second case, both the sum and the product of the roots are positive, *i.e.*, $-\dfrac{b}{a}$ and $\dfrac{c}{a}$ are both positive. It is possible when b and a are opposite in sign and c and a have the same sign, i.e., when a and c have the same sign and b is opposite to them in sign.

Here, the coefficient of x^2 and the absolute term are both positive, and the coefficient of x is opposite in sign.

Ex. 11. Show that the roots of the equation $(x+a)(x+b)=abx^2$ are always real, if a and b are real.

Here, $(x+a)(x+b)=abx^2$, or, $x^2+(a+b)x+ab-abx^2=0$,

or, $(1-ab)x^2+(a+b)x+ab=0$.

$\therefore$ the discriminant $=(a+b)^2-4ab\,(1-ab)$

$=(a+b)^2-4ab+4a^2b^2=(a-b)^2+(2ab)^2$, in which each term is a perfect square and both a and b are real, so the discriminant cannot be negative.

Hence the roots of the given equation must be always real.

Ex. 12. Prove that the roots of $(b+c)x^2-(a+b+c)x+a=0$ are rational, if a, b, c are real and rational. [U. U. '47]

Here the discriminant $=\{-(a+b+c)\}^2-4a(b+c)$

$=\{a+(b+c)\}^2-4a(b+c)=\{a-(b+c)\}^2$

$=(a-b-c)^2$, which is a perfect square, and besides this the coefficients of x^2 and x are rational.

Hence, the roots of the given equation are rational.

Ex. 13. Show that the roots of $ax^2+bx+c=0$ are rational, if $a+b+c=0$, where a, b, c are real and rational.

Here the discriminant $=b^2-4ac$

$=\{-(a+c)\}^2-4ac \quad [\because \; a+b+c=0, \; \therefore \; b=-(a+c)]$

$=(a+c)^2-4ac=(a-c)^2$, this is a perfect square and a and c are real.

Moreover, the coefficient of x^2 and x are rational.

Hence the roots of the given equation are real.

Ex. 14. If l, m, n are real, prove that the roots of the equation $(x-l)(x-m)+(x-m)(x-n)+(x-n)(x-l)=0$ are always real and cannot be equal unless $l=m=n$

Simplifying the given equation we have

$3x^2-2(l+m+n)x+(lm+mn+nl)=0.$

$\therefore$ the discriminant $=\{-2(l+m+n)\}^2-4\times3\times(lm+mn+nl)$

$$\begin{aligned}
&=4(l+m+n)^2-12(lm+mn+nl)\\
&=4\{(l+m+n)^2-3(lm+mn+nl)\}\\
&=4\{l^2+m^2+n^2-lm-mn-nl\}\\
&=2\{2l^2+2m^2+2n^2-2lm-2mn-2nl\}\\
&=2\{(l^2-2lm+m^2)+(m^2-2mn+n^2)+(n^2-2nl+l^2)\}\\
&=2\{(l-m)^2+(m-n)^2+(n-l)^2\}
\end{aligned}$$

$\because$ l, m, n are real, $\therefore$ $(l-m)^2$, $(m-n)^2$ and $(n-l)^2$ are not negative.

$\therefore$ for the real values of l, m and n, the discriminant of the given equation can never be negative.

Hence, its roots are always real.

Again, the roots will be equal, if the discriminant is zero, i. e., if $2\{(l-m)^2+(m-n)^2+(n-l)^2\}=0$. But since none of the terms on the left-hand side is negative, their sum cannot be zero, unless each of them is zero.

$$\therefore \quad \left.\begin{array}{l}(l-m)^2=0\\(m-n)^2=0\\(n-l)^2=0\end{array}\right\}, \quad \text{or,} \quad \left.\begin{array}{l}l-m=0\\m-n=0\\n-l=0\end{array}\right\}, \quad \text{or,} \quad \left.\begin{array}{r}l=m\\m=n\\n=l\end{array}\right\}$$

$$\therefore \quad l=m=n.$$

Hence the proposition is proved.

Ex. 15. If the roots of the equation $ax^2+2bx+c=0$ are real and unequal, then show that the roots of the equation $x^2+2(a+c)x+(a^2+2b^2+c^2)=0$ are imaginary, where a, b, c are real.

$\because$ the roots of the equation $ax^2+2bx+c=0$ are real and unequal,

$\therefore$ its discriminant is not negative, i. e., $4b^2-4ac \nless 0$, or, $4(b^2-ac) \nless 0$.

The discriminant of the second equation

$$=4(a+c)^2-4(a^2+2b^2+c^2)=4(2ac-2b^2)$$
$$=-8(b^2-ac)=-2\times 4(b^2-ac), \text{ which is negative}$$
$$[\because \quad 4(b^2-ac) \nless 0.]$$

Hence, the roots of the second equation are imaginary (its discriminant being negative).

Ex. 16. Determine the values of m for which $3x^2+4mx+2=0$ and $2x^2+3x-2=0$ may have a common root. [C. U. '34]

Let α be a common root of the given equations.

Then $3\alpha^2+4m\alpha+2=0$ and $2\alpha^2+3\alpha-2=0$, from which we have by the rule of cross multiplication

$$\frac{\alpha^2}{-8m-6}=\frac{\alpha}{4+6}=\frac{1}{9-8m}, \text{ or, } \left(\frac{\alpha}{10}\right)^2=\frac{\alpha^2}{-(8m+6)}\times\frac{1}{(9-8m)}$$

$$\therefore \quad -(8m+6)(9-8m)=10^2$$

or, $(8m+6)(8m-9)=100$, or, $64m^2-24m-154=0$,

or, $32m^2-12m-77=0$, or, $32m^2-56m+44m-77=0$,

or, $(4m-7)(8m+11)=0$,

$\therefore$ $m=\frac{7}{4}$ or $-\frac{11}{8}$.

Hence, the equations will have a common root, when $m=\frac{7}{4}$ or $-\frac{11}{8}$.

Ex. 17. Show that the equations $(b-c)x^2+(c-a)x+(a-b)=0$ and $(c-a)x^2+(a-b)x+(b-c)=0$ have a common root.

The first equation is $(b-c)x^2+(c-a)x+(a-b)=0$

or, $(b-c)x^2-\{(b-c)+(a-b)\}x+(a-b)=0$

or, $(b-c)x^2-(b-c)x-(a-b)x+(a-b)=0$

or, $(b-c)x(x-1)-(a-b)(x-1)=0$

or, $(x-1)\{(b-c)x-(a-b)\}=0$.

The second equation is $(c-a)x^2+(a-b)x+(b-c)=0$,

or, $(c-a)x^2-\{(c-a)+(b-c)\}x+(b-c)=0$

or, $(c-a)x(x-1)-(b-c)(x-1)=0$

or, $(x-1)\{(c-a)x-(b-c)\}=0$.

Hence the given equations have a common root 1

[$\because$ in both cases $x-1=0$].

Ex. 18. If the equations $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root, then either $a+b+c=0$, or, $a=b=c$.

If possible, let the equations have a common root α. Then both the equations are satisfied by α.

$\therefore$ $a\alpha^2+b\alpha+c=0$......(1)

and $b\alpha^2+c\alpha+a=0$......(2)

From (*i*) and (*ii*) we have by the rule of cross multiplication

$$\frac{\alpha^2}{ab-c^2}=\frac{\alpha}{bc-a^2}=\frac{1}{ca-b^2}.$$

$\therefore$ $(bc-a^2)^2=(ab-c^2)(ca-b^2)$,

or, $b^2c^2-2a^2bc+a^4=a^2bc-c^3a-ab^3+b^2c^2$,

or, $a(a^3+b^3+c^3-3abc)=0$.

Now, $\because\ a\neq 0,\ \therefore\ a^3+b^3+c^3-3abc=0,$

or, $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=0.$

$\therefore$ Either $a+b+c=0$, or, $(a-b)^2+(b-c)^2+(c-a)^2=0$...(2)

But (2) is possible, if $a=b=c$ [See Ex. 14]

Hence, if the equations have a common root, then either $a+b+c=0$ or, $a=b=c$.

Ex. 19. Let $f(x)\equiv ax^2+bx+c=(x-\alpha)Q+R$, where $f(x)$ is the quadratic expression defined above. Q is the quotient and R the remainder when $f(x)$ is divided by $x-\alpha$, α being a number.

(a) Show that $Q=a(x+\alpha)+b$ and $R=f(\alpha)$.

(b) If α is a root of the quadratic equation $f(x)=0$, deduce from the above values of Q and R the following :—

(i) $f(x)$ is exactly divisible by $x-\alpha$,

(ii) $f(x)=0$ has two roots, distinct or equal and that

(iii) the sum of the roots is $-\dfrac{b}{a}$. [C. U. '51]

(a) $\because\ f(x)\equiv ax^2+bx+c=(x-\alpha)Q+R$......(1)

$\therefore\ f(\alpha)=a\alpha^2+b\alpha+c=R.$

Again, from (1) we have $(x-\alpha)Q=ax^2+bx+c-R$

$=ax^2+bx+c-a\alpha^2-b\alpha-c=a(x^2-\alpha^2)+b(x-\alpha)$

$=(x-\alpha)\{a(x+\alpha)+b\}$

$\therefore\ Q=a(x+\alpha)+b$ and $R=f(\alpha)$.

(b) (i). α is a root of the quadratic equation $f(x)=0$,

so it will be satisfied by α. $\therefore\ f(\alpha)=0.$

But $R=f(\alpha)$, $\therefore\ R=0.$

Hence $f(x)$ is exactly divisible by $x-\alpha$.

(ii) $\because\ f(x)=(x-\alpha)Q+R$, $\therefore$ when $f(x)=0$,

then $(x-\alpha)Q+R=0$ and $f(\alpha)=0$.

But $R=f(\alpha)=0$ and $Q=a(x+\alpha)+b$.

$\therefore \quad (x-\alpha)\{a(x+\alpha)+b\}=0,$

from the second factor $a(x+\alpha)+b=0$, or, $ax=-a\alpha-b$.

$$\therefore \quad x=-\alpha-\frac{b}{a}.$$

$\therefore$ the roots of the equation $f(x)=0$ are α and $-\left(\alpha+\dfrac{b}{a}\right)$.

If $b\neq 0$, then the roots will be distinct, but if $b=0$, then the absolute values of the roots will be equal.

(iii) The sum of the roots $=\alpha+\left\{-\left(\alpha+\dfrac{b}{a}\right)\right\}=\alpha-\alpha-\dfrac{b}{a}=-\dfrac{b}{a}$.

Exercise 1(B)

1. Examine the nature of the roots of the following equations :—

(a) $x^2-7x+12=0$; (b) $4x^2=25$; (c) $4x^2-4x+1=0$; (d) $2x^2-x+2=0$ and (e) $x^2-6x+2=0$.

2. Show that the roots of the equation $4x^2+25x=81$ are real.

3. Show that the roots of the equation $63x^2-62x=221$ are rational.

4. Show that the roots of the equation $2x^2-3x+4=0$ are imaginary.

5. If $x^2+7x+p=0$ have equal roots, find p.

6. One root of $x^2+ax+8=0$ is 4, while the equation $x^2+ax+b=0$ has equal roots. Find b. [C. U. '58]

7. (a) For what value of p will the equation

$x^2-2(1+3p)x+7(3+2p)=0$ have equal roots ?

(b) For what value of m will the equation

$(m+1)x^2+2(m+3)x+(m+8)=0$ have equal roots ?

(P. U. '44)

8. Show that the roots of $x^2+2(3a+5)x+2(9a^2+25)=0$ are imaginary unless $a=\frac{5}{3}$. (P. U. 37)

9. Show that the roots of $(x-a)(x-b)=b^2$ are always real, if a and b be real.

10. Show that the roots of $(a-b+c)x^2+2cx+(b+c-a)=0$ are rational, where a, b, c are real and rational.

11. If the roots of the equation $x^2-2cx+ab=0$ be real and unequal, prove that the roots of $x^2-2(a+b)x+(a^2+b^2+2c^2)=0$ will be imaginary and vice versa.

12. Prove that the roots of the equation

$(h^2-ab)x^2+2(gh-af)x+(g^2-ca)=0$ will be equal, if $abc+2fgh-af^2-bg^2-ch^2=0$. (Raj. '45)

13. If the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ be equal, then prove that a, b c will be in $A.\ P.$ (Ajmer '50)

14. If l, m, n are rational and $l+m+n=0$, show that the roots of the equation $(m+n-l)x^2+(n+l-m)x+(l+m-n)=0$ are rational.

15. If $a\neq0$, find the value of c for which

$m^2x^2+2(mc-2a)x+c^2=0$ have equal roots.

16. If the roots of $(a^2-bc)x^2+2(b^2-ca)x+(c^2-ab)=0$ are equal, prove that either $b=0$ or, $a^3+b^3+c^3=3abc$.

17. Prove that if one root of $x^2+px+q=0$ be a root of $x^2-lx+m=0$, its other root is a root of $x^2+(2p+l)x+(p^2+pl+m)=0$.

18. Prove that the roots of $x^2+2(a+c)x+(a^2+2b^2+c^2)=0$ are imaginary, if those of $ax^2+2bx+c=0$ are real.

19. Show that if the roots of the equation $(a^2+b^2)x^2+2(bc+ad)x+(c^2+d^2)=0$ be real, then they are equal. [P. U. '37]

20. Are the roots of the equation $x^2-\sqrt{13}x-3=0$ rational? If not, why not?

21. Solve the equations (i) $x^2-2\sqrt{3}x-22=0$ and (ii) $x^2-10x+22=0$. Distinguish between the roots of the two equations and try to account for the difference.

22. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be either $(pq'-p'q)/(q-q')$ or $(q-q')/(p'-p)$. [C. U. '14]

23. If $x^2+px+q=0$ and $x^2+qx+p=0$, have a common root, show that either $p=q$, or, $p+q+1=0$. [H.S.'68 ; C.U.'39]

24. Prove that if the equations $x^2+lx+m=0$ and $x^2+mx+l=0$ have a common root, their other roots will be the roots of $x^2+x+lm=0$.

25. The equations $x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root. Prove that their other roots will satisfy the equation $x^2+ax+bc=0$. [C. U. '56, '58]

26. If the equations $ax^2+2bx+c=0$ and $a_1x^2+2b_1x+c_1=0$ have a common root, then show that the equation
$(b^2-ac)x^2+(2bb_1-ac_1-a_1c)x+(b_1^2-a_1c_1)=0$ has equal roots.

27. Prove that the roots of $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$ are always real and cannot be equal unless $a=b=c$.

28. Show that, if the roots of $x^2+px+q=0$ be real and different, then those of $2x^2-4(1+q)x+(p^2+2q^2+2)=0$ are imaginary.

29. Show that the roots of the equation
$(a^4+b^4)x^2+4abcdx+(c^4+d^4)=0$ cannot be different, if real. [Pat. '46]

30. Show that for all real values of k, the roots of the equation $\frac{1}{x-k}+\frac{1}{x}+\frac{1}{x-1}=0$ are real.

31. Find the condition so that $y=mx+c$ and $y^2=4ax$ may have equal values of x.

31. (*a*) If a, b, c are in G. P., then the roots of the equation $ax^2+2bx+c=0$ will be equal.

32. If the roots of $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, then show that $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$.

33. Show that the roots of $x^2+px+q=0$ will be rational, if $p=k+\frac{q}{k}$, where p, q and k are rational.

34. Obtain the relations between the roots and the coefficients of the equation $x^2-2px+q=0$. If the said equation has two equal roots, the equation $(1+y)x^2-2(p+y)x+(q+y)=0$ will have its roots real and distinct only when y is negative and p is not unity.

35. Let $f(x)\equiv ax^2+bx+c$, where a, b, c are real numbers.

(*a*) What can you say about the roots of the equation $f(x)=0$, in the following cases ?

(i) $b=0$, $ac\lesseqgtr 0$; (ii) $c=0$, $ab\neq 0$; (iii) $b=c=0$, $a\neq 0$;

(iv) $a=0$, $bc\neq 0$.

(*b*) If $a\neq 0$, show that the roots of $f(x)=0$ are either both real or both imaginary.

(*c*) If x be real, find the least value of $f(x)$ when $a=2$, $b=-4$ and $c=10$. [C. U. '52]

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6. *Factors of a Quadratic expression.*

ax^2+bx+c is the standard form of a quadratic expression in x.

Let α and β be the roots of the equation $ax^2+bx+c=0$.

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

$$\therefore \quad ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\{x^2-(\alpha+\beta)x+\alpha\beta\}$$

$$=a\{x(x-\alpha)-\beta(x-\alpha)\}=a(x-\alpha)(x-\beta).$$

7. *Factors of a Quadratic expression in two unknowns x and y.*

$ax^2+2hxy+by^2+2gx+2fy+c$ is the general form of a quadratic expression in x and y, where a, b, c, f, g, h are each a constant. Such expressions can be resolved into two linear factors under certain conditions relating to the coefficients.

Arranging $ax^2+2hxy+by^2+2gx+2fy+c=0$ as a quadratic in x we have $ax^2+2(hy+g)x+(by^2+2fy+c)=0$.

$$\therefore \quad x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}$$

$\therefore$ By Art. 6 we have $ax^2+2(hy+g)x+(by^2+2fy+c)$

$=a(x-$ one root of $x)(x-$ other root of $x)$

$$=a\left\{x+\frac{(hy+g)-\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}\right\}\times$$

$$\left\{x+\frac{(hy+g)+\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}\right\}$$

Hence, the two factors of the given expression obtained above will be both linear, if $(hy+g)^2-a(by^2+2fy+c)$ is a perfect square, i. e., if $(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$ is a perfect square.

Now, the condition for $(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$ being a perfect square is $\{2(gh-af)^2-4(h^2-ab)(g^2-ac)=0$,

or, $(gh-af)^2=(h^2-ab)(g^2-ac)$,

or, $g^2h^2+a^2f^2-2afgh=h^2g^2-ach^2-abg^2+a^2bc$,

or, $a^2bc+2afgh-a^2f^2-abg^2-ach^2=0$,

or, $abc+2fgh-af^2-bg^2-ch^2=0$, which is the condition required.

8. *The sign of a quadratic expression.*

ax^2+bx+c is the standard form of a quadratic expression. Its sign (positive or negative) is determined from the nature of the roots of the equation $ax^2+bx+c=0$.

We have already seen (Art. 6) that if α and β be the roots of $ax^2+bx+c=0$, then $ax^2+bx+c=a(x-\alpha)(x-\beta)$.

Now, the roots α and β may be (i) real and equal, or (ii) real and unequal, or (iii) imaginary.

(i) *Let α and β be real and equal*, i. e., $\alpha=\beta$,

Then, $ax^2+bx+c=a(x-\alpha)(x-\beta)$

$=a(x-\alpha)(x-\alpha)=a(x-\alpha)^2$

$=a\times$ a positive quantity [$\because$ $(x-\alpha)^2$ is positive]

Hence ax^2+bx+c will have the same sign as a.

(ii) *Let α and β be real and unequal.*

(a) If x is greater than α and β, then $(x-\alpha)$ and $(x-\beta)$ are both positive.

$\therefore$ in this case, $a(x-\alpha)(x-\beta)=a\times$ a positive quantity.

$\therefore$ ax^2+bx+c will have the same sign as a.

(b) If x is less than both α and β, then both $(x-\alpha)$ and $(x-\beta)$ will be negative and so their product will be positive.

$\therefore$ in this case, $ax^2+bx+c=a(x-\alpha)(x-\beta)$

$=a\times$ a positive quantity.

Hence ax^2+bx+c will have the same sign as a.

(c) If x is greater than α but less than β (i. e., if $\alpha<x<\beta$, i. e., if x lies between α and β), then $(x-\alpha)$ is positive and $(x-\beta)$ is negative and so their product $(x-\alpha)(x-\beta)$ is negative.

(d) If x is less than α but greater than β (i.e., if $\beta<x<\alpha$), then $(x-\alpha)$ is negative and $(x-\beta)$ is positive and so their product $(x-\alpha)(x-\beta)$ is negative.

Hence in both the cases (c) and (d)

$ax^2+bx+c=a(x-\alpha)(x-\beta)=a\times$ a negative quantity.

$\therefore$ In both the cases the sign of ax^2+bx+c is opposite to that of a.

(iii) *Let α and β be imaginary.*

Suppose $\alpha=p+iq$ and $\beta=p-iq$.

Now, $ax^2+bx+c=a(x-\alpha)(x-\beta)$

$=a\{x-(p+iq)\}\{x-(p-iq)\}$

$=a\{(x-p)-iq\}\{(x-p)+iq\}$

$=a\{(x-p)^2-(iq)^2\}=a\{(x-p)^2+q^2\}$ [$\because$ $(i)^2=-1$]

$=a\times$ a positive quantity.

$\therefore$ ax^2+bx+c has the same sign as a.

N. B. It is evident from the discussion above that ax^2+bx+c will have the same sign as a for *all real values of* x, but its sign will be opposite to that of a only when the roots of $ax^2+bx+c=0$ are real and unequal and the value of x lies between the roots.

9. *The maximum and the minimum values of the quadratic expression* ax^2+bx+c.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left\{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$$

$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \qquad \ldots \quad \ldots(1)$$

(i) *Suppose a is positive.*

Now, since $\left(x+\frac{b}{2a}\right)^2$ being a perfect square is not negative for all real values of x,

$\therefore \quad a\left(x+\frac{b}{2a}\right)^2 \nless 0$ (is not less than 0).

Hence from (1) we find that the value of ax^2+bx+c can never be less than $\frac{4ac-b^2}{4a}$ $\left[\because \quad a\left(x+\frac{b}{2a}\right)^2 \text{ is not negative.}\right]$

If; however, $x+\frac{b}{2a}=0$, then ax^2+bx+c will be equal to $\frac{4ac-b^2}{4a}$, but not less than it.

Hence, the *minimum or least value* of ax^2+bx+c is $\frac{4ac-b^2}{4a}$ and then $x=-\frac{b}{2a}$.

N.B. In this case there is no limit to the value of ax^2+bx+c and so its maximum value cannot be determined.

(ii) *Suppose a is negative.*

Here, since $\left(x+\frac{b}{2a}\right)^2$ is not negative, $a\left(x+\frac{b}{2a}\right)^2$ must be negative ($\because$ a is negative).

Hence we find from (1) that the value of ax^2+bx+c can never be greater than $\frac{4ac-b^2}{4a}$.

If $x+\frac{b}{2a}=0$, ax^2+bx+c will be equal to $\frac{4ac-b^2}{4a}$, but not greater than it.

Hence, the *maximum or greatest value* of ax^2+bx+c is $\frac{4ac-b^2}{4a}$ and then $x=-\frac{b}{2a}$.

N. B. In this case ax^2+bx+c may be as small as we like, so no definite least value of it can be found.

Examples (2)

Ex. 1. Find the sign of $2x^2-5x+6$ for real values of x.

$$2x^2-5x+6=2(x^2-\tfrac{5}{2}x+3)=2\{(x-\tfrac{5}{4})^2+3-\tfrac{25}{16}\}$$
$$=2\{(x-\tfrac{5}{4})^2+\tfrac{23}{16}\}$$

$\because$ x is real, $\therefore$ $(x-\frac{5}{4})^2$ is not negative.

$\therefore$ $(x-\frac{5}{4})^2+\frac{23}{16}$ is positive.

Hence, the given expression is positive for any real value of x.

Ex. 2. If x be real, show that the least value of $4x^2-4x+1$ is zero and the corresponding value of x is $\frac{1}{2}$. [C. U. '37]

$4x^2-4x+1=(2x-1)^2$, which is a perfect square.

$\therefore$ For real values of x, the value of $(2x-1)^2$ is either 0 or a positive quantity.

$\therefore$ Its least value $=0$.

Again, when $(2x-1)^2=0$, then $2x-1=0$, $x=\frac{1}{2}$.

Ex. 3. Prove that for real values of x, the expression $3x^2-6x+8$ can never be less than 5. [C. U. '35]

$$3x^2-6x+8=3(x^2-2x+1)+5=3(x-1)^2+5.$$

Now, $(x-1)^2$ is not negative for any real value of x.

$\therefore$ the value of $3(x-1)^2+5$ is never less than 5.

Hence, $3x^2-6x+8$ can never be less than 5, and when the least value is 5, then $x=1$.

Ex. 4. For what value of x is $2x^2+5x-3$ negative? [C. U. '50]

$$2x^2+5x-3=2x^2+6x-x-3=(x+3)(2x-1)=2(x+3)(x-\tfrac{1}{2}).$$

When $x<-3$, both the factors are negative, and so their product, i.e., the given expression, is positive.

Again, both the factors are positive when $x>\frac{1}{2}$, and so their product, i.e. the expression, is positive.

When $x=-3$ or $\frac{1}{2}$, the value of the expression is zero. Again, if x is greater than -3 but less than $\frac{1}{2}$, than $(x+3)$ is positive and $(x-\frac{1}{2})$ is negative. So their product, i.e., the expression is negative.

Hence, we find that the given expression is negative for any value of x between -3 and $\frac{1}{2}$.

Again, $2x^2+5x-3=2\{x^2+\frac{5}{2}x-\frac{3}{2}\}$

$$=2\{x^2+\tfrac{5}{2}x+(\tfrac{5}{4})^2-(\tfrac{5}{4})^2-\tfrac{3}{2}\}$$
$$=2(x+\tfrac{5}{4})^2-2\times\tfrac{25}{16}-3=2(x+\tfrac{5}{4})^2-\tfrac{49}{8}.$$

Here, $(x+\frac{5}{4})^2$ is not negative for any real value of x, so $2(x+\frac{5}{4})^2-\frac{49}{8}$ (i.e., the given expression) is never less than $-\frac{49}{8}$.

Hence, the required least value $=-\frac{49}{8}$.

Ex. 5. Show that $\dfrac{x^2-3x+4}{x^2+3x+4}$ lies between 7 and $\frac{1}{7}$, if x be real. [C. U. '40]

Suppose $y=\dfrac{x^2-3x+4}{x^2+3x+4}$.

Then $y(x^2+3x+4)=x^2-3x+4$,

or, $x^2(y-1)+3x(y+1)+4(y-1)=0$, it is a quadratic equation in x.

Since x is real, the discriminat of the equation is not negative.

Here the discriminant $=\{3(y+1)\}^2-16(y-1)^2$

$$=-(7y^2-50y+7)=-7(y-\tfrac{1}{7})(y-7)$$

$\therefore$ x being real, $-7(y-\frac{1}{7})(y-7)\nless 0$ (is not negative)

or, $7(y-\frac{1}{7})(y-7)\ngtr 0$ (is not positive).

Now, if y is less than $\frac{1}{7}$, both the factors $(y-\frac{1}{7})$ and $(y-7)$ are negative, and so their product is positive.

Again, if y is greater than 7, both the factors are positive, so their product is positive.

When $y=\frac{1}{7}$ or 7, then the discriminant is zero.

Again, if y is greater than $\frac{1}{7}$ but less than 7, then $(y-\frac{1}{7})$ is positive and $(y-7)$ is negative and so their product is negative.

$\therefore$ The given expression lies between 7 and $\frac{1}{7}$, if x be real.

Ex. 6. If x be real, prove that $\dfrac{2x^2-2x+4}{x^2-4x+3}$ cannot lie between 1 and -7. [C. U. '44]

Suppose, $y=\dfrac{2x^2-2x+4}{x^2-4x+3}$.

Then, $y(x^2-4x+3)=2x^2-2x+4$,

or, $x^2(y-2)-2x(2y-1)+(3y-4)=0$, this is a quadratic equation in x, and x being real, its discriminant is not negative.

Here, the discriminant $=4(2y-1)^2-4(y-2)(3y-4)$.

x being real $4(2y-1)^2-4(y-2)(3y-4)\nless 0$,

or, $4(y^2+6y-7)\nless 0$,

or, $4(y+7)(y-1)\nless 0$.

Now, if $y<-7$, both the factors $(y+7)$ and $(y-1)$ are negative and so their product is positive. If $y>1$, both the factors are positive and so their product is positive.

When $y=-7$ or 1, the product of the factors is zero.

Again, if y is greater than -7 but less than 1, then $(y+7)$ is positive and $(y-1)$ is negative and so their product is negative.

Hence, evidently, if x is real, the value of y, i.e., the given expression, cannot lie between -7 and 1.

Ex. 7. Show that for real values of x, $\dfrac{2x^2+4x+1}{x^2+4x+2}$ is capable of having all real values. [C. U. '47]

Let $y=\dfrac{2x^2+4x+1}{x^2+4x+2}$. Then $y(x^2+4x+2)=2x^2+4x+1$,

or, $x^2(y-2)+4x(y-1)+(2y-1)=0$, which is a quadratic equation in x.

$\because$ x is real, $\therefore$ the discriminant of the equation, i.e.

$16(y-1)^2-4(y-2)(2y-1)\nless 0$,

or, $8y^2-12y+8\nless 0$, or, $8(y^2-\frac{3}{2}y)+8\nless 0$,

or, $8\{y^2-\frac{3}{2}y+(\frac{3}{4})^2-\frac{9}{16}\}+8\nless 0$, or, $8(y-\frac{3}{4})^2+\frac{7}{2}\nless 0$.

Here $(y-\frac{3}{4})^2$, being a perfect square, is not negative and therefore it cannot be negative for any real value of y.

$\therefore$ $8(y-\frac{3}{4})^2+\frac{7}{2}$ is always positive for any real value of y.

Hence, if x be real, y or the given expression can have all real values.

Ex. 8. Find the limits between which $\dfrac{x^2-x+1}{x^2+x+1}$ must lie for the real values of x.

Suppose $y=\dfrac{x^2-x+1}{x^2+x+1}$. Then $y(x^2+x+1)=x^2-x+1$,

or, $x^2(y-1)+x(y+1)+(y-1)=0$, which is a quadratic equation in x.

$\therefore$ Its dicriminant is not negative for real values of x,

i.e., $(y+1)^2-4(y-1)^2 \nless 0$, or, $-3y^2+10y-3 \nless 0$,

or, $-\{3y^2-9y-y+3\} \nless 0$, or, $-(3y-1)(y-3) \nless 0$,

or, $-3(y-\frac{1}{3})(y-3) \nless 0$, or, $3(y-\frac{1}{3})(y-3) \ngtr 0$ (not positive).

Now, if $y<\frac{1}{3}$, both the factors $(y-\frac{1}{3})$ and $(y-3)$ are negative and so their product is positive.

If $y>3$, both the factors are positive and their product is positive.

When $y=\frac{1}{3}$ or 3, the discriminant is zero.

Again if y is greater than $\frac{1}{3}$ but less than 3, then $(y-\frac{1}{3})$ is positive and $(y-3)$ is negative and so their product is negative.

Hence we find that if x is real, y (i.e., the given expression) can have any value from $\frac{1}{3}$ to 3, which are the required limits.

Ex. 9. Show that the greatest and least values of $\dfrac{6x^2-22x+21}{5x^2-18x+17}$ for all real values of x, are $\frac{5}{4}$ and 1 corresponding to the values 1 and 2 respectively of x. [C. U. '42]

Let $y=\dfrac{6x^2-22x+21}{5x^2-18x+17}$,

then $y(5x^2-18x+17)=6x^2-22x+21$,

or, $x^2(5y-6)-2x(9y-11)+(17y-21)=0$......(1).

It is a quadratic equation in x. $\therefore$ for real values of x, the discriminant of this equation is not negative.

i.e., $4(9y-11)^2-4(5y-6)(17y-21)\nless 0$ (not negative),

or, $-4(4y-5)(y-1)\nless 0$, [simplifying and factorising]

or, $-16(y-\frac{5}{4})(y-1)\nless 0$,

or, $16(y-\frac{5}{4})(y-1)\ngtr 0$ (not positive)

Now, if $y<1$, both the factors $(y-\frac{5}{4})$ and $(y-1)$ are negative and so their product is positive.

Again, if $y>\frac{5}{4}$, both the factors are positive and so their produat is positive.

The discriminant becomes zero, when $y=1$ or $\frac{5}{4}$.

Again, if $y>1$ but $<\frac{5}{4}$, $(y-1)$ is positive and $(y-\frac{5}{4})$ is negative and so their product is negative.

$\therefore$ we find that for real values of x, y may have any value from 1 to $\frac{5}{4}$.

Hence, for real values of x, the minimum value of the given expression is 1 and the maximum value is $\frac{5}{4}$.

Again, putting $y=1$ in equation (1) we have $x^2-4x+4=0$,

or, $(x-2)^2=0$, $\therefore$ $x=2$;

and putting $y=\frac{5}{4}$ in equation-(1) we have $x^2-2x+1=0$,

or, $(x-1)^2=0$, $\therefore$ $x=1$.

$\therefore$ when $x=1$, the maximum value of the given expression is $\frac{5}{4}$, and when $x=2$, its minimum value is 1.

Ex. 10. If x be real and a have any value between 1 and 3, show that $\dfrac{ax^2+x-2}{a+x-2x^2}$ can have any real value.

Suppose $y=\dfrac{ax^2+x-2}{a+x-2x^2}$. Then, $y\ (a+x-2x^2)=ax^2+x-2$

or, $x^2(2y+a)-x(y-1)-(ay+2)=0$, it is a quadratic equation in x. So, x being real, its discriminant is not negative.

i.e., $(y-1)^2+4(2y+a)(ay+2) \nless 0$

or, $(8a+1)y^2+2(2a^2+7)y+(8a+1) \nless 0$,

or, $(8a+1)\{y^2+2.\frac{2a^2+7}{8a+1}y+1\} \nless 0$

or, $(8a+1)\left\{\left(y+\frac{2a^2+7}{8a+1}\right)^2+1-\left(\frac{2a^2+7}{8a+1}\right)^2\right\} \nless 0$.

The above condition will be satisfied, if the conditions $(8a+1) \nless 0$...(1) and $\left\{\left(y+\frac{2a^2+7}{8a+1}\right)^2+1-\left(\frac{2a^2+7}{8a+1}\right)^2\right\} \nless 0$...(2) are satisfied.

Now, from condition-(2) we find that for any value of y, $\left(y+\frac{2a^2+7}{8a+1}\right)^2$ is always $\geqslant 0$, for it is a perfect square.

$\therefore$ The second condition is satisfied when $1-\left(\frac{2a^2+7}{8a+1}\right)^2 \nless 0$

or, if $\left(\frac{2a^2+7}{8a+1}\right)^2 \ngtr 1$, or if $(2a^2+7)^2 \ngtr (8a+1)^2$,

or, if $(2a^2+7)^2-(8a+1)^2 \ngtr 0$,

or, if $(2a^2+8a+8)(2a^2-8a+6) \ngtr 0$,

or, if $4(a^2+4a+4)(a^2-4a+3) \ngtr 0$,

or, if $4(a+2)^2(a-1)(a-3) \ngtr 0$.

Here $(a+2)^2$ being a perfect square is not negative. So, the above condition will be satisfied when $(a-1)(a-3) \ngtr 0$.

Now, following the reasons in Ex. 9, we find that the value of a must lie between 1 and 3 to make the above condition satisfied.

We also find that if a lies between 1 and 3, the first condition, i.e., $(8a+1) \nless 0$ is also satisfied.

Hence, if x be real and if a have any value between 1 and 3, the given expression can have any real value.

Ex. 11. If x and y are two real quantities connected by the equation $x^2+12xy+4y^2-26x-44y+89=0$, then x cannot lie between 4 and 1 and y between $\frac{5}{2}$ and 1.

Expressing the given equation as a quadratic equation in x, we have $x^2+2x(6y-13)+(4y^2-44y+89)=0$.

$\because$ x is real, $\therefore$ the discriminant $4(6y-13)^2-4(4y^2-44y+89) \nless 0$,

or, $4(32y^2-112y+80) \nless 0$, or, $64(2y^2-7y+5) \nless 0$,

or, $64(2y-5)(y-1) \nless 0$, or, $128(y-\frac{5}{2})(y-1) \nless 0$.

[Here give the reasons as before]

Hence we find that if x is real, y will have no value between $\frac{5}{2}$ and 1.

Again, writing the given equation as a quadratic equation in y, we have $4y^2+4y(3x-11)+(x^2-26x+89)=0$.

$\because$ y is real, $\therefore$ the discriminant $16(3x-11)^2-16(x^2-26x+89) \nless 0$,

or, $16(8x^2-40x+32) \nless 0$, or, $128(x^2-5x+4) \nless 0$,

or, $128(x-4)(x-1) \nless 0$.

[Here give the reasons as before]

Hence we find that if y is real, x cannot lie between 4 and 1.

Ex. 12. If $p>1$, then for real values of x the expression $\frac{x^2-2x+p^2}{x^2+2x+p^2}$ lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$.

Let $y=\frac{x^2-2x+p^2}{x^2+2x+p^2}$.

Then $y(x^2+2x+p^2)=x^2-2x+p^2$,

or, $x^2(y-1)+2x(y+1)+p^2(y-1)=0$, it is a quadratic equation in x.

Since x is real, $\therefore$ its discriminant $4(y+1)^2-4p^2(y-1)^2 \nless 0$

or, $-4\{y^2(p^2-1)-2y(p^2+1)+(p^2-1)\} \nless 0$,

or, $-4(p^2-1)\{y^2-2\frac{p^2+1}{p^2-1}y+1\} \nless 0$,

or, $4(p^2-1)\left\{\left(y-\frac{p^2+1}{p^2-1}\right)^2+1-\left(\frac{p^2+1}{p^2-1}\right)^2\right\} \not> 0,$

or, $4(p^2-1)\left\{\left(y-\frac{p^2+1}{p^2-1}\right)^2+\frac{(p^2-1)^2-(p^2+1)^2}{(p^2-1)^2}\right\} \not> 0,$

or, $4(p^2-1)\left\{\left(y-\frac{p^2+1}{p^2-1}\right)^2+\frac{-4p^2}{(p^2-1)^2}\right) \not> 0,$

or, $4(p^2-1)\left\{\left(y-\frac{p^2+1}{p^2-1}\right)^2-\left(\frac{2p}{p^2-1}\right)^2\right\} \not> 0,$

or, $4(p^2-1)\left\{\left(y-\frac{p^2+1}{p^2-1}+\frac{2p}{p^2-1}\right)\left(y-\frac{p^2+1}{p^2-1}-\frac{2p}{p^2-1}\right)\right\} \not> 0,$

or, $4(p^2-1)\left\{y-\frac{(p-1)^2}{p^2-1}\right\}\left\{y-\frac{(p+1)^2}{p^2-1}\right\} \not> 0,$

or, $4(p^2-1)\left(y-\frac{p-1}{p+1}\right)\left(y-\frac{p+1}{p-1}\right) \not> 0,$

Now, $\because$ $p>1$, $\therefore$ p^2-1 is not negative and $\frac{p+1}{p-1}>\frac{p-1}{p+1}$.

[Here give the reasons as in previous examples]

Hence, if x is real, y (i. e., the expression) lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$ where $p>1$.

Ex. 13. Show that the expression $2x^2+xy-8x-6y^2+5y+6$ is resolvable into two linear factors.

$ax^2+2hxy+by^2+2gx+2fy+c$ is a standard quadratic expression and it can be resolved into two linear factors,

if $abc+2fgh-af^2-bg^2-ch^2=0$.

Here, in the given expression, $a=2$, $h=\frac{1}{2}$, $b=-6$,
$g=-4$, $f=\frac{5}{2}$ and $c=6$.

$\therefore$ Here, $abc+2fgh-af^2-bg^2-ch^2=(2\times-6\times6)$
$+(2\times\frac{5}{2}\times-4\times\frac{1}{2})-(2\times\frac{25}{4})-(-6\times16)-(6\times\frac{1}{4})$
$=-72-10-\frac{25}{2}+96-\frac{3}{2}=-96+96=0.$

Hence, the given expression can be resolved into two linear factors.

Ex. 14. For what value of p is the expression $3x^2+7xy-5x-py^2+7y-2$ capable of resolution into two linear factors?

In the given expression $a=3$, $h=\frac{7}{2}$, $b=-p$, $g=-\frac{5}{2}$,

$$f=\frac{7}{2},\ c=-2.$$

$$\therefore\quad abc+2fgh-af^2-bg^2-ch^2=(3\times-p\times-2)$$
$$+(2\times\tfrac{7}{2}\times-\tfrac{5}{2}\times\tfrac{7}{2})-3\times\tfrac{49}{4}+p\times\tfrac{25}{4}+2\times\tfrac{49}{4}$$
$$=6p+\tfrac{25}{4}p-\tfrac{245}{4}-\tfrac{147}{4}+\tfrac{49}{2}=\tfrac{49}{4}p-\tfrac{294}{4}.$$

Now, the given expression is resolvable into two linear factors, if $\frac{49}{4}p-\frac{294}{4}=0$.

$$\therefore\quad \tfrac{49}{4}p-\tfrac{294}{4}=0,\quad \text{or,}\quad \tfrac{49}{4}p=\tfrac{294}{4},\quad \therefore\quad p=6.$$

Ex. 15. Find the two linear factors of the expression $5x^2+13xy-6y^2-7x+13y-6$.

Here, $5x^2+13xy-6y^2=5x^2+15xy-2xy-6y^2$

$$=(x+3y)(5x-2y).$$

Now, suppose $5x^2+13xy-6y^2-7x+13y-6$

$$=(x+3y+a)(5x-2y+b).$$

$\because$ the terms of second degree $(5x^2+13xy-6y^2)$ are the same on both sides

$$\therefore\quad -7x+13y-6\equiv a(5x-2y)+b(x+3y)+ab$$
$$\equiv(5a+b)x+(3b-2a)y+ab$$

$\therefore\quad 5a+b=-7\ldots(1)$, $3b-2a=13\ldots(2)$, and $ab=-6\ldots(3)$

Now, the given expression cannot be resolved into factors unless all the equations (1), (2) and (3) are satisfied by the same values of a and b.

Here solving (1) and (2) we have $a=-2$, $b=3$ and these values satisfy the equation-(3) also.

$\therefore$ the required factors are $x+3y-2$ and $5x-2y+3$.

Exercise 2

If x be real, find the sign of :

1. $2x^2+5x+4$ 2. $12x-3x^2-15$ 3. $4x^2-3x+1$.

4. If x is real, between what values of x will the function $2x^2-11x+14$ be positive ?

5. Find the maximum value of $(1-x)(2+3x)$ for real values of x. [C. U. '46]

6. Prove that $\dfrac{x^2-2x+4}{x^2+2x+4}$ lies between 3 and $\frac{1}{3}$ for real values of x. [C. U. '55]

7. If x be real, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9. [C. U. '54 ; G. U. '48]

8. Find the maximum and the minimum values of $\dfrac{5(x^2-\frac{8}{5}+1)}{x^2+x+1}$ when x is real.

9. Find the limits between which the expression $\dfrac{x^2-4x+9}{x^2+4x+9}$ must lie for real values of x. [C. U. '48]

10. If x is real, prove that $\dfrac{3x^2+38x-85}{x^2+2x-7}$ can have no value between 7 any 11. [G. U. '49]

11. If x be real, show that $\dfrac{x}{x^2-5x+9}$ must lie between 1 and $-\frac{1}{11}$. [C. U. '53 ; P. U. '41]

12. If x be real, show that the value of $\dfrac{(x-2)(x+3)}{(x-3)(x+4)}$ cannot lie between $\frac{25}{49}$ and 1.

13. Show that the expression $\dfrac{p^2}{1-x}-\dfrac{q^2}{1+x}$ is capable of having all real values for real values of x.

14. Prove that $\dfrac{1}{2x+1}+\dfrac{1}{x+1}-\dfrac{1}{(x+1)(2x+1)}$ cannot lie between 1 and 9 for any real values of x.

15. For real values of x, find the greatest and least values of $\dfrac{x^2+14x+9}{x^2+2x+3}$. [H. S. '65 ; P. U. '40]

16. Show that the greatest and least values of $\dfrac{x^2+14x+9}{x^2+2x+3}$ for all real values of x are 4 and -5 corresponding to the values 1 and 2 respectively of x. [Pat. U. '40]

17. Show that the expression $2x^2+3xy+5x-2y^2+5y+3$ can be resolved into two linear factors.

18. Find the two linear factors of $3x^2+7xy-5x-6y^2+7y-2$.

19. Find the values of m which will make $2x^2+mxy+3y^2-5y-2$ equivalent to the product of two linear factors.

20. Find m so that x^2-7x+m and $x^2-13x+3m$ may have a common factor.

21. If the expressions px^2-qx+r and qx^2-rx+p have a common linear factor, prove that either $p=0$, or $p^3+q^3+r^3=3pqr$.

22. Show that x must lie between 3 and 7 and y must lie between -2 and 2, if the equation $y^2+x^2-10x+21=0$ is to be satisfied by real values of x and y.

23. Show that the expression $8x-15-x^2$ can be positive only for values of x which lie between certain limits ; and find these limits.

24. Find the condition that the expresssions $ax^2+2hxy+by^2$ and $a'x^2+2h'xy+b'y^2$ may have factors of the forms $y-mx$ and $my+x$ respectively.

25. Show that if $ax^2+by^2+cz^2+2ayz+2bzx+2cxy$ is resolvable into linear factors, then $a^3+b^3+c^3=3abc$.

26. Show that, for real values of x, the expression ax^2+bx+c has the same sign as a, except when the roots of the equation $ax^2+bx+c=0$ are real and unequal and x lies between them. [C. U. '53]

27. If x be real and p have any value between 1 and 7, show that $\dfrac{px^2+3x-4}{p+3x-4x^2}$ can have any real value.

PERMUTATIONS AND COMBINATIONS

10. Permutation: Each of the different *arrangements*, which can be made out of a given number of things, by taking *some* or *all of them* at a time is called their **permutation.**

Thus taking the letters a and b together, *two* arrangements ab and ba can be made (*i.e.*, first putting b after a and then putting a after b).

Again, let us see how many permutations may be made by taking *two letters* at a time out of the three letters a, b and c.

Here, six arrangements, namely ab, ba, ac, ca, bc, cb can be made.

So, the number of permutations in this case is 6. If arrangements be made taking all the three letters a, b, c at a time, evidently we have 6 permutations abc, acb, bca, bac, cab, cba.

11. Combination: Each of the *groups* or *selections* that can be made by taking at a time some or all of a given number of things (without regard to the *order* of the things in each group) is called a **combination.**

Thus only one selection ab can be made taking the two letters a and b together.

As combination does not depend on the *order* in which the things are arranged, we have only one group here. Thus ab and ba are but one combination.

The combinations of the letters a, b and c by taking two at a time are ab, bc, ca.

There will be only one combination abc by taking the three letters all at a time, for abc, acb, bac, bca, cab, cba are not different but the same combination.

12. Distinction between Permutation and Combination: It has been said before that in permutation the *order* in which the things in each group are arranged is taken into consideration, but in combination only groups are selected *without regard to the order*

of things in each group. In permutation, first the groups are made by taking two or three things at a time out of a number of things and then the things in each group are arranged in all *possible orders*, whereas in combination only the selections of the groups are to be made but the things in each group are not to be arranged in all possible orders.

Let us take the three letters a, b, c. If selections are made taking two at a time out of them, then we have only three different combinations, namely, ab, ac, bc.

But if arrangements (permutations) are to be made taking two at a time out of those three letters, then we first select the three groups ab, ac, bc and then arrange each group in different orders. Thus the group ab can be arranged as ab and ba. Hence we have two permutations ab and ba out of the single combination ab.

Similarly the permutations from the other two groups are ac, ca and bc, cb.

13. An important principle.

'*If one operation can be performed in* **m** *ways and (when it has been performed in any one of these ways) a second operation can then be performed in* **n** *ways, the number of ways of performing the two operations will be* $\mathbf{m \times n}$".

Suppose the first operation is performed in **any one** way (out of the m ways). We can then associate with this one way *any of the* n *ways* of performing the second operation and have n ways of performing the two operations. Thus corresponding to **one** way of performing the first operation, we have $\mathbf{1 \times n}$ ways of performing the two operations.

Similarly, corresponding to each of m ways of performing the first operation, we have n ways of performing the two operations.

Hence, associating the n ways of the second operation with the m ways of the first, we altogether have $m \times n$ ways of performing the two operations.

Example 1. Suppose there are 3 horses at the Howrah station and 4 cars at the Liluah station.

If a man goes from Howrah to Liluah on a horse and returns from Liluah to Howrah by a car, let us see in how many ways he can perform the journeys.

Evidently there are 3 ways of making the first journey, for he can go to Liluah taking any one of the three horses, and for *each* of these 3 ways there are 4 ways of returning to Howrah by any one of the 4 cars. Hence the number of ways of making the two journeys is 3×4 or 12.

[Since there are 3 horses at Howrah, he can perform his journey from Howrah to Liluah in 3 different ways and for each way of going to Liluah, he can perform the return journey in 4 different ways as there are 4 cars at Liluah. Hence the two operations can be done in 3×4 ways.]

N. B. The above principle can be used when there are more than two operations. We have seen that the first and second operations can be together performed in $m \times n$ ways. Now, if it is said that a third operation can be performed in p different ways for each of these $m \times n$ ways, then the number of ways of performing the three operations is $m \times n \times p$.

Example 2. There are 3 seats in a room. Let us see in how many ways the seats can be occupied by 5 men.

Here any one of the 5 men can occupy the first seat. So the first seat can be occupied in 5 ways. Then any one of the remaining 4 men can occupy the second seat. So for each way of occupying the first seat, the second can be occupied in 4 ways. Hence the first and the second seats can be occupied in 5×4 different ways.

Now the first two seats being occupied in any one of the 5×4 ways, there remain 3 men and one vacant seat. So any one of the 3 men can occupy the third seat, i. e., the third seat can be filled in 3 ways for each of the 5×4 ways in which the first two seats can be occupied.

Hence the total number of ways required $= 5 \times 4 \times 3 = 60$.

PERMUTATION OF THINGS ALL DIFFERENT

14. *To find the number of permutations of n different things taken r at a time (where $r<n$ or $r=n$).*

Suppose there are n marbles of different colours and r vacant holes.

The number of permutations of n different things taken r at a time will be the same as the number of ways in which we can fill up the r holes by n marbles. Let us see in how many ways this can be done.

Now, we can place *any one* of the n marbles in the first hole, so the first hole can be filled up in n different ways.

The first hole being filled up by any one of the n marbles, there remain $(n-1)$ marbles. So the second hole can be filled up by any one of these $(n-1)$ marbles and therefore the second hole can be filled up in $(n-1)$ ways. Since each way of filling up the first hole can be associated with each way of filling up the second, the first two holes can be filled up in $n(n-1)$ different ways. When the first two holes have been filled up, the third hole can be filled up by any one of the remaining $(n-2)$ marbles, so the third hole can be filled up in $(n-2)$ different ways.

Since each of these $(n-2)$ ways can be associated with each of the $n(n-1)$ ways of filling up the first two holes the first three holes can be filled up in $n(n-1)(n-2)$ ways.

So far three holes have been filled up and we find that there are as many factors in the number of ways as the number of holes filled up. We also notice that as we proceed filling up a new hole from the first, the number of factors goes on increasing by a new factor and each new factor is less by 1 than its preceding factor,

Hence the number of ways in which the r holes can be filled up $=n(n-1)(n-2)$......to r factors,

Here the rth factor $=\{n-(r-1)\}=n-r+1$.

$\therefore$ The required number of permutations

$$=n(n-1)(n-2)(n-3)\text{......to } r \text{ factors}$$
$$=n(n-1)(n-2)(n-3)\text{......}(n-r+1).$$

Corollary : In the above example, if all the n things be taken at a time, the number of premutations

$$=n(n-1)(n-2)......\text{to } n \text{ factors}=n(n-1)(n-2)......3.2.1.$$

[Here the nth factor $=n-(n-1)=1$.]

15. Symbol : (i) The product of the natural numbers 1, 2, 3, etc. up to n is denoted by the symbol $\underline{|n}$ (or, $n\,!$).

This symbol is read as '**factorial n**'.

Thus, $\underline{|5}=5.\ 4.\ 3.\ 2.\ 1$;

$6\,!=6.\ 5.\ 4.\ 3.\ 2.\ 1$, etc.

(ii) The number of permutations of n things taken r at a time is represented by the symbol ${}^nP_r$ or ${}_nP_r$. The number of permutations of 8 things taken 3 at a time is denoted by ${}^8P_3$. Similarly, the number of permutations of n things taken all at a time is denoted by ${}^nP_n$.

$\therefore\quad {}^nP_n=\underline{|n}.\quad {}^nP_r=n(n-1)(n-2)......(n-r+1).$

[N. B. ${}^nP_r$ is called an r-permutations as it denotes the number of permutations of n things taken r at a time. Similarly if 3 things are taken at a time, the arrangements are called 3-permutations. In the symbol ${}^nP_r$, r also represents the *number of factors* in the permutations.]

16. Some Corollaries :

(1) $\underline{|n}=n\underline{|n-1}.$

Proof : $\underline{|n}=n(n-1)(n-2)......3.2.1.$

Again, $\underline{|n-1}=(n-1)(n-2)......3.2.1.\quad \therefore\quad \underline{|n}=n\,\underline{|n-1}.$

Similarly $\underline{|n}=n(n-1)\underline{|n-2}$, etc.

(2) ${}^nP_r$ can be expressed by factorial symbol.

${}^nP_r=n(n-1)(n-2)......(n-r+1)$ [already proved]

Multiplying and dividing the righthand side by

$(n-r)(n-r-1)...3.2.1.$ we have

$${}^nP_r=\frac{\{n(n-1)(n-2)...(n-r+1)\}(n-r)(n-r-1)...3.2.1}{(n-r)(n-r-1)...3.2.1}.$$

Here, the numerator = the product of $1 \times 2 \times 3 \times \ldots\ldots$ to n factors $= \lfloor n$, and the denominator $= 1 \times 2 \times 3 \times \ldots \times (n-r) = \lfloor n-r$.

$$\therefore \quad {}^nP_r = \frac{\lfloor n}{\lfloor n-r}.$$

(3) **The meaning of $\lfloor 0$.**

Though strickly speaking $\lfloor 0$ has no meaning according to the definition of permutation, $\lfloor 0$ is regarded simply as a symbol having the value 1, *i.e.*, $\lfloor 0 = 1$.

Proof : From the corollary-(2) above we have

$${}^nP_n = \frac{\lfloor n}{\lfloor n-n} = \frac{\lfloor n}{\lfloor 0}. \quad \text{Again } {}^nP_n = \lfloor n \quad \text{[Proved]}$$

$$\therefore \quad \lfloor n = \frac{\lfloor n}{\lfloor 0}, \quad \therefore \quad \lfloor 0 = \frac{\lfloor n}{\lfloor n} = 1.$$

(4) **The meaning of $\dfrac{1}{\lfloor -r}$.**

Stricktly speaking $\dfrac{1}{\lfloor -r}$ is meaningless as per definition, but it is regarded as a symbol having the value 0.

Proof : $$\frac{\lfloor n}{\lfloor n-r} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)\lfloor n-r}{\lfloor n-r}$$
$$= n(n-1)(n-2)\ldots\ldots(n-r+1)\,;$$

Now putting 0 for n we have $\dfrac{\lfloor 0}{\lfloor -r} = 0$, but $\lfloor 0 = 1$, $\therefore \dfrac{1}{\lfloor -r} = 0$.

(5) $\mathbf{{}^nP_n = {}^nP_{n-1}}$.

Proof : ${}^nP_n = \lfloor n$ [proved]

Again, $${}^nP_{n-1} = \frac{\lfloor n}{\lfloor n-(n-1} = \frac{\lfloor n}{\lfloor 1} = \lfloor n. \quad \therefore \quad {}^nP_n = {}^nP_{n-1}.$$

(6) The number of permutations of n things taking at will 1, 2, 3,...up to n things at a time $= {}^nP_1 + {}^nP_2 + {}^nP_3 + \ldots\ldots + {}^nP_n$.

PREMUTATIONS OF THINGS NOT ALL DIFFERENT

17. *To find the number of permutations of n things taken all together when the things are not all different.*

Let n letters be the n given things. Suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be all different or unlike. Let x be the required number of permutations.

Now, if in any one of the x permutations, all the p number of a's be changed into p *unlike* letters *different from any of the rest* and only the arrangement of these p new letters be changed among themselves without altering the position of any of the remaining letters, then $\lfloor\underline{p}$ new permutations will be formed from each one of the x permutations.

Hence, if this change is made in all the x permutations, we have $x \times \lfloor\underline{p}$ permutations.

Similarly, if in each of the new $x\lfloor\underline{p}$ permutations, the q number of b's be changed into q new letters different from each other and from any of the rest and if the order of these new q letters only be changed without altering the order of any other letter, we have $\lfloor\underline{q}$ permutations from each of the $x\lfloor\underline{p}$ permutations. Hence the total number of permutations will then be $x \times \lfloor\underline{p} \times \lfloor\underline{q}$.

In the same manner, by changing the r number of c's into r unlike letters different from the rest, the total number of permutations will be $x \times \lfloor\underline{p} \times \lfloor\underline{q} \times \lfloor\underline{r}$.

Now, the p a's, q b's and r c's are all changed into all different letters and we now have n letters all different.

$\because$ the permutations of these n letters taken all at a time $= \lfloor\underline{n}$,

$$\therefore \quad x \times \lfloor\underline{p} \times \lfloor\underline{q} \times \lfloor\underline{r} = \lfloor\underline{n},$$

$$\therefore \quad x = \frac{\lfloor\underline{n}}{\lfloor\underline{p}\,\lfloor\underline{q}\,\lfloor\underline{r}}.$$

[N. B. The above method is also applicable if there be more than three kinds of things, i.e., in any case where the things are not all different.]

PERMUTATIONS INVOLVING REPETITIONS

[Each thing may be repeated once, twice, etc.]

18. *To find the total number of r-permutations of n differnt things taken r at a time, when each thing may be repeated up to r times in any arrangement.*

Let the n things be n different kinds of letters (there being not less than r of each kind). Suppose there are r blank places.

Here the required number of permutations will be equal to the number of ways in which the r places can be filled up by n kinds of different letters, each being taken once, twice, thrice,... up to r times.

The first place can be filled up by any one of the letters and so it can be filled up in n ways. When the first place has been filled up in any one way, the second place also can be filled up in n ways, for we may use the same letter again. Hence the first two places can be filled up in $n \times n$ or n^2 ways.

The third place also can be filled up in n ways and therefore the first 3 places can be filled up in n^3 ways.

Thus we find that at any stage the index of n is always the same as the number of places filled up.

Proceding in this way we have n^r as the number of ways in which the r places can be filled up.

$\therefore$ The required number of permutations $= \mathbf{n^r}$.

[**N. B.** Here, the number of things in any one kind of the different n kinds of things must not be less than r, since each thing is permitted to be repeated r times.]

PERMUTATION IN A CIRCLE

19. If a number of things be placed in a row, the arrangement is called a *linear* arrangement and if the things are placed in a circle (or ring), the arrangement is called a circular arrangement.

Distinction. A linear arrangement has two ends, while there are no such ends in a circular arrangement. Again, a linear arrangement depends on the *absolute* positions in which the things are placed, while a circular arrangement depends only on their *relative* positions, i.e., on the positions of the things relative to each other.

Thus *abcd*, *bcda*, *cdab*, *dabc* are four different linear arrangements, but there is no essential difference when they are regarded as circular arrangements. If the four letters a, b, c, d be read in *cyclic order* beginning from each letter successively, the above four arrangements are obtained. Hence, here we have four linear arrangements from each circular arrangement.

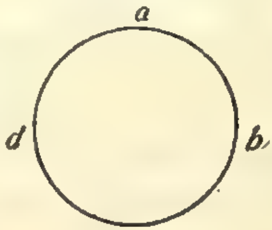


Fig. A

20. Two circular arrangements in which the same relative positions of the things are maintained, are regarded as the same. But they are regarded as different, if the relative positions of the things are different in them.

Thus, though the letters a, b, c, d in Fig. B are not placed exactly in the same places as in Fig. A, they maintain the same order in both, i.e., their relative positions are the same. For in both the figures, if the letters are read clockwise in a cyclic order, starting from any one of them (a or b or c or d), the letters will be obtained in the same order in both the cases. Hence the arrangements are the same here.

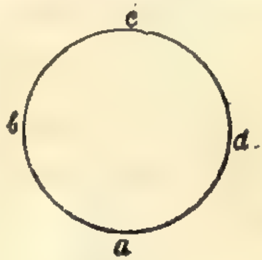


Fig. B

Again in fig. C, the relative positions of the letters a, b, c, d are not the same as in figs. A and B, for if the letters are read in cyclic order in fig. A starting from a, we have *abcd*, but from fig. C we have *adbc*. Hence the third arrangement is different from the first two.

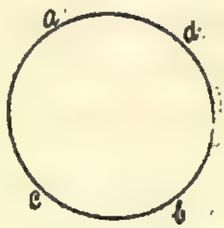


Fig. C

We, therefore, come to the conclusion that to arrange a number of things in a ring, one of the things may be placed in *any one* position and then the remaining things may be arranged in the other positions in all possible ways.

Suppose there are four different positions in a circle. Then keeping a fixed in any one of the positions, the letters b, c, d can fill the remaining positions in $\underline{|3}$ ways. Hence the number of permutations $= \underline{|3}$.

Similarly, if n different things be arranged in a circle, the number of permutations $= \underline{|n-1}$.

[**N. B.** In revolving round a circle there are two directions, the clockwise and the anti-clockwise. In the above example distinction has been made between the two directions and we have $\underline{|n-1}$ permutations. But if there be no such distinction then the number of permutations is halved and we have $\frac{1}{2}\underline{|n-1}$ permutations.

As for examples : (1) Suppose there will be a round table conference with 20 persons. In how many ways can they be seated round the table ?

(2) In how many ways can 20 boys sit in a ring (or merry-go-round) ?

(3) In how many ways can 20 beads of different colours form a necklace ?

Now in (1), the arrangements donot depened on the positions of the persons relative to each other but only on the positions with respect to the table.

Hence the number of permutations $= \underline{|20}$.

(2) In (2) we are concerned with the positions of the boys relative to each other. Now, if we fix one boy in one position and arrange the remaining 19 boys with respect to him, we have $\underline{|19}$ arrangements.

Hence the number of permutations here $= \underline{|19}$.

(3) It is also a question of relative arrangements, but there is a difference between (2) and (3). In (3), if we keep one bead fixed and the necklace be formed arranging the other 19 beads in different ways with respect to the fixed bead, either in the clockwise or in the anti-clockwise direction, we have the same kind of necklace and there will be no distinction. For, corresponding to any such arrangement, if the necklace be turned on its other side, it will look the same necklace. Hence, the total number of permutations will be haved, i.e. we have $\frac{1}{2}\lfloor 19$ permutations.]

Examples (3)

Ex. 1. Find the numerical value of

$$\text{(i)}\ \frac{\lfloor 6}{\lfloor 4} \text{ and (ii) } {}^7P_4 \div {}^8P_3.$$

$$\text{Now, (i)}\ \frac{\lfloor 6}{\lfloor 4} = \frac{6\times 5\times \lfloor 4}{\lfloor 4} = 6\times 5 = 30\ ;$$

$$\text{and (ii)}\ {}^7P_4 \div {}^8P_3 = \frac{7\times 6\times 5\times 4}{8\times 7\times 6} = \frac{5}{2}.$$

Ex. 2. If ${}^{n+1}P_4 : {}^{n-1}P_3 = 72 : 5$, find n.

$$\because \frac{{}^{n+1}P_4}{{}^{n-1}P_3} = \frac{72}{5}, \quad \therefore \frac{(n+1)(n)(n-1)(n-2)}{(n-1)(n-2)(n-3)} = \frac{72}{5},$$

$$\text{or, } \frac{n(n+1)}{n-3} = \frac{72}{5}, \quad \text{or, } 5n^2 + 5n = 72n - 216,$$

$$\text{or, } 5n^2 - 67n + 216 = 0, \quad \text{or, } (n-8)(5n-27) = 0, \quad \therefore n = 8.$$

Here, the other value of n, obtained from the other factor, not being an integer is rejected.

Ex. 3. Show that ${}^{n-1}P_r = (n-r)\ {}^{n-1}P_{r-1}$.

$${}^{n-1}P_r = (n-1)(n-2)(n-3)\cdots(n-r+1)(n-r).$$

Again, $^{n-1}P_{r-1}=(n-1)(n-2)(n-3)\cdots(n-r+1)$

$\therefore\quad (n-r).^{n-1}P_{r-1}=(n-1)(n-2)(n-3)\ldots(n-r+1)(n-r)$

$\therefore\quad ^{n-1}P_r=(n-r)\,^{n-1}P_{r-1}.$

Ex. 4. Two boys enter a hall in which there are only 5 vacant seats. In how many different ways can they seat themselves ?

The required number of ways = the number of permutations of 5 things taken 2 at a time $= {^5P_2}=5\times4=20$.

[**Otherwise** ; The first boy can sit in any one of the 5 seats. So he can sit in 5 ways. Now, the first boy having seated himself in any one seat, the second boy can sit in any one of the remaining 4 seats. So for each way of the sitting of the first boy, the second boy can sit in 4 ways. $\therefore$ They can altogether seat themselves in 5×4 or 20 ways.]

Ex. 5. In how many way can one consonant and one vowel be chosen out of the letters of the word *neighbour* ?

There are 4 vowels and 5 consonants in the given letter. One vowel can be taken out of 4 vowels in 4 ways.

Corresponding to each way of choosing one vowel, one consonant can be chosen out of 5 in 5 ways.

$\therefore$ The required number of ways $=4\times5=20$.

[N. B. The reqd. number of ways $= {^4P_1}\times{^5P_1}=4\times5=20$.]

Ex. 6. In how many ways can the letters of the word '*balloon*' be arranged ?

There are 2 *l*'s and 2 *o*'s in the given word consisting of 7 letters.

$\therefore$ The required number of arrangements

$$=\frac{\underline{|7}}{\underline{|2}\,\underline{|2}}=\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times2\times1}=1260.$$

Ex. 7. In how many ways can the letters of the word '*daughter*' be arranged so that the vowels may never be separated?
[C. U. '46]

There are 3 vowels (a, u, e) and 5 consonants (d, g, h, t, r) in the given word. As the vowels can never be separated, they may be regarded as one letter. Then there will be 6 letters (a, u, e), d, g, h, t, r. These 6 letters can be arranged in $\lfloor 6$ or 720 ways.

Again, in each of these arrangements, the 3 vowels can be arranged in $\lfloor 3$ or 6 ways among themselves.

$\therefore$ The total number of ways $= 6 \times 720 = 4320$.

Ex. 8. In how many ways can the letters of the word '*cotton*' be arranged, so that the two *t*'s donot come together?

The given word consists of 6 letters, of which there are two *t*'s and two *o*'s.

$\therefore$ The number of arrangements without any restriction

$$= \frac{\lfloor 6}{\lfloor 2 \lfloor 2} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 180.$$

Now, to determine the number of arrangements in which the two *t*'s come together, the two *t*'s are to be considered as one letter. Then we have 5 letters including two *o*'s.

$\therefore$ The number of such arrangements

$$= \frac{\lfloor 5}{\lfloor 2} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60.$$

$\therefore$ The required number of arrangements in which the two *t*'s donot come together $= 180 - 60 = 120$.

Ex. 9. How many different words can be formed out of the letters of the word '*accommodation*'? How many of them will have the three *o*'s as consecutive letters?

There are 13 letters in the given word, namely, two *a*'s, two *c*'s, two *m*'s, three *o*'s and 4 other different letters.

$\therefore$ The reqd. number of different words $= \dfrac{\lfloor 13}{\lfloor 2 \times \lfloor 2 \times \lfloor 2 \times \lfloor 3}$

$= 129729600.$

Now, to find the number of words in which the three o's may be consecutive letters, the three o's are to be considered as one letter. In this case there will be 11 letters including 2 a's, 2 c's and 2 m's.

$\therefore$ The number of such words $=\dfrac{\lfloor 11}{\lfloor 2 \lfloor 2 \lfloor 2}=4989600.$

Ex. 10. Show that the letters of the word '*anticipation*' can be arranged in thrice as many ways as the letters of the word '*commencement*'.

In the word 'anticipation', there are 2 a's, 2 n's, 2 t's and 3 i's, the other letters being different. There are 12 letters altogether.

$\therefore$ The number of arrangements $=\dfrac{\lfloor 12}{\lfloor 2 \lfloor 2 \lfloor 2 \lfloor 3}$ ...(1)

Again, in the word 'commencement' there are 2 c's, 3 m's, 3 e's and 2 n's, the other letters being different and there are altogether 12 letters.

$\therefore$ The number of arrangements $=\dfrac{\lfloor 12}{\lfloor 2 \lfloor 3 \lfloor 3 \lfloor 2}$ ...(2)

Now, dividing (1) by (2) we have

$$\frac{\lfloor 12}{\lfloor 2 \lfloor 2 \lfloor 2 \lfloor 3}\times\frac{\lfloor 2 \lfloor 3 \lfloor 3 \lfloor 2}{\lfloor 12}=\frac{\lfloor 3}{\lfloor 2}=\frac{3\times 2\times 1}{2\times 1}=3.$$

$\therefore$ The number of arrangements in the first case is thrice as many as in the second case.

Ex. 11. In how many of the permutations of 10 things taken 4 at a time will one particular thing (i) always occur, (ii) never occur ?

The total number of permutations of 10 things taken 4 at a time without restriction $={}^{10}P_4=10\times 9\times 8\times 7=5040.$

(ii) As in the second case one particular thing will never occur, arrangements will have to be made of the remaining $(10-1)$ or 9 things taking 4 thinges at time.

$\therefore$ The required number of such permutations

$$={}^{9}P_4=9\times 8\times 7\times 6=3024.$$

(i) Hence the reqd. number of permutations in which a particular thing will always occur $= 5040 - 3024 = 2016$.

Ex. 12. In how many ways can the letters of the word *normal* be arranged, so that the vowels may occupy only odd positions ?

There are two vowels (a, o) and 3 odd positions in the given word. Since the vowels will occupy only the odd poitions, they can be arranged in $^3P_2$ or 6 ways. Corresponding to each of these ways, there remain 4 positions to be filled up by the 4 consonants. This can be done in $\underline{|4}$ or 24 ways.

$\therefore$ The required number of arrangements $= 6 \times 24 = 144$.

Ex. 13. Find in the how many ways can the letters of the word '*purpose*' be re-arranged (i) keeping the positions of the vowels fixed, (ii) without changing the relative order of the vowels and consonants.

(i) There are 7 letters in the given word. Three of them are vowels and the remaning 4 are consonants.

As in the first case, the positions of the vowels are fixed, so the arrangements will have to be made with the 4 consonants including 2 p's.

$\therefore$ The number of arrangements $= \dfrac{\underline{|4}}{\underline{|2}} = 12$.

Since the given word (purpose) also is included in these 12 arrangements, the reqd. number of *re-arrangements* $= 12 - 1 = 11$.

(ii) In the given word, the consonants occupy the first, third, fourth and sixth places, while the vowels are in the second, fifth and seventh places. By the problem the relative positions of the consonants and the vowels are to be maintained here.

Now, the 3 vowels can be arranged in the 3 fixed places in $\underline{|3}$ ways.

As there are 2 p's, the 4 consonants can be arranged in their 4 fixed places in $= \dfrac{\underline{|4}}{\underline{|2}}$ ways.

$\therefore$ The total number of arrangements $= \lfloor 3 \times \frac{\lfloor 4}{\lfloor 2} = 72$, in which the given word also is included.

$\therefore$ The reqd. number of re-arrangements $= 72 - 1 = 71$.

Ex. 14. How many numbers, each lying between 100 and 600, can be formed with the digits 1, 2, 3, 4, 5 each of the digits occurring once in each number ?

As the numbers lie between 100 and 600, each number will consist of 3 digits. We have 5 digits here. So we have to find the number of ways in which 3 places can be filled by 5 digits.

$\therefore$ The required number $= {}^5P_3 = 5 \times 4 \times 3 = 60$.

Here all these numbers will lie between 100 and 600, since the digit in the hundreds' place in all cases does not exceed 5.

Ex. 15. How many of the numbers formed by using all the digits 1, 2, 3, 4, 5, 6 only once are even ?

Here the number of the digits is 6. Evidently the numbers ending with the digits 2, 4, 6 will be even.

Now, the number of permutations ending with 2 = the number of ways in which the remaining 5 places can be filled up by the remaining 5 digits $= {}^5P_5 = 120$.

Similarly, the number of permutations ending with 4 and 6 is 120 each.

Hence, the required number of even numbers $= 120 \times 3 = 360$.

Ex. 16. How many odd numbers of five significant figures can be formed with the digits 0, 2, 3, 4, 5 ?

Evidently the odd numbers must end with the digit 3 or 5.

Here, we have to find how many permutations can be made ending with 3 and how many among them begin with 0.

The number of arrangements ending with $3 = {}^4P_4 = 24$

[$\because$ here the remaining 4 places are filled by the remaining 4 digits taken all at a time.]

Now, the number of arrangements beginning with 0 and ending with $3 = {}^{3}P_{3} = 6$ [$\because$ here the remaining 3 places are filled up by the remaining 3 digits taken all at a time.]

$\therefore$ The number of odd numbers of 5 significant digits ending with $3 = 24 - 6 = 18$.

Similarly, the number of odd numbers of 5 significant digits ending with $5 = 24 - 6 = 18$.

Hence, the reqd. number of odd numbers $= 18 + 18 = 36$.

Ex. 17. How many numbers less than 1000 and divisible by 5 can be formed with the digits 0, 1, 2, 3, 4, 5, 6 each digit occurring only once in each number ?

Since the numbers are divisible by 5 here, the digit in them must be either 0 or 5. Again, since the numbers are less than 1000, they must consist of 1, 2 or 3 digits.

(i) Now, there will be only one number (*i. e.* 5) consisting of 1 digit and ending with 5,

(ii) There will be ${}^{6}P_{1}$ or 6 numbers consisting of 2 digits and ending with 0.

Now, to find how many numbers consisting of 2 significant digits and ending with 5 but not beginning with 0 can be formed, we have to find the number of arrangements that can be made with 5 digits (0 and 5 being excluded) taken one at a time.

The number of such numbers $= {}^{5}P_{1} = 5$.

Hence, there will be $(6+5)$ or 11 numbers consisting of 2 digits and divisible by 5.

(iii) The number of numbers consisting of 3 digits and ending with $0 =$ the number of ways in which the first two places can be filled up by the remaining 6 digits taken 2 at a time

$$= {}^{6}P_{2} = 30.$$

Again, the number of numbers of 3 digits ending with $5 = {}^{6}P_{2}$ $= 6 \times 5 = 30$, in which the numbers beginning with 0 are also included and they should be excluded.

Now, the number of numbers of 3 digits of which the first is 0 and the last is 5 = 5 [$\because$ the middle place is filled by the remaining 5 digits in 5 ways.]

$\therefore$ The number of numbers ending with 5 is (30 − 5) or 25.

Hence, the number of the required numbers

$$=1+11+30+25=67.$$

Ex. 18. In how many ways can the letters of the word '*player*' be arranged ? How many of these arrangements begin with p ? How many begin with p but do not end with r ?

(i) There are six letters all different in the given word.

$\therefore$ The letters can be arranged in $\underline{|6}$ or **720** ways.

(ii) In the arrangements that begin with p, the remaining five places are filled up by the remaining 5 letters taken all at a time.

$\therefore$ The number of such arrangements $= \underline{|5} =$ **120**.

(iii) Here the total number of arrangements is 720 of which 120 begin with p.

$\therefore$ The number of arrangements that do not begin with $p = 720 - 120 = 600$.

(iv) Now, in the arrangements that begin with p and end with r, the remaining 4 places between them are filled by the remaining 4 letters.

$\therefore$ The number of such arrangements $= \underline{|4} = 24$.

$\because$ The number of arrangements beginning with $p = 120$.

$\therefore$ The required number of arrangements that begin with p and do not end with $r = 120 - 24 =$ **96**.

Ex. 19. In how many ways can 3 prizes, one for good conduct, one for regular attendance and one for sports, be given away to 20 boys ?

$\because$ Each prize can be given to any one of the boys,

$\therefore$ any one of the prizes can be given in 20 ways.

Then any one of the remaining prizes can also be given in 20 ways, as it may be awarded even to the boy who has already got a prize.

$\therefore$ Two prizes can be given away in 20×20 or 20^2 ways.

Similarly, the three prizes can be given away in 20^3 or 8000 ways.

Ex. 20. In how many ways can 6 boys form a ring ?

Here, it is a question of relative positions.

Suppose one boy is fixed in one position. So, the remaining 5 boys can be arranged in $\lfloor\underline{5}$ ways.

$\therefore$ The required number of ways $= \lfloor\underline{5} = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Ex. 21. In how many ways can 6 pearls of different colours be strung on a necklace ?

[See the reasoning given in question (3) at the end of Art. 20.]

Here, the required number of ways $= \frac{1}{2}\lfloor\underline{5} = \frac{1}{2} \times 120 = 60$.

Ex. 22. In how many ways can 6 persons be seated in a round table ?

Here, the arrangements are in relation to the table but not with respect to the relative positions of the persons.

$\therefore$ The reqd. number of ways $= \lfloor\underline{6} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Ex. 23. In how many ways 5 boys and 5 girls sit at a round table so that no two girls will be in consecutive positions ?

(i) [*For positions relative to the table*] : Keeping one boy fixed in one position the other 4 boys can be seated round the table in $\lfloor\underline{4}$ or 24 ways.

Now, if a girl be seated in between two boys, no two girls will be in consecutive positions. As there are 5 positions in between the 5 boys, they can be filled up by the 5 girls in $\lfloor\underline{5}$ ways.

Again, in fixing the one boy in one position, he can be seated in any of the 10 vacant positions.

$\therefore$ The total number of ways $= 10 \times \lfloor\underline{4} \times \lfloor\underline{5} = 28800$.

(ii) [*For positions relative to each other*] : The required number $= \lfloor\underline{4} \times \lfloor\underline{5} = 2880$.

(iii) If no distinction of clockwise and anti-clockwise arrangements is made, the required number $= \frac{1}{2} \times 2880 = 1440$.

Ex. 24. Find the number of different arrangements that can be made of bars of seven prismatic colours (violet, indigo, blue, green, yellow, orange and red) so that the blue and green, shall never come together. [C. U. '55]

The 7 colours can be arranged in $\lfloor 7$ ways. Now, if the blue and the green colours be taken together as one colour, we have 6 colours which can be arranged in $\lfloor 6$ ways. But the two colours, the blue and the green, can be arranged in $\lfloor 2$ or 2 different ways among themselves.

$\therefore$ The number of arrangements in which the two colours come together $= 2\lfloor 6$.

$\therefore$ The required number of arrangements in which these two colours will not come together

$$= \lfloor 7 - 2 \times \lfloor 6 = 7\lfloor 6 - 2\lfloor 6 = \lfloor 6(7-2)$$
$$= 5 \times \lfloor 6 = 3600.$$

Ex. 25. Find the total number of permutations of n dissimilar things taken r things at a time, in which a particular thing always occurs.

Let the different letters $a_1, a_2, \ldots\ldots, a_n$ be the n dissimilar things. We have to determine only the number of those permutations (out of the permutations of n things taken r at a time without restriction) in each of which one particular letter (say a_1) always occurs.

Let the particular letter a_1 be taken out. Then the number of permutations of the remaining $n-1$ letters taken $r-1$ letters at a time will be ${}^{n-1}P_{r-1}$. Now, if a_1 be put in the first place in each of these ${}^{n-1}P_{r-1}$ permutations, we then have the number of permutations of n letters taken r letters at a time in each of which the particular letter a_1 occurs. Evidently this number is ${}^{n-1}P_{r-1}$.

Similarly we also have ${}^{n-r}P_{r-1}$ permutations by putting a_1 in the second place. Since here r things are to be taken at a time, there are evidently r positions (places) where the letter a_1 may be placed. If a_1 is put once in each of these r places, each time we have ${}^{n-1}P_{r-1}$ permutations.

$\therefore$ The total number of such permutations $= r. {}^{n-1}P_{r-1}$.

Corollary. (i) The number of permutations in which a particular thing will never occur $= {}^{n-1}P_r$.

(ii) ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$.

Exercise 3

1. Find the numerical values of :—

   $\lfloor 8$; $\dfrac{\lfloor 7}{\lfloor 4}$; $7!$; ${}^6P_4$; ${}^5P_5$.

2. If ${}^{n+1}P_3 = 10 \times {}^{n-1}P_2$ find n.

3. If ${}^{2r+1}P_{r-1} : {}^{2r-1}P_r = 3 : 5$, find r.

   Show that :—

4. ${}^{n+1}P_{r+1} = (n+1) \times {}^nP_r$.

5. ${}^nP_r = n \times {}^{n-1}P_{r-1}$.

6. $\lfloor 2r = \lfloor r.2^r\{1.3.5 \ldots\ldots (2r-1)\}$

7. ${}^nP_{r-1} = {}^{n-1}P_{r-1} + (r-1).{}^{n-1}P_{r-2}$.

B 8. Two persons go in a railway carriage where there are six vacant seats. In how many different ways they may seat themselves? [C. U. '10]

9. Find the number of permutations of the letters of the word 'Paresh' taken all together.

10. Find the number of permutations of the letters in *India*. [C. U. '20]

11. There are 8 hospitals in a town. In how many ways can 3 patients be sent to hospital, so that no two of them may be in the same hospital?

12. There are 20 stations on a certain railway line. How many different single third class tickets must be printed so that it may be possible to travel from one station to another?

13. If there be 30 stations on a Railway line, how many different kinds of third class tickets will be necessary to make it possible to book from any one station to any other station?

B 14. In how many ways can the letters of the word '*laughter*' be arranged so that the vowels may never be separated?

15. Find the number of permutations which can be made with the letters of the word '*approximation*'? How many of them will begin with '*p*'?

16. How many different words can be formed out of the letters of the word *multiple*? In how many of these will the two *l*'s be consecutive?

17. In how many other ways can the letters of the word *struggle* be arranged (i) without changing the order of the vowels, (ii) without changing the relative order of the vowels and consonants, (iii) keeping the positions of the vowels unaltered?

18. Show that the letters of the word '*Calcutta*' can be arranged in twice as many ways as the letters of the word '*America*'. [C. U. '44]

19. In how many ways can the letters of the word '*pointed*' be arranged so that the vowels may occupy only odd positions?

20. How many numbers lying between 100 and 1000 can be formed with the digits 5, 6, 7, 8, 9, each of the digits occurring only once in each number?

21. How many numbers between 3000 and 4000 can be formed with the digits 1, 2, 3, 4, 5, 6?

22. How many odd numbers of five significant digits can be formed with the digits 0, 3, 4, 5, 6?

23. How many of the numbers formed by using all the digits 3, 4, 5, 6, 7, 8, 9 only once are even?

24. How many even numbers, each of 7 digits, can be formed with the digits 7, 5, 4, 7, 6, 5, 7?

25. How many different numbers, each of 5 digits, can be formed by means of the digits of the number 32302?

26. How many numbers less than 1000 and divisible by 5 can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number? [C. U. '42]

27. Find the number of ways in which n different books can be arranged on a shelf so that two particular books are not together. [C. U. '47]

B 28. In how many of the permutations of 12 things taken 3 at a time will one particular thing (i) always occur, (ii) never occur ?

B 29. Find how many words can be formed of the letters in the word '*failure*', the four vowels always coming together. [C.U.'40]

30. In how many ways can 4 prizes, one for recitation, one for sports, one for smartness and one for general proficiency be given away to 8 boys ? [C. U. '49]

31. There are 36 candidates for an examination, 20 boys and 16 girls. In how many ways can they be seated in a line so that no two girls may occupy consecutive positions ?

32. In how many ways can 7 examination papers be arranged so that the best and the worst papers never come together ? [C. U. '51]

33. In how many ways can the letters of the word '*blossom*' be arranged, so that the two *o*'s do not come together ?

34. In how many ways can the letters of the word '*Friday*' be arranged ? How many of these arrangements do not begin with '*F*' ? How many begin with *F* and do not end with *y* ?

B 35. A man has to post 4 letters and there are 3 letter-boxes, in how many ways can he post the letters ?

36. In how many ways can 10 examination papers be arranged so that the best and the worst papers never come together ? [C. U. '53]

37. In how many ways can 10 chidren sit in a merry-go-round relatively to one another ? [C. U. '27]

38. In how many ways can 8 boys form a ring ?

39. In how many ways can 8 different pearls be strung on a necklace ?

40. Find how many different words can be formed with 5 given letters of which 3 are consonants and 2 are vowels, no two consonants coming together. [C. U. '29]

41. In how many ways can 8 persons sit at a round table so that all shall not have the same neighbours in any two arrangements ?

42. In how many ways can 6 teachers and 6 students sit at a round table, so that no two students be in consecutive positions ?

43. In how many ways can 5 Indians and 5 Englishmen be arranged alternately at a round table ?

44. In how many ways can 15 I.Sc. and 12 B.Sc. candidates be arranged in a line so that no two B. Sc. candidates may occupy consecutive positions ? [P. U. '42]

45. In how many ways can 5 I.Sc. and 2 B.Sc. students be arranged at a round table if the two B.Sc. students (i) sit together, (ii) are separated ?

46. Find the number of ways in which 5 boys and 5 girls can be placed alternately in a ring.

47. In a library there are 4 copies of one book, 5 copies each of two books, 7 copies of each of three books and single copies of 6 books. In how many ways can all the books be arranged ?

48. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books, and single copies of eight books. In how many ways can all the books be arranged ? [C. U. '34]

49. How many words can be formed taking together 2 consonants out of 7 and one vowel out of 3 so that the vowel is always in between the two consonants ? [C. U. '22]

50. There are two works of three volumes and two works each of two volumes ; in how many ways can the ten books be placed on a shelf so that the volumes of the same work are not separated ? [P. U. '46]

51. Show that the total number of permutations (with repetitions) of n different things, not more than p being taken at a time, is $\dfrac{n(n^p-1)}{n-1}$.

52. How many different arrangements can be made out of the letters in the expression $x^5y^3z^2$ when written at full length ?

53. If words be formed by taking only 5 at a time out of the letters of the word 'sulphanomide', in how many of them does the letter 'h' occur ?

COMBINATIONS

21. We have given the definition of *combination* in Art. 11 and discussed the distinction between *permutation* and *combination* in Art. 12.

Each of the groups or selections that can be made by taking at a time some or all of a given number of things (without regard to the order of the things in each group) is called a combination.

You have already seen that in combination only the selections or groups of things are made, but it has nothing to do with arranging the things in each group among themselves in all possible ways.

Symbol : The number of combinations of n different things taken r at a time is represented by the symbol ^nC_r or $_nC_r$.

Combinations of things all different

22. To find the number of combinations of n different things taken r at a time $(r \leqslant n)$.

Let x be the required number of combinations. There are r things in each combination. If the r things in each combination be arranged among themselves in all possible ways, we have $\lfloor r$ permutations in each case.

$\therefore$ From x combinations we have $x \times \lfloor r$ permutations. It is also evident that when the r things in each of the x combinations are arranged among themselves in all possible ways, we have the total number of permutations of n things taken r at a time.

$\therefore \quad x \times \lfloor r = {}^nP_r$,

$\therefore \quad {}^nC_r \times \lfloor r = n(n-1)(n-2)(n-3)......(n-r+1)$.

$$\therefore \quad {}^nC_r = \frac{n(n-1)(n-2)(n-3)......(n-r+1)}{\lfloor r} ......(1)$$

Now, to express (1) in factorial form, we multiply both its numerator and denominator by $\underline{|n-r}$. Thus we have

$$^nC_r=\frac{\{n(n-1)(n-2)\ \ldots..(n-r+1)\}\times(n-r)(n-r-1)\ldots\ldots3.2.1}{\underline{|r}\times\underline{|n-r}}$$

$$=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

Alternative Proof : This can be proved without assuming the formula for the number of permutations. Let the total number of combinations be ^nC_r.

The combinations in which a particular thing occurs can be obtained if this particular thing be associated with each of the combinations that can be made taking $(r-1)$ things at a time out of the remaining $(n-1)$ things.

$\therefore$ The number of combinations in which a particular thing occurs $={}^{n-1}C_{r-1}$.

$\therefore$ Each of the n things occurs $^{n-1}C_{r-1}$ times in all the combinations of n things taken r at a time.

$\therefore$ The total number of letters in those combinations

$$=n\times{}^{n-1}C_{r-1}.$$

But as the number of combinations is here assumed to be ^nC_r and as there are r things in each combination, the total number of things in all the combinations is $r\times{}^nC_r$.

$$\therefore\quad r\times{}^nC_r=n\times{}^{n-1}C_{r-1},$$

$$\therefore\quad {}^nC_r=\frac{n}{r}\times{}^{n-1}C_{r-1}.$$

Similarly, $^{n-1}C_{r-1}=\dfrac{n-1}{r-1}\times{}^{n-2}C_{r-2}$,

$$^{n-2}C_{r-2}=\frac{n-2}{r-2}\times{}^{n-3}C_{r-3},$$

$$\ldots\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots$$

$$^{n-r+2}C_2=\frac{n-r+2}{2}\times{}^{n-r+1}C_1$$

and $^{n-r+1}C_1=\dfrac{n-r+1}{1}$

Now, taking the products of the left hand and the right hand sides separately and cancelling the common factors in the two products we have

$$^nC_r=\frac{n}{r}\times\frac{n-1}{r-1}\times\frac{n-2}{r-2}\times\ldots\ldots\times\frac{n-r+1}{1}$$

$$=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots2.1}$$

$$=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}\quad\ldots\ldots(1)$$

$$=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}$$

$$=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\quad\ldots\ldots(2).$$

Corollary. (i) $^nC_1=n$.

Proof. $\because\ ^nC_r=\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$, $\therefore\ ^nC_1=\dfrac{\underline{|n}}{\underline{|1}\,\underline{|n-1}}=\dfrac{n\underline{|n-1}}{\underline{|n-1}}=n$.

(ii) $^nC_n=1$.

Proof. $\because\ ^nC_r=\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$, $\therefore\ ^nC_n=\dfrac{\underline{|n}}{\underline{|n}\,\underline{|n-n}}=\dfrac{\underline{|n}}{\underline{|n}}=1$,

(iii) $^nP_r=\underline{|r}\times{}^nC_r$.

Proof. $^nP_r=\dfrac{\underline{|n}}{\underline{|n-r}}=\underline{|r}\times\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}=\underline{|r}\times{}^nC_r$.

(iv) $^nC_0=1$.

Proof. Putting $r=0$, we have $^nC_0=\dfrac{\underline{|n}}{\underline{|0}\,\underline{|n}}=1$.

COMPLEMENTARY COMBINATIONS

23. The number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $(n-r)$ at a time.

If r things are taken from n things, $(n-r)$ things are left. So every time r things are selected from n things, $(n-r)$ things are also separately selected along with it. Hence the second operation can be done in as many ways as the first.

$\therefore \quad {}^nC_r = {}^nC_{n-r}$.

Otherwise : [Proof with the help of formula]

$$\because \quad {}^nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \ldots\ldots \quad (1)$$

$\therefore$ Putting $(n-r)$ for r in (1) we have

$${}^nC_{n-r} = \frac{\lfloor n}{\lfloor n-r \, \lfloor n-(n-r)} = \frac{\lfloor n}{\lfloor n-r \, \lfloor r},$$

$\therefore \quad {}^nC_r = {}^nC_{n-r}$.

Corollary : (i) If ${}^nC_p = {}^nC_q$, then either $p=q$, or, $p+q=n$.

Proof : Here, the number of combinations of n things taken p things at a time is equal to the number of combinations of n things taken q things at a time [$\because$ here ${}^nC_p = {}^nC_q$]. $\therefore \; p=q$.

Again, $\because \; {}^nC_p = {}^nC_{n-p}$ and $\because \; {}^nC_p = {}^nC_q$ (given),

$\therefore \quad {}^nC_{n-p} = {}^nC_q, \quad \therefore \quad n-p=q, \quad \therefore \quad p+q=n.$

(ii) $\because \quad {}^nC_{n-r} = {}^nC_r$ (proved),

$\therefore$ Putting n for r we have ${}^nC_{n-n} = {}^nC_n$, or, ${}^nC_0 = {}^nC_n = 1$.

24. To prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$. [H. S. 1966]

Evidently the number of combinations of $(n+1)$ things taken r at a time = the number of combinations including a particular thing + the number of combinations excluding that particular thing.

Here the number of things $=n+1$, and r things are to be taken at a time.

$\therefore$ the number of combinations including the particular thing $=$the number of combinations of the remaining n things taken $(r-1)$ at a time$={}^nC_{r-1}$.

Again, the number of combinations excluding that particular thing$=$the number of combinations of n things taken r at a time $={}^nC_r$.

Hence ${}^{n+1}C_r={}^nC_{r-1}+{}^nC_r$.

[N. B. To find the number of combinations in which a particular thing occurs, suppose we first take out that paticular thing from $(n+1)$ things. Then n things are left and we have to take $(r-1)$ things from the remaining n things. The particular thing, already taken out, together with these $(r-1)$ things will give r things.

Again, to find the number of things in which that particular thing does not occur, suppose that particular thing is removed from the $(n+1)$ things as it will never be taken. Then n things are left over and r things are to be taken from these n things.]

Alternative Proof :

$${}^nC_r+{}^nC_{r-1}=\frac{\lfloor\underline{n}}{\lfloor\underline{r}\ \lfloor\underline{n-r}}+\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\ \lfloor\underline{n-(r-1)}}$$

$$=\frac{\lfloor\underline{n}}{\lfloor\underline{r}\ \lfloor\underline{n-r}}+\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\ \lfloor\underline{n-r+1}}.$$

Now, $\because$ $\lfloor\underline{r}=r\lfloor\underline{r-1}$ and $\lfloor\underline{n-r+1}=(n-r+1)\lfloor\underline{n-r}$,

$$\therefore\quad {}^nC_r+{}^nC_{r-1}=\frac{\lfloor\underline{n}}{r\lfloor\underline{r-1}\ \lfloor\underline{n-r}}+\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\times(n-r+1)\lfloor\underline{n-r}}$$

$$=\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\ \lfloor\underline{n-r}}\left(\frac{1}{r}+\frac{1}{n-r+1}\right)=\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\ \lfloor\underline{n-r}}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$$

$$=\frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\ \lfloor\underline{n-r}}\times\frac{n+1}{r(n-r+1)}=\frac{(n+1)\lfloor\underline{n}}{r\lfloor\underline{r-1}\,.\,(n-r+1)\lfloor\underline{n-r}}$$

$$=\frac{\lfloor\underline{n+1}}{\lfloor\underline{r}\ \lfloor\underline{n-r+1}}={}^{n+1}C_r.$$

25. *To find the number of combinations of n things taken r at a time in which p particular things always occur.*

Here evidently $r>p$. Suppose the p particular things are first kept separate from the n things. Then $(n-p)$ things remain. Now all possible selections of $(r-p)$ things are made from the $(n-p)$ things. If these selections be then added to the p particular things already kept aside, we have all the combinations in which the p particular things occur.

Since $(r-p)$ things are taken at a time from $(n-p)$ things,

$\therefore$ the required number of combinations $= {}^{n-p}C_{r-p}$.

26. *To find the number of combinations of n things taken r at a time in which p particular things never occur.*

Here $r>p$. As these p particular things are never to be included in any selection, they are removed from the n things. Now, if selections be made from the remaining $(n-p)$ things taken r things at a time, then the p particular things do not occur in any one of the selections.

$\therefore$ The required number of combinations $= {}^{n-p}C_r$.

27. *To find the total number of combinations of n different things taken any number at a time.*

Here selections are to be made out of n things all different. Evidently there may be two kinds of operations with regard to each thing, for it may be either selected or left out entirely.

Since there will be two operations with each of the n things, and since either operation with any one thing may be associated with each of the other $(n-1)$ things,

$\therefore$ the total number of operations with all the n things $=2\times2\times2\times\ldots$to n factors$=2^n$.

Now these 2^n operations include the case in which all the n things are left out. *But as there cannot be any selection in which not a single thing occurs, this case is inadmissible.

$\therefore$ The required number of combinations $= 2^n - 1$.

[*N. B. (i) If the operation in which the first thing has been excluded be associated with the operation in which the second thing has been excluded, we have an operation in which both the first and the second things have been excluded. Again, this operation will once be combined with the operation in which the third thing also has been excluded. Proceeding in this way, we have an operation in which all the n things are excluded. So, that single operation is not admissible.

(ii) Here selections are made from n different things taking at will any number of them (say 1, 2, 3,......up to n things) at a time.

If 1 thing be taken at a time, the number of combinations is ${}^nC_1$, if 2 things be taken at a time, we have ${}^nC_2$ combinations, and so on.

$\therefore$ ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$.]

28. *To find the total number of ways in which a selection may be made by taking any number (some or all) out of $p+q+r+\ldots$ things, of which p are alike of one kind, q alike of a second kind, r alike of a third kind, and so on.*

The p like things may be disposed of in $(p+1)$ ways ; for we may take 1, 2, 3,...or, p of them at a time (this can be done in p ways) or in one operation we may take none of then (so there are $p+1$ ways). Similarly q things may be disposed of in $(q+1)$ ways, the r things in $(r+1)$ ways, and so on.

Now, each of the $(p+1)$ selections of the first kind may be combined with each of the $(q+1)$ selections of the second kind. So p and q things may be disposed of in $(p+1)(q+1)$ ways.

Hence the number of ways in which all the things $(p+q+r+\cdots)$ may be disposed of is $(p+1)(q+1)(r+1)\ldots$.

But this includes the case in which none of the things are taken and this case should be rejected.

Hence, the total number of combinations

$$=\{(p+1)(q+1)(r+1)\ldots\}-1.$$

Division into groups

29. *To find the number of ways in which $(m+n)$ things can be subdivided into two groups containing m and n things respectively.*

If a group of m things be selected out of $(m+n)$ things, then with each of these selections n things are left out to make up the second group of n things.

Now, the number of ways in which m things can be selected out of $(m+n)$ things $={}^{m+n}C_m$; and each time m things are selected, the other group can be selected by taking n things out of the n things left out. The latter selection can be made in ${}^nC_n$ or 1 way.

$\therefore$ The required number of ways $={}^{m+n}C_m\times 1$

$$=\frac{\underline{|m+n}}{\underline{|m}\ \underline{|m+n-m}}=\frac{\underline{|m+n}}{\underline{|m}\ \underline{|n}}.$$

Corollary. (i) If here $n=m$, the groups are equal (m things being in each group) and hence it is possible to interchange the two groups without having a new subdivision.

Hence, the number of **different** ways of subdivision

$$=\frac{\underline{|m+m}}{\underline{|m}\ \underline{|m}\ \underline{|2}}=\frac{\underline{|2m}}{\{\underline{|m}\}^2\ \underline{|2}}.$$

Illustration ; Suppose the four letters A, B, C, D are to be divided into two equal groups. Evidently there will be 2 letters in each group, and we have the following groups :

(i)	(ii)	(iii)	(iv)	(v)	(vi)
AB	AC	AD	BC	BD	CD
CD	BD	BC	AD	AC	AB

Here of course we have $\dfrac{\underline{|4}}{\underline{|2}\ \underline{|2}}=6$ groups as per formula in Art. 29, but the groups (i) and (vi) are the same and so also are the groups (ii) and (v) and the groups (iii) and (iv).

Hence in this case the number of different groups $=\dfrac{6}{\underline{|2}}=3.$

Here we have to divide the letters into **two** groups and find that $\underline{|2}$ number of groups have been identical and so the required number of groups has been found by dividing the result, obtained as per Art. 29, by $\underline{|2}$.

Hence, in such cases the factorial of the number of groups into which the things are to be divided will be identical and the result obtained from the formula of Art. 29 is to be divided by that factorial.

In the above illustration the division is to be made into 2 parts and the value of 2 and $\underline{|2}$ being the same, it may be asked why the result is divided here by $\underline{|2}$ and not by 2.

To be free from doubt let us take another illustration.

Dividing the 6 letters A, B, C, D, E and F as before into 3 equal groups, we have 90 groups, but of these 6 or $\underline{|3}$ groups will be the same, so to have the number of different groups, the result (90) is to be divided by $\underline{|3}$.

It should be noted, however, that if $2m$ things are to be distributed equally among two men, it can be done in $\dfrac{\underline{|2m}}{(\underline{|m})^2}$ ways.

(ii) If $(m+n+p)$ things are to be divided into three groups containing m, n and p things respectively, it can be done in $\dfrac{\underline{|m+n+p}}{\underline{|m}\;\underline{|n}\;\underline{|p}}$ ways.

Now, if $m=n=p$, then the number of ways$=\dfrac{\underline{|3m}}{\{\underline{|m}\}^3\;\underline{|3}}$.

But if $3m$ things are to be distributed equally among three men, then this can be done in $\dfrac{\underline{|3m}}{(\underline{|m})^3}$ ways.

(iii) The results will be similar for any number of groups.

30. If there are different things in each of the different groups, *i.e.*, if there be m different things in the first group, n different things in the second, p different things in the third, and so on, and if selections are to be made taking r things at a time from the first group, p things from the second, t things from the third and so on, then the total number of combinations will be ${}^mC_r \times {}^nC_q \times {}^pC_t \times \ldots\ldots$

Proof. From the m things in the first group r things can be selected in ${}^mC_r$ ways. Then q things can be selected out of n different things of the second group in ${}^nC_q$ ways. Hence combining the two kinds of operations, the total number of combinations will be ${}^mC_r \times {}^nC_q$. After the two kinds of operations, t things can be selected out of the p different things of the third group in ${}^pC_t$ ways.

$\therefore$ When the three operations are done together, the number of combinations will be ${}^mC_r \times {}^nC_q \times {}^pC_t \times \ldots$and so on.

Hence the total number of combinations $= {}^mC_r \times {}^nC_q \times {}^pC_t \times \ldots$

[N.B. Now let us see what will be the number of permutations in such cases. When all the three kinds of operations have been done, there will be $(r+q+t)$ things in each combination. The $(r+q+t)$ things can be arranged among themselves in $\lfloor\underline{r+q+t}$ ways. Hence the total number of permutations from all the combinations is ${}^mC_r \times {}^nC_q \times {}^pC_t \times \lfloor\underline{r+q+t}$.

Hence, generally the number of permutations

$$= {}^mC_r \times {}^nC_q \times {}^pC_t \times \ldots\ldots \times \lfloor\underline{r+q+t+\ldots\ldots}.$$

Greatest value of ${}^nC_r$

31. For a given value of n, what value of r will make ${}^nC_r$ greatest ?

Or, To find for what value of r the number of combinations of n things taken r at a time is greatest.

We have already shown shat

$${}^nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}\ldots(1), \quad \text{and} \quad {}^nC_{r-1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}\ldots$$

From (1) we have $^nC_r = \dfrac{\lfloor n}{r\,\lfloor r-1\,\lfloor n-r}$.

and from (2) we have $^nC_{r-1} = \dfrac{\lfloor n}{\lfloor r-1 \times (n-r+1)\,\lfloor n-r}$.

$\therefore \quad {}^nC_r = \dfrac{n-r+1}{r} \times {}^nC_{r-1}$.

Hence, $^nC_r >, =$ or $< {}^nC_{r-1}$,

if $\dfrac{n-r+1}{r} > =$ or <1,

i.e., if $n-r+1 > =$ or $<r$,

i.e., if $n+1 > =$ or $<2r$,

i.e., if $r < =$ or $> \dfrac{n+1}{2}$.

Here the value of r must be a positive integer.

(*i*) Now, if n be even, then $\dfrac{n+1}{2}$ must be a fraction $(n+1)$ being then an odd number). So, in this case, the greatest value of r may be $\frac{n}{2}$. For, if r be greater than $\frac{n}{2}$ by at least 1, then $\left(\frac{n}{2}+1\right)$ will be greater than $\dfrac{n+1}{2}$ $\left(i.e., \frac{n}{2}+\frac{1}{2}\right)$ and consequently ^nC_r will be less than $^nC_{r-1}$.

$\therefore$ In this case ^nC_r is greatest when $r=\frac{n}{2}$ and the greatest value is $^nC_{\frac{n}{2}}$.

(*ii*) If n be odd, then $\dfrac{n+1}{2}$ will be an integral number. So in this case $^nC_r = {}^nC_{r-1}$, if $r=\dfrac{n+1}{2}$, but if the value of r exceeds $\dfrac{n+1}{2}$ by at least 1, then ^nC_r will be less than $^nC_{r-1}$.

$\therefore$ In this case ^nC_r is greatest when $r=\dfrac{n+1}{2}$ or $\dfrac{n-1}{2}$, the same result being obtained for both the values, and the greatest value $= {}^nC_{\frac{n+1}{2}}$ and $^nC_{\frac{n-1}{2}}$.

Examples (4)

Ex. 1. Find the value of :

(i) ${}^7C_3$. (ii) ${}^8C_4 \times \lfloor 5$, (iii) $\dfrac{16\,!}{14\,!\,\lfloor 2}$.

Now, (i) $${}^7C_3 = \frac{\lfloor 7}{\lfloor 3 \lfloor 7-3} = \frac{\lfloor 7}{\lfloor 3 \lfloor 4} = \frac{7.6.5.\lfloor 4}{\lfloor 3 \lfloor 4} = \frac{7\times 6\times 5}{3\times 2\times 1} = 35.$$

(ii) $${}^8C_4 \times \lfloor 5 = \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} \times \lfloor 5 = 70 \times \lfloor 5 = 70 \times 120 = 8400.$$

(iii) $$\frac{16\,!}{14\,!\,\lfloor 2} = \frac{16\times 15\times \lfloor 14}{\lfloor 14 \times 2\times 1} = 120.$$

Ex. 2. If ${}^nC_4 = 21 \times {}^{\frac{n}{2}}C_3$, find n.

$${}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{4\times 3\times 2\times 1},$$

and $$21 \times {}^{\frac{n}{2}}C_3 = 21 \times \frac{\frac{n}{2}(\frac{n}{2}-1)\,(\frac{n}{2}-2)}{3\times 2\times 1} = 21 \times \frac{n(n-2)(n-4)}{8\times 3\times 2\times 1}.$$

Now, from the given condition we have

$$\frac{n(n-1)(n-2)(n-3)}{4\times 3\times 2\times 1} = \frac{21\times n(n-2)(n-4)}{8\times 3\times 2\times 1},$$

or, $(n-1)(n-3) = \frac{21}{2}(n-4)$,

or, $2n^2 - 8n + 6 = 21n - 84$, or, $2n^2 - 29n + 90 = 0$,

or, $(n-10)(2n-9) = 0$, or, $n = 10, \frac{9}{2}$.

$\therefore$ the required value of $n = 10$,

($\because$ the fractional value is inadmissible).

Ex. 3. If ${}^nC_{10} = {}^nC_8$, find ${}^nC_{16}$.

$\because$ ${}^nC_{10} = {}^nC_8$, $\therefore$ $n = 10 + 8 = 18$,

$\therefore$ ${}^nC_{16} = {}^{18}C_{16} = {}^{18}C_2 = \frac{18\times 17}{2\times 1} = 153$.

Ex. 4. How many committees each consisting of 6 members can be formed from 9 men ?

Here the number of committees will be equal to the number of combinations of 9 things taken 6 at a time.

$\therefore$ the reqd. number of committees $= {}^9C_6 = {}^9C_3 = \frac{9\times 8\times 7}{3\times 2\times 1} = 84$.

Ex. **5.** How many words can be made taking 3 consonants and 2 vowels out of 13 consonants and 4 vowels ?

Here selections are made from 13 consonants taken 3 at time.

$\therefore$ the number of selections $= {}^{13}C_3$.

Again, the number of ways of choosing 2 vowels out of $4 = {}^4C_2$.

$\therefore$ The number of combined groups, each consisting of 3 consonants and 2 vowels $= {}^{13}C_3 \times {}^4C_2$.

Since formation of 'word" means different arrangements of letters the different 5 letters in each group are to be arranged in all possible ways among themselves.

Now, the 5 letters in each group can be arranged among themselves in $\underline{|5}$ ways.

$\therefore$ The required number of words $= {}^{13}C_3 \times {}^4C_2 \times \underline{|5}$

$$= \frac{13 \times 12 \times 11}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} \times 5 \times 4 \times 3 \times 2 = 205920.$$

[**N. B.** In forming **words** and **numbers**, the question of permutation always arises with each combination, for formation of words or numbers means different arrangements. Thus the number of possible combinations are to be determined first and then the letters or the digits (as the case may be) in each combination are to be arranged among themselves in order to find the total number of words or numbers.

But in forming a '**committee**', only the combinations are to be determined, for a committee remains the same in whatever different ways the members may take their seats.]

Ex. **6.** From 12 things in how many ways can a selection of 4 be made (i) when one particular thing is always included and (ii) when a particular thing is always excluded ?

(i) $\because$ a particular thing is to be taken in each selection,

$\therefore$ only 3 things are to be selected from the remaining 11 things and each selection is to be added to that particular thing. This can be done in ${}^{11}C_3$ ways

$\therefore$ the required number of combinations

$$= {}^{11}C_3 = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165.$$

(ii) Here a particular thing is not to be taken in any selection. So keeping it aside, selections are to be made from the remaining 11 things taken 4 at a time.

$\therefore$ The required number of combinations $= {}^{11}C_4$

$$= \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

Ex. 7. From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done to include at least one lady ? [C. U. '48]

Since of the 5 persons forming the committee at least 1 must be a lady, the committee can be formed in the following ways :—

(*a*) with 1 lady and 4 gentlemen,

(*b*) with 2 ladies and 3 gentlemen,

(*c*) with 3 ladies and 2 gentlemen,

or (*d*) with 4 ladies and 1 gentleman.

(*a*) In this case 1 lady out of 4 and 4 gentlemen out of 7 are to be selected.

$\therefore$ the number of selections $= {}^4C_1 \times {}^7C_4 = {}^4C_1 \times {}^7C_3$

$$= 4 \times \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 140.$$

(*b*) In this case 2 ladies out of 4 and 3 men out of 7 are to be selected.

$\therefore$ the number of selections $= {}^4C_2 \times {}^7C_3 = \frac{4 \times 3}{2 \times 1} \times \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 210.$

(*c*) Similarly in this case, the number of selections

$$= {}^4C_3 \times {}^7C_2 = {}^4C_1 \times {}^7C_2 = 4 \times \frac{7 \times 6}{2 \times 1} = 84.$$

and (*d*) here the number of selections $= {}^4C_4 \times {}^7C_1 = 1 \times 7 = 7.$

$\therefore$ the required number of committees

$$= 140 + 210 + 84 + 7 = 441.$$

Ex. 8. How many words can be formed taking 2 consonants and one vowel out of 7 consonants and 3 vowels so that the vowel is always between the consonants ? [C. U. '22]

Here, the number of permutations of 7 consonants taken 2 at a time $= {}^7P_2 = 7 \times 6 = 42.$

Again, a word will be formed by placing anyone of the 3 vowels in between the two consonants in each permutation.

$\therefore$ 3 words will be formed from each permutation.

$\therefore$ The required number of words $= 42 \times 3 = 126.$

Ex. 9. How many different triangles can be formed by joining the angular points of a polygon of 14 sides ? Find also the number of the diagonals of the polygon.

As the polygon has 14 sides, it must have 14 angular points. A triangle is obtained by joining any three of the 14 points.

Hence, there will be as many triangles as the number of selections of 14 points taken 3 at a time.

$\therefore$ The required number of triangles $= {}^{14}C_3 = \dfrac{14.13.12}{3.2.1} = 364.$

Again, a straight line is obtained by joining any two of the 14 points.

$\therefore$ The number of straight lines $= {}^{14}C_2 = \dfrac{14.13}{2.1} = 91$, but this number (91) includes the 14 sides of the polygon, which are not its diagonals.

$\therefore$ The required number of diagonals $= 91 - 14 = 77.$

Ex. 10. A man has 5 friends, in how many ways may he invite one or more of them to a feast ?

Out of the 5 friends, either 1 or 2 or 3 or 4 or 5 can be invited at a time.

$\therefore$ The required number of ways
$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31.$

Otherwise : The number of selections of 5 friends taken any number at a time $= 2^5 - 1 = 32 - 1 = 31.$

Ex. 11. A man has 4 sovereigns, 3 guineas and 5 shillings. In how many ways can he subscribe to a poor fund ?

Here the 4 sovereigns are alike coins of one kind, the 3 guineas of another kind and the 5 shillings of a third kind.

$\therefore$ The required number $= (4+1)(3+1)(5+1) - 1 = 119.$

[Vide Art. 28]

Ex. 12. In how many ways can 6 pens be equally divided among 3 boys ?

Let X, Y, Z be the 3 boys. Here each boy will get 2 pens.

Now, 2 pens can be selected for X from the 6 given pens in $^6C_2$ ways, and after each such selection, 4 pens will remain. Then 2 pens can be selected for Y from the remaining 4 pens in $^4C_2$ ways. Evidently corresponding to each selection for X and Y, the two remaining pens can be given to Z in one way.

$\therefore$ The required number of ways $= {}^6C_2 \times {}^4C_2 \times 1 = \frac{6.5}{2.1} \times \frac{4.3}{2.1} = 90.$

Ex. 13. In how many ways can 12 things be divided in three groups of 3, 4 and 5 things respectively ?

3 things can be selected from 12 things in $^{12}C_3$ ways. After each selection 9 things remain. 4 things can be selected from the 9 things in $^9C_4$ ways. After each of these second selections 5 things will remain and 5 things can be selected from these remaining 5 things in $^5C_5$ ways.

$\therefore$ The required number of groups

$$= {}^{12}C_3 \times {}^9C_4 \times {}^5C_5 = \frac{12.11.10}{3.2.1} \times \frac{9.8.7.6}{4.3.2.1} \times 1 = 27720.$$

Ex. 14. How many different factors may 1155 have ?

3, 5, 7 and 11 are the four prime factors of 1155. Each of these factors may be dealt with in two ways in forming a new factor of 1155, for it may be either selected or left out. Again, one factor or more than one can be taken at a time from the 4 factors, for each of the prime factors as well as the products of more than one of them may be the factors of 1155.

Hence, neglecting the case in which none of the 4 factors occur, we have the required number of factors $= 2^4 - 1 = 15$.

[**Otherwise** : There will be as many factors as the number of combinations of 4 factors taken any number at a time.

$\therefore$ The requd. number of factors $= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15.$]

Ex. 15. In a group of 15 boys there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (i) exactly 6 boy-scouts, (ii) at least 6 boy-scouts ? [C. U. '43]

Out of the 15 boys, 7 are boy-scouts and 8 are other boys.

In case (i), as 6 scouts must be taken, so 6 other boys must be taken to have a selection of 12 boys.

The number of selections of 7 scouts taken 6 at a time $= {}^7C_6$,

and the number of selections of 6 boys from 8 other boys $= {}^8C_6$.

Hence, the required number of selections

$$= {}^7C_6 \times {}^8C_6 = {}^7C_1 \times {}^8C_2 = 7 \times \frac{8.7}{2.1} = 196.$$

In case (ii), as the number of scouts cannot be less than 6, the selections can be made either with,

(*a*) 6 scouts and 6 other boys, or, (*b*) 7 scouts and 5 other boys.

Now (*a*), the number of selections of 7 scouts taken 6 at a time is ${}^7C_6$, and that of 8 other boys taken 6 at a time is ${}^8C_6$;

Again, (*b*) 7 scouts can be selected in ${}^7C_7$ ways and 5 other boys can be selected in ${}^8C_5$ ways.

$\therefore$ The required total number of combinations

$$= {}^7C_6 \times {}^8C_6 + {}^7C_7 \times {}^8C_5 = {}^7C_1 \times {}^8C_2 + {}^7C_7 \times {}^8C_3$$
$$= 7 \times \tfrac{8 \times 7}{2 \times 1} + 1 \times \tfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 196 + 56 = 252.$$

Ex. 16. A boy is required to answer 8 questions out of 2 groups, each group containing 7 questions, and he is not permitted to attempt more than 5 questions from any group. In how many different ways can he choose them ?

Let G_1 and G_2 be the two groups, each containing 7 questions. The boy has to answer 8 questions, but not more than 5 from any group.

Hence he can select the questions as follows :—

(1) 3 questions from group G_1 and 5 from group G_2,

or, (2) 4 from G_1 and 4 from G_2.

or, (3) 5 from G_1 and 3 from G_2.

Now, from (1) the number of selections

$$= {}^7C_3 \times {}^7C_5 = {}^7C_3 \times {}^7C_2 = \frac{7.6.5}{3.2.1} \times \frac{7.6}{2.1} = 735.$$

from (2) the number of selections

$$={}^7C_4 \times {}^7C_4 = {}^7C_3 \times {}^7C_3 = \frac{7.6.5}{3.2.1} \times \frac{7.6.5}{3.2.1} = 1225,$$

and from (3) the number of selections $= {}^7C_5 \times {}^7C_3 = 735.$

$\therefore$ The required number of ways in which he can choose

$$= 735 + 1225 + 735 = 2695.$$

Ex. 17. In how many ways can a crew of 8 be arranged if 2 of them can only row on stroke side and 1 only on the bow side ?

Let X and Y be two of the crew who can row only on stroke side and Z be the one who can row only on the bow side. So the remaining 5 of the crew can row on both sides.

Here 4 sailors must be on each side. So 2 of the 5 men, who can row on both sides, are to be placed on the side of X and Y. This can be done in ${}^5C_2$ or 10 ways. Now, for each of these 10 ways, the 4 men on the side of X and Y can be arranged among themselves in $\lfloor 4$ ways.

Similarly the 4 men on the other side may also be arranged in $\lfloor 4$ ways.

Hence, associating each arrangement on one side with each arrangement on the other side, we have $\lfloor 4 \times \lfloor 4$ arrangements.

$\therefore$ for the above 10 ways of selections the total number of arrangements $= 10 \times \lfloor 4 \times \lfloor 4 = 5760.$

Ex. 18. Find the number of ways in which (*a*) a selection (*b*) an arrangement of 4 letters can be made from the letters of the word '*successive*'.

In the given word there are 10 letters of 6 different kinds, viz,, (*s, s, s*), (*c, c*), (*e, e*), *u, v, i.*

The selections of 4 letters can be made in the following ways:—

(1) Three alike letters and one different,
(2) Two alike and two others alike of another kind,
(3) Two alike and two others different,
(4) All four different.

Now in case (1), there can be 5 selections, for here any one of the 5 different letters c, i, u, v, e can be taken with the 3 alike letters, i. e., the 3 s's.

In case (2), selections can be made from the 3 different pairs (s, s), (c, c) and (e, e), taken two pairs at a time.

$\therefore$ The number of selections $= {}^3C_2 = \frac{3\times2}{2\times1} = 3$.

In case (3), selections are to be made of one pair of like letters out of the three and any two of the 5 different letters.

$\therefore$ The number of selections $= {}^3C_1 \times {}^5C_2 = 3 \times \dfrac{5.4}{2.1} = 30$.

In case (4), selections are to be made from the 6 different letters s, c, i, u, v, e taken 4 at a time.

$\therefore$ the number of selections $= {}^6C_4 = {}^6C_2 = \frac{6\times5}{2\times1} = 15$.

Hence, the required total number of selections

$$=5+3+30+15=53.$$

(*b*) To determine the total number of permutations, the 4 letters in each of the above selections in the different cases are to be arranged in all possible ways.

Now, from (1) the arrangements $= 5 \times \dfrac{\lfloor 4}{\lfloor 3} = 20$,

,, (2) ,, ,, $= 3 \times \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} = 18$

[$\because$ of the 4 letters 2 are alike, the other two are alike],

,, (3) the arrangements $= 30 \times \dfrac{\lfloor 4}{\lfloor 2} = 360$,

and ,, (4) ,, ,, $= 15 \times \lfloor 4 = 360$.

Hence, the required total number of arrangements

$$=20+18+360+360=758.$$

Exercise 4

1. Find the numerical values of :

(i) ${}^{6}C_{2}$ (ii) ${}^{10}C_{7}$ (iii) ${}^{30}C_{28}$ (iv) $\dfrac{18\,!}{16\,!\ \underline{|3}}$.

2. Find n, if ${}^{30}C_{n+5} = {}^{30}C_{n-3}$.

3. (i) Find r, if $2 \times {}^{r}C_{4} = 35 \times {}^{\frac{r}{2}}C_{3}$.

(ii) If ${}^{n}C_{12} = {}^{n}C_{8}$, find ${}^{22}C_{n}$.

(iii) If $m = {}^{n}C_{2}$, show that ${}^{m}C_{2} = 3.{}^{n+1}C_{4}$. [C. U. '12]

4. If ${}^{2n}C_{r} = {}^{2n}C_{r+2}$, find r. [C. U. '30]

5. How many different selections can be made of 5 members out of 8 ?

6. Out of 9 Swarajists and 5 Ministerialists how many different committees can be formed, each consisting of 6 Swarajists and 2 Ministerialists ? [C. U. '31]

7. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady ? [C. U. '37]

8. In a Municipal Corporation there are 20 Councillors and 8 Aldermen. How many committees can be formed consisting of 5 Councillors and 3 Aldermen ? [C. U. '33]

9. How many words of 2 vowels and 3 consonants can be formed from an alphabet of 5 vowels and 17 consonants, the letters of the word being all different ? [C. U. '32]

10. A basket contains 10 mangoes. Find how many different selections you can make of 3 mangoes so as to always include a particular mango. [C. U. '21]

11. From a company of 15 men how many selections of 9 men can be made so as to exclude three particular men ? [C. U. '54]

12. A committee of 6 is to be made from 7 teachers and 4 students. In how many ways can this be done, (i) if the committee contains exactly 2 students, (ii) at least 2 students ?

13. A candidate is required to answer 6 out of 10 questions which are divided into 2 groups, each containing 5 questions and he is not permitted to attempt more than 4 from any group. In how many different ways can he make his choice ? [C. U. '32]

14. At an election there are 5 candidates and 3 members are to be elected and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways may a voter choose a vote ? [C. U. '35]

15. How many different triangles can be formed by joining the angular points of a hexagon ? Find also the number of the diagonals of the hexagon.

16. There are 16 points in a plane, no three of which are in the same straight line. Find the number of straight lines which can be formed by joining them. [C. U. 1909]

17. There are 10 points in a plane, 4 of which are collinear. Find the number of (i) straight lines, (ii) of triangles which result from joining them.

18. There are n points in a plane of which no three are in a straight line except m, which are all in a straight line. Find the number of (i) different straight lines ; (ii) different triangles formed by joining the points. [C. U. 1928]

19. A certain council consists of a chairman, two vice-chairmen and 12 other members. How many different committees consisting of 6 can be formed including always the chairman and only one vice-chairman ? [C. U. '14]

20. If ${}^nC_r = 20$ and ${}^nP_r = 120$, find the n and r.

21. A man has 6 friends ; in how many ways may he invite one or more of them to dinner ? [C. U. '50]

22. A cricket team consisting of 11 players is to be selected from two groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of six shall contribute no fewer than 4 players ?

[C. U. '38]

22. (a) A cricket team is to be selected from 15 players of whom only 5 can bowl. In how many ways can the team be formed so as to include at least 3 bowlers ?

23. Find (i) the number of different straight lines that can be had by joining 15 different points on a plane, no three of which are collinear excepting 4 points which are collinear ; find also (ii) the number of triangles formed by joining them.

24. I have a money bag containing a rupee, an eight-anna piece, a four-anna piece and a two-anna piece. In how many ways is it possible for me to contribute a sum to a relief fund ?

25. How many combinations can be formed of eight counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even in each combination ? [C. U. '41]

26. In how many ways can 22 people be divided into 2 cricket teams to play against each other in a friendly game ? [C. U. '50]

27. Given n points in space no three of which are collinear and no four of which are coplanar. For what value of n will the number of st. lines be equal to the number of planes obtained by connecting these points ?

28. In how many ways can one or more of 8 patients be sent to a hospital ?

29. In how many ways can 12 things be equally divided among 4 persons ? [P. U. '46]

30. Find the number of ways in which p positive signs and n negative signs may be placed in a row so that no two negative signs shall be together, given $p+1 \geqslant n$.

31. How many different factors can 2310 have ?

32. I invite 8 friends to a party and place 4 at one round table and 4 at another. In how many ways can I arrange the friends ?

33. There are 7 gentlemen and 3 ladies contesting for 2 vacancies ; an elector can vote for any number of candidates not exceeding the number of vacancies. In how many ways is it possible to vote ? [P. U. '43]

34. If the total number of combinations of $4n$ different things : the total number of combinations of $2n$ different things $=257:1$, find n.

35. Show that the greatest value of ${}^{2p}C_a$: the greatest value of ${}^{2p-1}C_a = 2:1$.

36. In how many ways can 68 cards be divided (i) into four sets of 17 each, (ii) equally among four players ?

37. In how many ways can 9 things be divided in three groups of 2, 3, 4 things respectively ?

38. In how many ways can 12 different things be divided into four groups of 3 each ?

39. Find the number of (i) combinations, (ii) of permutations that can be made from the letters of the word *impression*, taken 4 at a time.

40. A boat's crew consists of 12 men, 4 of whom can only row on one side and 2 only on the other side. In how many ways can the crew be arranged ?

41. Show that in $({}^{2n}{}_n)(={}^{2n}C_n)$, the number of combinations in which a particular thing occurs is equal to the number of combinations in which a particular thing does not occur.

42. Find the number of (i) selections and (ii) arrangements that can be made taking 4 letters from the word '*alliteration*'.

[C. U. 1887]

43. Prove that the total number of selections that can be made out of the letters 'daddy did a deadly deed' is 1919.

[B. U. '10]

44. Show that the number of all possible selections of one or more questions from eight given questions, each question having an alternative, is 3^8-1. [C. U. 1928]

45. At an election three districts are to be canvassed by 12, 16 and 23 men respectively. If 51 men volunteer, in how many ways can they be allotted to the different districts ?

46. There are 4 boys of class IX, 3 of class X and 2 of class XI. Find the total number of selections that can be made with them so that no two of the same class may be included in any selection.

47. Find the number of different ways of dividing pr things into r equal groups.

48. From 3 mangoes, 4 oranges and 2 apples, how many selections of fruit can be made, taking at least one of each kind ? [Here fruits of the same kind are of different shapes.]

49. In how many ways can five things be divided between two boys.

50. There are 10 male and 8 female patients seeking admission into a hospital having 8 beds of which four are reserved for male and three for female patients and the remaining bed for either a male or a female patient. In how many ways can the admissions be made ?

THE BINOMIAL THEOREM

32. **Binomial expression :** Any expression consisting of two terms only is called a binomial expression.

Thus $x+a$, $3a+4b$, etc. are binomial expressions.

Binomial Theorem : The algebraical formula by which any binomial expression can be raised to any assigned power (*i. e.*, the expansion of any power of a binomial expression can be found) is called the Binomial Theorem.

This theorem was discovered by Sir Issac Newton.

33. Multiplying the four binomial expressions $(x+a)$, $(x+b)$, $(x+c)$, $(x+d)$, we have $(x+a)(x+b)(x+c)(x+d)$

$$=x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$+(abc+abd+acd+bcd)x+abcd.$$

It appears from this product that—

(*ii*) the first term x^4 is obtained by taking the product of the 4 x's out of the 4 given factors (none of the letters a, b, c, d has been taken here) ;

(*ii*) the second term involving x^3 is formed by taking x from any three factors in all possible ways and one of the letters a, b, c, d from the remaining factor ;

(*iii*) the third term involving x^2 is formed by taking x from *any two* factors in every possible ways and two of the letters a, b, c, d out of the remaining factors ;

(*iv*) the fourth term involving x is obtained by taking x out of *any one* factor and three of the letters a, b, c, d out of the remaining factors ;

(*v*) The last term, which is independent of x, is the product of all the letters a, b, c, d.

If we put $a=b=c=d$ in the above factors, we have

$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4.$$

This process is generally followed to prove the Binomial Theorem.

Binomial Theorem for a Positive Integral Index

34. *To find the expansion of $(a+x)^n$ when n is a positive integer.*

Or, *To prove that when n is a positive integer*

$$(a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + x^n \ldots(1)$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2}a^{n-2}x^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}a^{n-r}x^r + \ldots + x^n \ \ldots\ldots(2)$$

We know that $(a+x)^n = (a+x)(a+x)(a+x)\ldots$to n factors. In the continued product of the n factors on the right hand side each term is obtained by a product formed by multiplying together n number of letters, taken *one* from each of the factors. Hence each term is of n dimensions, *i.e.*, the sum of the powers of a and x is n in each term.

In case of the first term, only the letter a (and not any x) is taken from each of the n factors. We have n a's and their product is a^n. So the first term is $\mathbf{a}^n$.

Similarly, the last term is formed by taking only the letter x (and not any a) from each of the n factors and having their product. So the last term is $\mathbf{x}^n$.

The second term also is the product of n letters. The second term is formed by taking one a less, *i. e.*, taking a from any $(n-1)$ of the factors and one x from the remaining factor. Then their product is $a^{n-1}x$. The number of selections in this case is the coefficient of the second term. So ${}^nC_1$ or n is the coefficient. $\therefore$ The second term is ${}^nC_1a^{n-1}x$ or $na^{n-1}x$.

In the third term one more a is dropped. Thus a is taken from *any* $(n-2)$ of the factors and two x's from the remaining factors to make the dimension n. This selection can be made in ${}^nC_2$ or $\dfrac{n(n-1)}{\lfloor 2}$ ways. Thus the third therm is ${}^nC_2a^{n-2}x^2$

or $\dfrac{n(n-1)}{\lfloor 2}a^{n-2}x^2$.

Thus it is found that in any general term, if $(n-r)$ number of a's be taken, then r x's are to be taken with them to make the dimension n. Now the selection of n x's taken r x's at a time can be made in ${}^nC_r$ ways.

Hence, any general term is of the form ${}^nC_r a^{n-r} x^r$.

So, the terms in the expansion of $(a+x)^n$ can be obtained, putting 0, 1, 2, 3,......up to n for r in the general term. Hence it is proved that $(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \ldots$

$$\ldots + {}^nC_r a^{n-r} x^r + \ldots + x^n \quad [\because \quad {}^nC_0 = {}^nC_n = 1].$$

Or, $$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2}a^{n-2}x^2 + \ldots\ldots$$

$$+ \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r}a^{n-r}x^r + \ldots\ldots + x^n.$$

N. B. (1) The above formula is called the Binomial Theorem.

(2) The series on the right side of the formula is called the expansion of $(a+x)^n$.

(3) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_r, \ldots\ldots, {}^nC_n$ are the coefficients of the terms of the expansion. These are called the *Binomial coefficients.*

(4) It is evident from the formula-(1) that the expansion is *finite* and the number of terms is $n+1$, *i.e.*, 1 more than the index of $(a+x)$.

(5) In each term the sum of the indices of a and x is n and in each term the index of x is one less than the serial number of the term [*i.e.*, the index of x in the first term is $(1-1)$ or 0, in the second term it is $(2-1)$ or 1, and so on]. This index of x is also equal to the suffix of C.

(6) It is clear from the formula-(2) that in each term the number of factors of both the numerator and the denominator is ess by 1 than the serial number of the term.

35. Proof of Binomial Theorem by Induction

By actual multiplication we have

$(a+x)^2=a^2+2ax+x^2=a^2+{}^2C_1ax+x^2$,

$(a+x)^3=a^3+3a^2x+3ax^2+x^3=a^3+{}^3C_1a^2x+{}^3C_2ax^2+x^3$.

Hence we find that the theorem is true when $n=2$ and 3.

Let us assume the theorem to be true for a certain particular value of n (say m) *i.e.* let us assume that

$$(a+x)^m=a^m+{}^mC_1a^{m-1}x+{}^mC_2a^{m-2}x^2+\ldots\ldots$$
$$+{}^mC_ra^{m-r}x^r+\ldots+x^m \quad \ldots\ldots(1)$$

Multiplying both sides of (1) by $a+x$ we have

$$(a+x)^{m+1}=(a+x)\{a^m+{}^mC_1a^{m-1}x+\ldots+{}^mC_ra^{m-r}x^r+\ldots\ldots+x^m\}$$
$$=a^{m+1}+({}^mC_1+1)a^mx+({}^mC_2+{}^mC_1)a^{m-1}x^2+\ldots$$
$$+({}^mC_r+{}^mC_{r-1})a^{m-r+1}x^r+\ldots+x^{m+1}.$$

Now, $\because$ ${}^mC_1+1=m+1={}^{m+1}C_1$, and generally

$${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r,$$

$$\therefore \quad (a+x)^{m+1}=a^{m+1}+{}^{m+1}C_1a^mx+{}^{m+1}C_2a^{m-1}x^2+\ldots$$
$$+{}^{m+1}C_ra^{m-r+1}x^r+\ldots+x^{m+1}.$$

Hence, it is found that if the thoerem is true when $n=m$, it is also true when $n=m+1$. We have already seen that the theorem is true, when $n=3$, so it must be true when $n=4$. Again, since it is true for $n=4$, it is also true for $n=5$, and so on.

Hence, it is proved that the theorem is true for *all positive integral values of n.*

36. When n is a positive integer, prove that

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots\ldots+{}^nC_rx^r+\ldots\ldots+x^n\ldots\ldots(1)$$
$$=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}x^r$$
$$+\ldots\ldots+x^n\ldots\ldots(2)$$

By multiplication we have

$$(1+x)^2=1+2x+x^2=1+{}^2C_1x+x^2,$$

$$(1+x)^3=1+3x+3x^2+x^3=1+{}^3C_1x+{}^3C_2x^2+x^3.$$

Thus, it is evident that the theorem is true when $n=2$ and 3.

So let us assume that the theorem is true for some particular value of n (say m), *i.e.*, let us assume that

$$(1+x)^m=1+{}^mC_1x+{}^mC_2x^2+\ldots+{}^mC_rx^r+\ldots+x^m.$$

Multiplying both sides by $(1+x)$ we have

$$(1+x)^{m+1}=(1+x)\{1+{}^mC_1x+{}^mC_2x^2+\ldots+{}^mC_rx^r+\ldots+x^m\}$$
$$=1+({}^mC_1+1)x+({}^mC_2+{}^mC_1)x^2+\ldots$$
$$+({}^mC_r+{}^mC_{r-1})x^r+\ldots+x^{m+1}.$$

Now, $\because\ {}^mC_1+1=m+1={}^{m+1}C_1$ and ${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r$,

$$\therefore\ (1+x)^{m+1}=1+{}^{m+1}C_1x+{}^{m+1}C_2x^2+\ldots+{}^{m+1}C_rx^r+\ldots+x^{m+1}.$$

Thus, we find that if the theorem is true for $n=m$, it is also true for $n=m+1$.

It has been shown before that the theorem is true when $n=3$, it is therefore true when $n=4$.

Again, it being true for $n=4$, it must be true for $n=5$, and so on.

Hence, it is proved that the theorem is true for all integral values of n.

[N. B. We obtain the expansion of $(1+x)^n$, putting $a=1$ in the expansion of $(a+x)^n$. $(1+x)^n$ is therefore a particular form of $(a+x)^n$, the binomial coefficients being the same in both cases.]

37. To find the expansion of $(a+x)^n$ from that of $(1+x)^n$.

$$(a+x)^n=\left\{a\left(1+\frac{x}{a}\right)\right\}^n=a^n\left(1+\frac{x}{a}\right)^n=a^n(1+y)^n \text{ [putting } y \text{ for } \tfrac{x}{a}\text{]}$$
$$=a^n(1+{}^nC_1y+{}^nC_2y^2+\ldots\ldots+y^n)$$
$$=a^n\left(1+{}^nC_1\frac{x}{a}+{}^nC_2\frac{x^2}{a^2}+\ldots+\frac{x^n}{a^n}\right)$$
$$=a^n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+\ldots\ldots+x^n.$$

Hence the expansion of $(a+x)^n$ is obtained from that of $(1+y)^n$, *i.e.*, of $(1+x)^n$.

38. To find the expansion of $(a-x)^n$ and $(1-x)^n$.

$\because$ the expansions of $(a+x)^n$ and $(1+x)^n$ are true for all values of a and x,

$\therefore$ putting $-x$ for x in those two expansions we have

(*i*) $$(a-x)^n = a^n + {}^nC_1 a^{n-1}(-x) + {}^nC_2 a^{n-2}(-x)^2 + \dots + (-x)^n$$
$$= a^n - {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 - \dots + (-1)^n x^n.$$

(*ii*) $$(1-x)^n = (1)^n + {}^nC_1(-x) + {}^nC_2(-x)^2 + \dots + (-x)^n$$
$$= 1 - {}^nC_1 x + {}^nC_2 x^2 - \dots + (-1)^n x^n.$$

[N. B. If n is even in (*i*) and (*ii*) above, then $(-x)^n$ is positive, *i.e.*, $(-x)^n = x^n$. If n is odd, then $(-x)^n = -x^n$. As it is not known in (*i*) and (*ii*) whether n is even or odd, $(-1)^n x^n$ is put for $(-x)^n$. $(-1)^n$ will be positive or negative according as n is even or odd.]

It appears from (*i*) and (*ii*) that the numerical values of the terms in the expansions of $(a-x)^n$ and $(1-x)^n$ are the same as those of the terms in the expansions of $(a+x)^n$ and $(1+x)^n$ respectively, only they are alternately positive and negative. The first term in both cases is positive and the last term is positive or negative according as n is even or odd.]

39. To find the general Term.

The $(r+1)$th term in the expansion of $(a+x)^n$ is usually referred to as the general term, for any required term in the expansion can be found by putting a suitable value for r in the $(r+1)$th term.

Thus t_{10} or the tenth term can be found by putting 9 for r.

Just as the first term is represented by T_1 or t_1 and the second term by T_2 or t_2, so also the $(r+1)$th term is respresented by T_{r+1} or t_{r+1}. From the expansion of $(a+x)^n$ we have

$$\text{first term} = (1+0)\text{th term} = {}^nC_0 a^n = a^n,$$
$$\text{second term} = (1+1)\text{th term} = {}^nC_1 a^{n-1}x,$$
$$\text{third term} = (2+1)\text{th term} = {}^nC_2 a^{n-2}x^2,$$
$$\text{fourth term} = (3+1)\text{th term} = {}^nC_3 a^{n-3}x^3,$$

Similarly $(r+1)$th term $= {}^nC_r a^{n-r} x^r$,

$\therefore$ The **general term** in the expansion of $(a+x)^n$

$$= {}^nC_r a^{n-r} x^r \ldots\ldots(1)$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r} a^{n-r} x^r \ldots\ldots(2)$$

Again, similarly, the *general term* in the expansion of

$$(1+x)^n = {}^nC_r x^r \ldots\ldots(3)$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r} x^r \ldots\ldots(4);$$

The general term in the expansion $(a-x)^n$

$$= {}^nC_r a^{n-r}(-x)^r = (-1)^r\ {}^nC_r a^{n-r} x^r \ldots\ldots(5)$$

and the general term in the expansion of $(1-x)^n$

$$= {}^nC_r (-x)^r = (-1)^r\ {}^nC_r x^r \ldots\ldots(6).$$

40. To find the middle term.

We have already seen that the number of terms in the expansion of any power of a binomial expression is one more than its index. Evidently there will be one middle term, if the index of the power (say n) is even, but two terms will be the middle terms, when the index (n) is odd.

Let the index n be even, i.e., let $n=2m$. Then the number of terms in the expansion is $2m+1$ (which is an odd number). Hence the $(m+1)$ *th* term or $(\frac{n}{2}+1)$ *th* term is the only middle term here.

Again, let the index n be odd, i.e., $n=2m+1$.

Then the number of terms in the expansion $= n+1 = 2m+2$ (which is an even number). Hence we have two middle terms here, the $(m+1)$ *th* or $\frac{n+1}{2}$*th* term and the $(m+2)$*th* or $(\frac{n+1}{2}+1)$*th* term being the two middle terms.

Example. To find the middle term in the expansion of $(1+x)^n$.

(i) If n be even, suppose $n=2m$, so $m=\frac{n}{2}$.

Here the total number of terms in the expansion of $(1+x)^n$ is $n+1$ i.e., $2m+1$ (which is odd). So $(m+1)th$ term, i.e., $(\frac{n}{2}+1)$th term will be the only middle term here.

$$\therefore \quad \text{The middle term} = {}^nC_{\frac{1}{2}n} x^{\frac{1}{2}n} = \frac{\lfloor n}{(\lfloor \frac{1}{2}n)^2} x^{\frac{1}{2}n}.$$

(ii) If n be odd, let $n=2m+1$ so $m=\frac{1}{2}(n-1)$.

Here the total number of terms in the expansion is $2m+2$ (which is even). So there will be two middle terms here, $(m+1)$th and $(m+2)$th terms, i.e., $\{\frac{1}{2}(n-1)+1\}$th and $\{\frac{1}{2}(n+1)+1\}$th terms being the two middle terms.

$$\therefore \quad \text{the middle terms} = {}^nC_{\frac{1}{2}(n-1)} x^{\frac{1}{2}(n-1)}$$

$$\text{and } {}^nC_{\frac{1}{2}(n+1)} x^{\frac{1}{2}(n+1)}$$

$$Or, = \frac{\lfloor n}{\lfloor \frac{1}{2}(n-1) \cdot \lfloor \frac{1}{2}(n+1)} x^{\frac{1}{2}(n-1)}$$

$$\text{and } \frac{\lfloor n}{\lfloor \frac{1}{2}(n+1) \cdot \lfloor \frac{1}{2}(n-1)} x^{\frac{1}{2}(n+1)}.$$

N. B. In (ii) above, the numerical coefficients of the two middle terms are equal.]

41. Equidistant terms

If the serial order of a term from the beginning of a series be the same as that of another term from the end (i.e., counting from the end to the left), then the two terms are said to be equidistant terms of the series.

Thus the fifth term from the beginning and the fifth term from the end (counting towards left) are equidistant terms. Generally, the $(r+1)$th term from the beginning and the $(r+1)$th term from the end are equidistant terms.

42. *To prove that in the expansion of $(a+x)^n$ or $(1+x)^n$ the coefficents of terms equidistant from the beginning and the end are equal.*

In both the expansions the coefficient of $(r+1)$th term from the beginning is ${}^nC_r$.

Now, the $(r+1)$th term from the end has $\{(n+1)-(r+1)\}$ or $(n-r)$ number of terms preceding it, as the total number of terms in each of these two expansions is $(n+1)$.

Hence, the $(r+1)$th term from the end is the $(n-r+1)$th term counting from the beginning.

$\therefore$ the coefficient of the $(r+1)$th term from the end

$=$the coefficient of $(n-r+1)$th term from the beginning

$={}^nC_{n-r}={}^nC_r.$ $[\because {}^nC_{n-r}={}^nC_r]$

Hence, the coefficient of the $(r+1)$th term from the beginning $=$the coefficient of the $(r+1)$th term from the end.

Similarly it can be proved that the coefficients of any two equidistant terms are equal.

43. Properties of the Binomial coefficients.

${}^nC_0, {}^nC_1, {}^nC_2,\ldots,{}^nC_r,\ldots,{}^nC_n$ are binomial coefficients and they are often briefly written as $C_0, C_1, C_2\ldots,C_r,\ldots,C_n$ respectively.

So we have

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots+x^n \quad \ldots(1)$$

$$=C_0+C_1x+C_2x^2+\ldots+C_nx^n \quad \ldots(2)$$

$[\because {}^nC_0={}^nC_n=1]$

I. Now putting $x=1$ in (2) above we have

$2^n=C_0+C_1+C_2+\ldots+C_n$ $[\because$ any power of $1=1]$

$=$the sum of all the coefficients.

Hence, we find that **the sum of all the coefficients in the expansion of $(1+x)^n=2^n$.**

II. Again, putting $x=-1$ in (2) we have

$(1-1)^n=C_0-C_1+C_2-\ldots+(-1)^nC_n,$

or, $0=C_0-C_1+C_2-\ldots+(-1)^nC_n$

$\therefore$ $C_0+C_2+C_4+\cdots=C_1+C_3+C_5+\cdots$ [by transposition]

Hence, in the expansion of $(1+x)^n$, **the sum of the coefficients of the odd terms=the sum of the coefficients of the even terms.**

III. Since the sum of all the coefficients $= 2^n$,

$\therefore$ the sum of the coefficients of the odd terms
$=$ the sum of the coefficients of the even terms
$= \frac{1}{2} \times$ sum of all the coefficients $= \frac{1}{2} \times 2^n = 2^{-1} \times 2^n = 2^{n-1}$.

IV. $\because C_0 + C_1 + C_2 + \ldots + C_n = 2^n$,

$\therefore \quad 1 + C_1 + C_2 + \ldots + C_n = 2^n, \quad \therefore \; C_1 + C_2 + \cdots + C_n = 2^n - 1$.

[N. B. It can easily be proved by putting $a = x = 1$ and $a = x = -1$ that the numerical coefficients in the expansion of $(a+x)^n$ have the same properties.]

44. Greatest coefficient.

The coefficient of $(r+1)$th term in each of the expansions of $(1+x)^n$ and $(a+x)^n$ is ${}^nC_r$. It has been shown in Art. 31 (in the chapter 'combination') that

(i) When n is even ${}^nC_r$ is greatest, if $r = \frac{1}{2}n$;

and (ii) when n is odd ${}^nC_r$ is greatest, if $r = \frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$.

Hence, it is proved that (i) **when n is even,** the $(r+1)$th, i. e., $(\frac{1}{2}n+1)$th term, i.e., the middle term of the expansion has the greatest coefficient ;

and (ii) **when n is odd,** the two middle terms of the expansion, i. e., the $\{\frac{1}{2}(n-1)+1\}$th and $\{\frac{1}{2}(n+1)+1\}$th terms have the greatest coefficients, these two coefficients being equal.

45. To find the greatest term.

Suppose we have to find the greatest term in the expansion of $(a+x)^n$. Let the rth term be denoted by t_r.

In the expansion of $(a+x)^n$, $t_r = {}^nC_{r-1} a^{n-r+1} x^{r-1}$ ...(1)

and $t_{r+1} = {}^nC_r a^{n-r} x^r$ ...(2).

Dividing (2) by (1) we have

$$\frac{t_{r+1}}{t_r} = \frac{n-r+1}{r} \cdot \frac{x}{a}, \quad \therefore \; t_{r+1} = t_r \times \left(\frac{n-r+1}{r} \cdot \frac{x}{a} \right).$$

Hence, $t_{r+1} >=$ or $< t_r$,

if $(n-r+1)x >=$ or $< ra$,

i.e., if $(n+1)x >=$ or $< ra+rx$,

i.e., if $(n+1)x >=$ or $< (a+x)r$,

i.e., if $r < =$ or $> \frac{n+1}{a+x}.x$.

(i) Now, let $\frac{n+1}{a+x}.x$ be an integer denoted by m.

Hence, so long as $r < m$, $t_{r+1} > t_r$, i.e., each term will be greater than its preceding term, and thus the terms go on increasing up to t_m. When $r=m$, then $t_{r+1}=t_r$ i.e., $t_{m+1}=t_m$.

Again, when $r > m$ (i.e., r is $m+1$ or more), $t_{r+1} < t_r$ and the succeeding terms go on diminishing.

Hence t_{m+1} and t_m will be the greatest terms and $t_{m+1}=t_m$.

(ii) If, however $\frac{n+1}{a+x}.x$ be not an integer,

let it $= p +$ a positive proper fraction.

Hence, for values of r up to p, $r < \frac{n+1}{a+x}.x$, and so $t_{r+1} > t_r$.

But if r is equal to or greater than $(p+1)$, then $t_{r+1} < t_r$ and the succeeding terms go on diminishing.

Hence, t_{p+1} is the greatest term here.

[N. B. (1) The greatest term in the expansion of $(1+x)^n$ can be derived in the same way, putting 1 for a.

(2) As we are only concerned with the *numerically* greatest term (i.e., with the absolute value of the term), the greatest term in the expansion of $(a-x)^n$ will be the same as the greatest term in the expansion of $(a+x)^n$. So to find the numerically greatest term, the sign of the second term of the binomial may be ignored. The greatest term in the expansion of $(1-x)^n$ can be similarly obtained].

Examples (5)

Ex. 1. Expand $(a+3b)^6$.

Putting $3b$ for x in the formula of Art. 34, we have

$$(a+3b)^6=a^6+{}^6C_1a^5(3b)+{}^6C_2a^4(3b)^2+{}^6C_3a^3(3b)^3$$
$$+{}^6C_4a^2(3b)^4+{}^6C_5a(3b)^5+{}^6C_6(3b)^6.$$

Now, ${}^6C_1=6$, ${}^6C_2=\frac{6\times5}{2\times1}=15$, ${}^6C_3=\frac{6\times5\times4}{3\times2}=20$,
${}^6C_4={}^6C_2=15$, ${}^6C_5={}^6C_1=6$ and ${}^6C_6=1$.

$$\therefore\quad (a+3b)^6=a^6+6a^5.3b+15a^4.9b^2+20a^3.27b^3+15a^2.81b^4$$
$$+6a.243b^5+729b^6$$
$$=a^6+18a^5b+135a^4b^2+540a^3b^3+1215a^2b^4$$
$$+1458ab^5+729b^6.$$

Ex. 2. Find the expansion of $\left(a-\frac{1}{a}\right)^7$.

$$\left(a-\frac{1}{a}\right)^7=a^7-{}^7C_1a^6.\frac{1}{a}+{}^7C_2a^5.\frac{1}{a^2}-{}^7C_3a^4.\frac{1}{a^3}+{}^7C_4a^3.\frac{1}{a^4}$$
$$-{}^7C_5a^2.\frac{1}{a^5}+{}^7C_6a.\frac{1}{a^6}-{}^7C_7.\frac{1}{a^7}$$
$$=a^7-7a^5+21a^3-35a+\frac{35}{a}-\frac{21}{a^3}+\frac{7}{a^5}-\frac{1}{a^7}.$$

Ex. 3. Expand $(\frac{1}{2}a-3b)^5$.

$$(\tfrac{1}{2}a-3b)^5=\left\{\frac{a}{2}\left(1-\frac{6b}{a}\right)\right\}^5=\left(\frac{a}{2}\right)^5\left(1-\frac{6b}{a}\right)^5$$
$$=\frac{a^5}{32}\left\{1+{}^5C_1\left(-\frac{6b}{a}\right)+{}^5C_2\left(-\frac{6b}{a}\right)^2+{}^5C_3\left(-\frac{6b}{a}\right)^3\right.$$
$$\left.+{}^5C_4\left(-\frac{6b}{a}\right)^4+{}^5C_5\left(-\frac{6b}{a}\right)^5\right\}$$
$$=\frac{a^5}{32}\left\{1-5\left(\frac{6b}{a}\right)+\frac{5.4}{2.1}\left(\frac{6b}{a}\right)^2-\frac{5.4.3}{3.2.1}\left(\frac{6b}{a}\right)^3+\frac{5}{1}\left(\frac{6b}{a}\right)^4-\left(\frac{6b}{a}\right)^5\right\}$$
$$=\frac{a^5}{32}\left(1-\frac{30b}{a}+\frac{360b^2}{a^2}-\frac{2160b^3}{a^3}+\frac{6480b^4}{a^4}-\frac{7776b^5}{a^5}\right)$$
$$=\frac{a^5}{32}-\frac{15}{16}a^4b+\frac{45}{4}a^3b^2-\frac{135}{2}a^2b^3+\frac{405}{2}ab^4-243b^5.$$

Ex. 4. Find the expansion of $(x^2+x+1)^4$.

Taking $x+1$ as one term we have

$$(x^2+x+1)^4=\{x^2+(x+1)\}^4$$
$$=(x^2)^4+{}^4C_1(x^2)^3.(x+1)+{}^4C_2(x^2)^2(x+1)^2+{}^4C_3x^2(x+1)^3+{}^4C_4(x+1)^4$$
$$=x^8+4x^6(x+1)+6x^4(x^2+2x+1)+4x^2(x^3+3x^2+3x+1)+x^4+4x^3+6x^2+4x+1$$
$$=x^8+4x^7+4x^6+6x^6+12x^5+6x^4+4x^5+12x^4+12x^3+4x^2+x^4+4x^3+6x^2+4x+1$$
$$=x^8+4x^7+10x^6+16x^5+19x^4+16x^3+10x^2+4x+1.$$

Ex. 5. Find the value of $(a-\sqrt{1-a^2})^4+(a+\sqrt{1-a^2})^4$.

Here putting x for $\sqrt{1-a^2}$ we have to find the sum of the expansions of $(a-x)^4$ and $(a+x)^4$, in which the numerical values of the terms are equal, but the second and fourth terms in the first expansion are negative.

Hence, the second and fourth terms of the first expansion will cancel those of the second expansion.

$\therefore$ The required value $=2(a^4+{}^4C_2a^2x^2+{}^4C_4x^4)$

$$=2\{a^4+6a^2(1-a^2)+1(1-a^2)^2\}$$
$$=2(a^4+6a^2-6a^4+1-2a^2+a^4)=2+8a^2-8a^4.$$

Ex. 6. Find the $5th$ term in the expansion of $(x-5y)^9$.

Here the fifth term will be positive.

$$\therefore \quad t_5={}^9C_4x^5(-5y)^4=126x^5\times 625y^4=78750x^5y^4.$$

Ex. 7. Find the term containing x^{18} in the expansion of $(x^3-3x)^{10}$.

$$(x^3-3x)^{10}=\left\{x^3\left(1-\frac{3}{x^2}\right)\right\}^{10}=x^{30}\left(1-\frac{3}{x^2}\right)^{10}.$$

$$\because \quad x^{30}\times\frac{1}{x^{12}}=x^{18}\ ;$$

$\therefore$ the term containing x^{18} will be obtained if the term containing $\frac{1}{x^{12}}$ in the expansion of $\left(1-\frac{3}{x^2}\right)^{10}$ be multiplied by x^{30}.

$\because$ the term of $\left(1-\frac{3}{x^2}\right)^{10}$ in which $\left(\frac{3}{x^2}\right)^6$ occurs will contain $\frac{1}{x^{12}}$,

$\therefore$ the required term $=x^{30}\times{}^{10}C_6\left(-\frac{3}{x^2}\right)^6$

$$=x^{30}\times\frac{10\times9\times8\times7}{4\times3\times2\times1}\times3^6\times\frac{1}{x^{12}}=153090x^{18}.$$

Ex. 8. Find the coefficient of x^{15} in the expansion of $\left(x^3+\frac{a}{x^2}\right)^{10}$.

Suppose the $(r+1)$th term in the expansion contains x^{15}.

Here $t_{r+1}={}^{10}C_r(x^3)^{10-r}\left(\frac{a}{x^2}\right)^r={}^{10}C_r x^{30-3r}.x^{-2r}.a^r$

$$={}^{10}C_r x^{30-5r}.a^r.$$

Now, $\because$ x^{15} occurs in $(r+1)$th term,

$\therefore$ $x^{30-5r}=x^{15}$, $\therefore$ $30-5r=15$, $\therefore$ $r=3$.

$\therefore$ $(r+1)$th or the 4th term contains x^{15}.

$\therefore$ The required coefficient $={}^{10}C_3a^3=\frac{10\times9\times8}{3\times2\times1}a^3=120a^3$.

Ex. 9. Find the coefficient of a^{-11} in the expansion of $\left(a^3-\frac{1}{a^2}\right)^{13}$.

Suppose the $(r+1)$th term contains a^{-11}.

$(r+1)$th term in the expansion of $\left(a^3-\frac{1}{a^2}\right)^{13}$

$={}^{13}C_r(a^3)^{13-r}\left(-\frac{1}{a^2}\right)^r$

$={}^{13}C_ra^{39-3r}\times\frac{(-1)^r}{a^{2r}}=(-1)^r\times{}^{13}C_ra^{39-3r}\times a^{-2r}$

$=(-1)^r\times{}^{13}C_r\times a^{39-5r}$.

Now, if a^{-11} occurs in this term, then the index of a's power here is -11, *i.e.*, $39-5r=-11$, or, $5r=50$, $\therefore$ $r=10$.

$\therefore$ The required coefficient $=(-1)^r\times{}^{13}C_r=(-1)^{10}.{}^{13}C_{10}$

$$=1\times{}^{13}C_3=\frac{13\times12\times11}{3\times2\times1}=286.$$

Ex. 10. Find the term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$. [H. S. '65 ; C. U. '34]

Suppose the $(r+1)$th term to be independent of x.

Here $t_{r+1} = {}^{12}C_r(x^2)^{12-r}.\dfrac{1}{x^r} = {}^{12}C_r x^{24-2r}.x^{-r}$

$$= {}^{12}C_r x^{24-3r}$$

Now, this term will be independent of x, if the index of x here is zero, i.e., if $24-3r=0$. $\because$ $24-3r=0$, $\therefore$ $r=8$.

Hence $(8+1)$th or 9th term is independent of x.

$$\therefore \text{ the required term} = {}^{12}C_8 = {}^{12}C_4 = \frac{12.11.10.9}{4.3.2.1} = 495.$$

Ex. 11. Obtain the term free from x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$. [C. U. '31]

Suppose $(r+1)$th term to be independent of x.

Here, $t_{r+1} = {}^{2n}C_r(x)^{2n-r}.\left(\dfrac{1}{x}\right)^r = {}^{2n}C_r x^{2n-r}.x^{-r} = {}^{2n}C_r x^{2n-2r}$.

This term will be independent of x if $2n-2r=0$, or, if $r=n$.
$\therefore$ the $(n+1)$th term is independent of x.

$$\therefore \text{ the required term} = {}^{2n}C_n = \frac{\lfloor 2n}{\lfloor n \, \lfloor 2n-n} = \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

Ex. 12. Find the middle term in the expansion of $\left(3x-\dfrac{1}{2x}\right)^8$.

Here, the number of terms in the expansion is 9 and so the 5th term will be the middle term.

$$\therefore \text{ the required middle term} = {}^8C_4(3x)^4\left(-\frac{1}{2x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1} \times 3^4x^4.\left(\frac{1}{2}\right)^4.\left(\frac{1}{x}\right)^4$$

$$= \frac{2835}{8}x^4 \times \frac{1}{x^4} = \frac{2835}{8}.$$

Ex. 13. Find the middle term in the expansion of $\left(a-\frac{1}{a}\right)^7$.

Evidently the number of terms in the given expansion is 8 and it will have two terms as the middle terms. These two terms are the $(\frac{1}{2}\times 8)$ or 4th and $(\frac{1}{2}\times 8+1)$ or 5th terms of the expansion.

The 4th term $= {}^7C_3(a)^4\left(-\frac{1}{a}\right)^3 = \frac{7.6.5}{3.2.1}a^4 \times -\frac{1}{a^3} = -35a$;

and the 5th term $= {}^7C_4a^3\left(-\frac{1}{a}\right)^4 = \frac{7.6.5}{3.2.1}a^3.\frac{1}{a^4} = \frac{35}{a}$.

$\therefore$ $-35a$ and $\frac{35}{a}$ are the required middle terms.

Ex. 14. Expand $\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1}$, giving in particular the general term and the two middle terms. [C. U. '32]

$$\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1} = \left(\frac{a}{b}\right)^{2n+1} + {}^{2n+1}C_1\left(\frac{a}{b}\right)^{2n}\left(\frac{b}{a}\right) + {}^{2n+1}C_2\left(\frac{a}{b}\right)^{2n-1}\left(\frac{b}{a}\right)^2 + \ldots + \left(\frac{b}{a}\right)^{2n+1}$$

$$= \left(\frac{a}{b}\right)^{2n+1} + (2n+1)\left(\frac{a}{b}\right)^{2n-1} + \frac{(2n+1)2n}{2.1}\left(\frac{a}{b}\right)^{2n-3} + \ldots + \left(\frac{b}{a}\right)^{2n+1},$$

this is the required expansion.

The general term $=(r+1)$th term

$$= {}^{2n+1}C_r\left(\frac{a}{b}\right)^{2n-r+1}\left(\frac{b}{a}\right)^r = {}^{2n+1}C_r\left(\frac{a}{b}\right)^{2n-r+1}\left(\frac{a}{b}\right)^{-r}$$

$$= \frac{(2n+1).2n.(2n-1)\ldots(2n+1-r+1)}{\underline{|r}}\left(\frac{a}{b}\right)^{2n-r+1-r}$$

$$= \frac{(2n+1).2n.(2n-1)\ldots(2n+2-r)}{\underline{|r}}\left(\frac{a}{b}\right)^{2n+1-2r}$$

Here the number of terms in the expansion being $2n+2$, the $(n+1)$th and $(n+2)$th terms will be the middle terms.

$\therefore$ One middle term $=(n+1)$th term

$$=\frac{(2n+1).2n.(2n-1)\ldots(2n+1-n+1)}{\lfloor n}\left(\frac{a}{b}\right)^{2n+1-n}\left(\frac{b}{a}\right)^{n}$$

$$=\frac{(2n+1).2n.(2n-1)\ldots(n+2)}{\lfloor n}.\frac{a}{b}\left[\because\ \left(\frac{b}{a}\right)^{n}=\left(\frac{a}{b}\right)^{-n}\right]$$

The other middle term $=(n+2)$th term

$$=\frac{(2n+1).2n.(2n-1)\ldots(2n+1-n)}{\lfloor n+1}\left(\frac{a}{b}\right)^{2n+1-n-1}\left(\frac{b}{a}\right)^{n+1}$$

$$=\frac{(2n+1)2n(2n-1)\ldots(n+1)}{\lfloor n+1}.\frac{b}{a}\left[\because\ \left(\frac{a}{b}\right)^{n}=\left(\frac{b}{a}\right)^{-n}\right]$$

Ex. 15. Find which is the greatest term in the expansion of $(1-2a)^9$ when $a=\frac{1}{3}$.

Here $\dfrac{t_{r+1}}{t_r}=\dfrac{9-r+1}{r}.\ 2a$ (numerically)

$$=\frac{10-r}{r}\times 2\times\frac{1}{3}\quad(\because\ a=\tfrac{1}{3})=\frac{2(10-r)}{3r}.$$

$\therefore\quad t_{r+1}>=\text{or}<t_r$, if $2(10-r)>=\text{or}<3r$

i.e., if $20>=$ or $<5r$, i.e., if $r<=\text{or}>4$.

$\therefore\quad t_4>t_3>t_2\ldots\ldots,\ t_5=t_4$ and $t_5>t_6>t_7\ldots\ldots,$

$\therefore$ Here, the 4th and the 5th terms are equal and the greatest.

Ex. 16. Find the value of the greatest term in the expansion of $(2a+3x)^n$ when $n=13$, $a=9$, $x=4$.

Here $\dfrac{t_{r+1}}{t_r}=\dfrac{n-r+1}{r}.\dfrac{3x}{2a}=\dfrac{14-r}{r}.\dfrac{12}{18}=\dfrac{28-2r}{3r}$

$\therefore\quad t_{r+1}>=\text{or}<t_r$, if $28-2r>=\text{or}<3r$,

i.e., if $5r<=\text{or}>28$, i.e., if $r<=\text{or}>5\frac{3}{5}$,

$\therefore$ here t_6 is the greatest term.

$\therefore$ the greatest term $={}^{13}C_5(2a)^8(3x)^5={}^{13}C_5.2^8.3^5.a^8x^5$.

Now find the value of this term putting 9 for a and 4 for x.

Ex. 17. Find the numerically greatest coefficient in the expansion of (i) $(1+a)^{11}$, (ii) $(3-5x)^8$.

(i) Here $\frac{t_{r+1}}{t_r}=\frac{11-r+1}{r}$.

$\therefore \quad t_{r+1}>t_r$, if $\frac{12-r}{r}>1$, or, if $r<6$.

Hence, the coefficients of the 6th and 7th terms will be equal and the greatest.

$\therefore$ the required greatest coefficient

$$={}^{11}C_5=\frac{11\times10\times9\times8\times7}{5\times4\times3\times2}=462.$$

(ii) Here $\frac{t_{r+1}}{t_r}=\frac{8-r+1}{r}\cdot\frac{5}{3}$ [only the numerical value is to be taken into consideration and not the sign].

$\therefore \quad t_{r+1}>t_r$, if $(9-r).5>3r$,

i.e., if $45>8r$, or, if $r<5\frac{5}{8}$

$\therefore \quad t_6$ has the greatest coefficient.

$\therefore$ the reqd. coefficient$={}^8C_5(3)^3(5)^5=\frac{8\times7\times6}{3\times2}\times27\times3125$

$$=4725000.$$

Ex. 18. If three successive coefficients in the expansion of $(1+x)^n$ be 252, 210 and 120; find n.

Let the coefficients of the $(r+1)$th, $(r+2)$th and $(r+3)$th terms be the successive coefficients.

$\therefore$ Here, ${}^nC_r=252$, ${}^nC_{r+1}=210$, ${}^nC_{r+2}=120$.

$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r+1}}=\frac{252}{210}, \quad \text{or,} \quad \frac{\dfrac{\lfloor n}{\lfloor r\,\lfloor n-r}}{\dfrac{\lfloor n}{\lfloor r+1\,\lfloor n-r-1}}=\frac{252}{210}, \quad \text{or,} \quad \frac{r+1}{n-r}=\frac{6}{5},$$

or, $6n-6r=5r+5$, or, $6n-11r=5$......(1).

Again, $\frac{{}^nC_{r+1}}{{}^nC_{r+2}}=\frac{210}{120}$, or, $\frac{r+2}{n-r-1}=\frac{210}{120}=\frac{7}{4}$,

or, $7n-7r-7=4r+8$, or, $7n-11r=15$......(2).

Now, solving (1) and (2) we have $n=10$.

Ex. 19. Find the coefficient of a in the expansion of $(1-2a^3+3a^5)\left(1+\frac{1}{a}\right)^8$.

The given exp. $=(1-2a^3+3a^5)\times$

$$\left(1+{}^8C_1\frac{1}{a}+{}^8C_2\frac{1}{a^2}+{}^8C_3\frac{1}{a^3}+{}^8C_4\frac{1}{a^4}+\ldots\ldots+\frac{1}{a^8}\right)$$

$$=\left(1-2a^3+3a^5\right)\left(1+\frac{8}{a}+\frac{28}{a^2}+\frac{56}{a^3}+\frac{70}{a^4}+\cdots\right)$$

Now, in the multiplication

$$-2a^3\times\frac{28}{a^2}=-56a,\ \text{ and }\ 3a^5\times\frac{70}{a^4}=210a,$$

and a has no other coefficient in the product.

$\therefore$ the required coefficient $=-56+210=154$.

Ex. 20. Prove that $3^{2n}-8n-1$ is divisible by 64 for all positive integral values of n greater than 1.

$$3^{2n}-8n-1=9^n-8n-1=(1+8)^n-8n-1$$

$$=(1+n.8+{}^nC_2.8^2+{}^nC_3.8^3+\ldots\ldots)-8n-1$$

$$={}^nC_2.8^2+{}^nC_3.8^3+\ldots\ldots$$

Now, since each term on the right hand side is a multiple of 8^2 or 64, $\therefore$ each term is divisible by 64.

$\therefore$ $3^{2n}-8n-1$ also is divisible by 64.

Ex. 21. Apply the Binomial Theorem to find the value of $(\cdot999)^4$ correct to 3 places of decimals.

$$(\cdot999)^4=(1-\cdot001)^4=1-{}^4C_1\times\cdot001+{}^4C_2\times(\cdot001)^2-\ldots$$

$$=1-4\times\cdot001+6\times\cdot000001-\ldots$$

$$=1-\cdot004=\cdot996 \text{ (correct to 3 places of decimals).}$$

Ex. 22. Show that the coefficient of x^p io the expansion of $\left(x+\frac{1}{x}\right)^n$ is $\dfrac{\lfloor n}{\lfloor \frac{1}{2}(n+p)\ \lfloor \frac{1}{2}(n-p)}$.

Suppose x^p occurs in the $(r+1)$ th term in the expansion.

Now, $t_{r+1}={}^nC_rx^{n-r}\left(\frac{1}{x}\right)^r={}^nC_rx^{n-r}x^{-r}={}^nC_rx^{n-2r}$.

This term will contain x^p, if $n-2r=p$. $\therefore$ $r=\frac{1}{2}(n-p)$.

$\therefore$ the required coefficient

$$={}^nC_r={}^nC_{\frac{1}{2}(n-p)}=\frac{\lfloor n}{\lfloor \frac{1}{2}(n-p)\ \lfloor \frac{1}{2}(n+p)}.$$

Ex. 23. If in the expansion of $(1+x)^{2n+1}$, the coefficients of x^r and x^{r+1} be equal, find r.

Here, the coefficient of $x^r={}^{2n+1}C_r$, and that of $x^{r+1}={}^{2n+1}C_{r+1}$.

$\because$ here ${}^{2n+1}C_r={}^{2n+1}C_{r+1}$ (hyp.)

$\therefore$ either $r=r+1$ or $r+(r+1)=2n+1$,

but r can never be equal to $r+1$.

$\therefore$ here $r+(r+1)=2n+1$, or, $2r+1=2n+1$, $\therefore$ $r=n$.

Ex. 24. If n is a positive integer, show that

$$(1+a)^n-2na(1+a)^{n-1}+\frac{2n(2n-2)}{\lfloor 2}a^2(1+a)^{n-2}-\ldots=(1-a)^n.$$

$$\text{L. H. S.}=(1+a)^n-n.(1+a)^{n-1}(2a)^1+\frac{n(n-1)}{\lfloor 2}(1+a)^{n-2}.(2a)^2-\ldots$$

$$=(1+a)^n-{}^nC_1(1+a)^{n-1}(2a)^1+{}^nC_2(1+a)^{n-2}.(2a)^2-\ldots$$

$$=\{(1+a)-2a\}^n=(1-a)^n.$$

Ex. 25. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that $C_0^2+C_1^2+C_2^2+C_3^2+C_4^2\ldots+C_n^2=\frac{(2n)\,!}{n\,!\,n\,!}$. [C. U. '41]

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n\ldots(1)$$

and $(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n\ldots(2)$

Now, multiplying (1) and (2) we have

$$(1+x)^{2n}=(C_0+C_1x+C_2x^2+\ldots+C_nx^n)\times(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n).$$

As this is an identity, the coefficients of x^n on the two sides must be equal.

$\because$ The coefficient of x^n on the L. H. S. $= {}^{2n}C_n = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}$,

and that of x^n on the R. H. S. $= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$.

$\therefore \quad C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \dfrac{(2n)\,!}{n\,!\,n\,!}$.

Exercise 5

Expand the following binomials :—

1. $(a+3)^5$ **2.** $(x-y)^6$ **3.** $(2x+3y)^5$ **4.** $\left(x-\dfrac{3}{y}\right)^7$

5. Expand and simplify : $(\sqrt{3}+1)^6 - (\sqrt{3}-1)^6$.

6. Find the value of $(1+\sqrt{1-a^2})^5 + (1-\sqrt{1-a^2})^5$.

7. Expand the trinomial $(a^2-a-2)^3$.

Write down the following terms :—

8. 8th term in the expansion of $\left(1+\dfrac{1}{x}\right)^{17}$.

9. 5th term in the expansion of $(x-5y)^8$.

10. nth term in the expansion of $\left(a+\dfrac{1}{a}\right)^{2n}$.

10. (*a*) Find the p th term from the beginning and the pth term from the end in the expansion of $(1+3x)^n$.

Find the *middle term* (or terms) in the expansion of :—

11. $\left(a-\dfrac{1}{a}\right)^{12}$ **12.** $\left(x-\dfrac{1}{x}\right)^9$ **13.** $\left(x-\dfrac{1}{x}\right)^{10}$ [C. U. '10]

14. $\left(3x-\dfrac{1}{2x}\right)^8$ **15.** (i) $\left(x-\dfrac{1}{x}\right)^{2n}$ (ii) $\left(\dfrac{a}{x}+\dfrac{x}{a}\right)^{10}$

[C. U. '43]

16. Show that the middle term in the expansion of

$\left(x+\dfrac{1}{x}\right)^{2n}$ is $\dfrac{1.\,3.\,5\ldots\ldots(2n-1)}{\lfloor n}2^n$. [P. U. '42]

17. Find the coefficient of x^{15} in the expansion of $(x-x^2)^{10}$. [C. U. '25]

18. Find the coefficient of x^{16} in the expansion of $(2x^2-x)^{10}$. [C. U. '47]

19. Find the coefficient of x^{-10} in the expansion of $\left(x^4-\frac{1}{x^2}\right)^{14}$.

20. Find the coefficient of x^{10} in the expansion of $(1+x+x^2)(1-x)^{15}$. [M. U. '20]

21. Find the coefficient of x in the expansion of $(1-2x^2+3x^4)\left(1+\frac{1}{x}\right)^{10}$, and that of x^3 in the expansion of $\left(x-\frac{1}{x}\right)^7$.

22. Find the coefficient of x^{2r+1} in the expansion of $\left(x-\frac{1}{x}\right)^{2n+1}$.

23. Show that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in tne expansion of $(1+x)^{2n-1}$.

Find the term *independent of* x in the following expansions :—

24. $\left(x^2+\frac{1}{x}\right)^{12}$ [C. U. '34] 25. $\left(2x+\frac{1}{3x^2}\right)^9$ [C. U. '36]

26. $\left(x+\frac{1}{x}\right)^{2n}$ [P. U. '48] 27. $(1-x)^2\left(x+\frac{1}{x}\right)^7$.

28. (a) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

(b) Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$. [C. U. '10. '21]

29. Obtain the term free from x in the expansion of $(1+x)^m\left(1+\frac{1}{x}\right)^n$. [U. U. '47]

30. Find the greatest term in the expansion of $(x-4y)^9$, when $x=\frac{1}{2}$ and $y=\frac{1}{3}$.

31. Which is the greatest term in the expansion of $(2x-3y)^n$ when $n=13$, $x=9$, $y=4$?

32. Find the value of the greatest term in the expansion of $(1+2a)^9$ when $a=\frac{1}{3}$.

33. Find the numerically greatest coefficient in the expansion of $(1+\frac{2}{3}a)^{12}$.

34. Find the sum of the coefficients of $(1-2x)^5$.

35. Find the sum of the coefficients of $(2x-3y)^7$.

36. Prove that in the expansion of $(a+b)^n$ the coefficients of terms equidistant from the two ends are equal.

37. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$ when n is a positive integer. [C. U. '41]

38. The third, fourth and fifth terms in the expansion of $(x+a)^n$ in descending powers of x are 84, 280 and 560 respectively ; find x, a and n. [C. U. '55]

39. If three successive coefficients in the expansion of $(1+x)^n$ be 28, 56 and 70, find n.

40. If three successive coefficients in the expansion of $(1+x)^n$ be 165, 330 and 462, find n. [P. U. '45]

41. In the expansion of $(1+x)^{m+n}$ where m and n are positive integers, prove that the coefficients of x^m and x^n are equal. [C. U. '35]

42. Show that the sum of the coefficients of the odd terms in the expansion of $(1+x)^{2n}$ is 2^{2n-1}. [C. U. '17]

43. If the rth term in the expansion of $(x+1)^{20}$ has its coefficient equal to that of $(r+4)$th term, find r. [C. U. '46]

44. In the expansion of $(1+x)^{10}$, the coefficient of the $(4r+5)$th term is equal to that of $(2r+1)$th term, find r. [C. U. '49]

45. Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$. [C. U. '18]

46. If a, b, c, d be the *3rd, 4th, 5th* and *6th* terms respectively in the expansion of $(x+A)^n$, n being a positive integer, show that $\dfrac{b^2-ac}{c^2-bd}=\dfrac{5a}{3c}$. [C. U. '57]

47. Apply the Binomial theorem to find the value of $(99)^3$ and of $(\cdot 99)^4$ correct to 2 places of decimals.

48. Show that $2^{2r}-3r-1$ is divisible by 9 for all positive integral values of r greater than 1.

49. Show that for all integral values of n greater than 1, $6^{2n}-35n-1$ is divisible by 1225.

50. In the expansion of $(1+x)^n$, if $C_0, C_1, C_2, \ldots, C_n$ be the coefficients, show that $C_1+2\,.\,C_2+3\,.\,C_3+\ldots+n.C_n=n\,.\,2^{n-1}$. [C. U. '38]

51. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that $C_1-2C_2+3C_3-\ldots+(-1)^{n-1}.n.C_n=0$. [C. U. '42]

52. In the expansion of $(1+x)^n$, the successive coefficients are $a_0, a_1, a_2, a_3, \ldots, a_n$; show that

$$a_0+2a_1+3a_2+\ldots+(n+1)a_n=2^n+n\,.\,2^{n-1}.$$ [C. U. '29]

If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that :—

53. $\dfrac{C_0}{1}+\dfrac{C_1}{2}+\dfrac{C_2}{3}+\ldots+\dfrac{C_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$. [C. U. '45]

54. $C_0-\dfrac{C_1}{2}+\dfrac{C_2}{3}-\ldots+(-1)^n.\dfrac{C_n}{n+1}=\dfrac{1}{n+1}$. [P. U. '44]

55. $C_0+2C_1+3C_2+\ldots+(n+1)C_n=(n+2)2^{n-1}$.

[H. S. 1966 ; B. U. E. '63 ; Rangoon, '50]

56. $C_k=C_{n-k}$.

57. $C_0C_n+C_1C_{n-1}+C_2C_{n-2}+\ldots+C_nC_0=\dfrac{\lfloor 2n}{(\lfloor n)^2}$.

58. $C_0+\frac{1}{2}C_1+\frac{1}{3}C_2+\ldots+\frac{C_n}{n+1}=\frac{(2^{n+1}-1)}{n+1}$. [Delhi '50]

59. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$,

show that $a_0+a_2+a_4+\ldots+a_{2n}=\frac{3^n+1}{2}$.

60. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, find the value of :—

(i) $2C_0-3C_1+4C_2-5C_3+\ldots$ to $(n+1)$ terms.

(ii) $C_0C_1+C_1C_2+C_2C_3+\ldots+C_{n-1}C_n$.

(iii) $\frac{C_1}{C_0}+\frac{2C_2}{C_1}+\frac{3C_3}{C_2}+\ldots+\frac{nC_n}{C_{n-1}}$.

61. Find the middle term in the expansion of $(1+x)^n$ when n is a positive integer.

62. Express $(1+x)^n$ in the form of a series when n is a positive integer and calculate the sum of the coefficients when $n=6$.

63. If the 2nd, 3rd and 4th terms in the expansion of $(a+x)^n$ be 240, 720 and 1080 respectively, find a, x and n. [G. U. '48]

64. If a, b, c be three consecutive coefficients in the expansion of a power of $(1+x)$, prove that the index of the power is $\frac{2ac+b(a+c)}{b^2-ac}$ and that the number of the term of which a is the coefficient is $\frac{a(b+c)}{b^2-ac}$. [P. U. '41]

65. If a_1, a_2, a_3, a_4 are any four consecutive coefficients in the expansion of $(1+x)^n$, show that,

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$ [P. U. '50]

BINOMIAL THEOREM FOR FRACTIONAL OR NEGATIVE INDEX

46. We know from the Binomial Theorem that when n is any positive integer,

$$(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots\ \ldots(1)$$

But we have not yet seen whether the theorem holds good when n is fractional or negative.

The Binomial Theorem can be proved when n is fractional or negative, but as this proof is beyond our scope, it is not discussed here. Let us see what will be the difference in the Binomial Theorem when n is fractional or negative.

We have seen that the general term of the Binomial series is $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r$ and the number of terms in the expansion of $(1+x)^n$ is $(n+1)$. Here the series terminates after $(n+1)$ terms, and hence the series is *finite*. For if the value of r be greater than n, one of the factors in the numerator of the coefficient of the general term becomes zero (thus when $r=n+1$, then $n-r+1=0$) and so the term vanishes. Hence the series ends with the $(n+1)th$ term where the index of x is n.

Now let us consider what happens if n be fractional or negative.

In the series (1) above, r must be a positive integer, for the number of terms cannot be fractional or negative. As r is a positive integer and n is fractional or negative, none of the factors in the numerator of the coefficient $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ of the general term can be zero and so the term does not vanish. Thus the series becomes *infinite*, as none of its terms vanishes, however large the power of x may be.

By actual division we find that $(1-x)^{-2}=\dfrac{1}{(1-x)^2}$

$=1+2x+3x^2+4x^3+\ldots$, and the process of division will never end.

Hence the number of terms in this series is unlimited.

Again, by putting -2 for n and $-x$ for x in the binomial theorem, we have

$$(1-x)^{-2}=1+(-2)(-x)+\frac{(-2)(-3)}{2.1}(-x)^2$$
$$+\frac{(-2)(-3)(-4)}{3.2.1}(-x)^3+\ldots$$
$$=1+2x+3x^2+4x^3+\ldots$$

Thus the Binomial Theorem holds good in this case.

Now, if we put 2 for x in $(1-x)^{-2}$, we have

$$(1-2)^{-2}=1+2.2+3.2^2+4.2^3+\ldots\text{to }\infty,$$

i.e., $(-1)^{-2}=1+2.2+3.2^2+4.2^3+\ldots$, which is not possible; for here the left hand side is not equal to the right hand side.

Similarly it can be shown that the Binomial Theorem does not hold good when $x=-1$ or 1.

Thus we find that the Binomial Theorem cannot be established for any value of x. It holds good when x is greater than -1 but less than 1, *i.e.*, when $-1<x<1$.

We, therefore, conclude that if n is fractional or negative,

$$(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\ldots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r+\ldots\text{to }\infty,$$

when x is numerically less than unity.

[N. B. The symbol $|x|<1$ is used to denote that x is less than 1.]

47. General term

You know that $(r+1)$-th term is taken as the general term. Let it be denoted by t_{r+1}.

$$\because \quad 1+x)^n = 1+nx+\frac{n(n-1)}{\lfloor\underline{2}}x^2+\frac{n(n-1)(n-2)}{\lfloor\underline{3}}x^3+\ldots\ldots,$$

$$\therefore \quad t_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor\underline{r}}x^r.$$

Thus, (1). in the expansion of $(1-x)^n$ the general term,

$$t_{r+1}=\frac{n(n-1)(n-2)\ldots(n+r+1)}{\lfloor\underline{r}}.\left(-x\right)^r$$

$$=(-1)^r.\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor\underline{r}}x^r.$$

(2). In the expansion of $(1+x)^{-n}$,

$$t_{r+1}=\frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{\lfloor\underline{r}}x^r$$

$$=(-1)^r.\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor\underline{r}}x^r.$$

[Here the number of factors in the numerator is r, and $-n=-1\times n$, $(-n-1)=-1\times(n+1)\ldots$;

$\therefore \quad (-1)\times(-1)\times(-1)\times\ldots$to r factors$=(-1)^r$.]

(3). In the expansion of $(1-x)^{-n}$,

$$t_{r+1}=\frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{\lfloor\underline{r}}\left(-x\right)^r$$

$$=(-1)^r.\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor\underline{r}}\left(-1\right)^r x^r$$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor\underline{r}}x^r$$

$[\because (-1)^r\times(-1)^r=(-1)^{2r}=1.]$

The above results may be summed up as follows :

As θ increases from 0° to 90°, $\cos\theta$ diminishes from 1 to 0.

As θ increases from 90° to 180°, $\cos\theta$ diminishes from 0 to -1

As θ increases from 180° to 270°, $\cos\theta$ increases from -1 to 0.

As θ increases from 270° to 360°, $\cos\theta$ increases from 0 to 1.

122. Changes in tangent.

$$\text{Tangent} = \frac{\text{perpendicular}}{\text{base}}.$$

In the first quadrant, as θ increases, the perpendicular (PM) increases but the base (OM) decreases, both remaining positive. So the tangent increases gradually.

When $\theta = 0^\circ$, the hypotenuse OP coincides with OX and then the perpendicular $= 0$ and the base $=$ OP, and therefore $\tan\theta = \frac{0}{OP} = 0$.

When $\theta = 90^\circ$, OP coincides with OY and so the perpendicular $=$ OP and the base $= 0$, and therefore $\tan 90^\circ = \frac{OP}{0} = \infty$.

Hence in this quadrant $\tan\theta$ increases from 0 to ∞.

In the second quadrant, the perpendicular (PM) gradually diminishes, but the base is negative and increases numerically. Here we observe that as soon as the radius vector OP passes OY and enters the second quadrant, the value of $\tan\theta$ suddenly passes from ∞ to $-\infty$, *i.e.*, there is a sudden break in the value of $\tan\theta$ which changes from a very large positive to a very large negative value.

As OP coincides with OX′ when θ is 180°, the perpendicular becomes zero and the base $=$ OP. Then $\tan\theta = \frac{0}{OP} = 0$. Hence, in the second quadrant the value of the tangent numerically diminishes from ∞ to 0 (*i.e.*, really increases from $-\infty$ to 0).

In the third quadrant, the perpendicular increases numerically from 0 to OY′, while the base decreases numerically from OX′ to 0, both being negative. So tan θ is positive. Since OP coincides with OY′ when θ is 270°, the base = 0 and the perpendicular = OP, and therefore tan $270° = \frac{OP}{0} = \infty$. Hence, in the third quadrant tan θ increases from 0 to ∞.

In the fourth quadrant the base is positive but the perpendicular is negative and so the tangent is negative. Again, in this quadrant, the perpendicular gradually diminishes while the base increases and therefore the tangent diminishes numerically. It is seen here that as soon as the radius vector (OP) passes OY′, tan θ suddenly passes from ∞ to $-\infty$. So here also there is a sudden break in the value of tan θ. When $\theta = 360°$, OP and OX coincide, the perpendicular (PM) = 0 and the base is equal to OP, $\therefore$ tan $360° = \frac{0}{OP} = 0$. Hence in this quadrant the tangent gradually increases from $-\infty$ to 0.

To state briefly :

As θ increases from 0° to 90°, tan θ increases from 0 to ∞, but as θ passes through 90°, tan θ suddenly changes from ∞ to $-\infty$.

As θ increases from 90° to 180°, tan θ increases from $-\infty$ to 0.

As θ increases from 180° to 270°, tan θ increases from 0 to ∞, but as θ passes through 270°, tan θ suddenly changes from ∞ to $-\infty$.

As θ increases from 270° to 360°, tan θ increases from $-\infty$ to 0.

123. Changes in cotangent.

The value of $\cot\theta$ is the reciprocal of the value of $\tan\theta$, *i.e.*, $\cot\theta=\dfrac{1}{\tan\theta}$. Hence the changes in $\cot\theta$ can be obtained from the changes in $\tan\theta$ as follows :

As θ increrases from 0° to 90°, $\cot\theta$ decreases from ∞ to 0 ;

As θ increases from 90° to 180°, $\cot\theta$ diminishes from 0 to $-\infty$, and as θ passes through 180°, $\cot\theta$ suddenly changes from $-\infty$ to $+\infty$;

As θ increases from 180° to 270°, $\cot\theta$ diminishes from ∞ to 0 ;

As θ increases from 270° to 360°, $\cot\theta$ diminishes from 0 to $-\infty$, but as θ passes through 360°, $\cot\theta$ suddenly changes from $-\infty$ to $+\infty$.

124. Changes in secant.

As $\sec\theta=\dfrac{1}{\cos\theta}$, the changes in secant can be determined from those in cosine as follows :

As θ increases from 0° to 90°, $\sec\theta$ increases from 1 to ∞ and immediately after $\sec\theta$ suddenly changes from $+\infty$ to $-\infty$.

As θ increases from 90° to 180°, $\sec\theta$ increases from $-\infty$ to -1.

As θ increases from 180° to 270°, $\sec\theta$ diminishes from -1 to $-\infty$ and immediatelv after $\sec\theta$ suddenly changes from $-\infty$ to $+\infty$.

As θ increases from 270° to 360°, $\sec\theta$ diminishes from ∞ to 1.

125. Chages in cosecant.

As cosec $\theta = \frac{1}{\sin\theta}$, the changes in cosecant can be obtained from the changes in sine as follows :

As θ increases from $0°$ to $90°$, cosec θ diminishes from ∞ to 1;

As θ increases from $90°$ to $180°$, cosec θ increases from 1 to ∞ and then cosec θ suddenly changes from $+\infty$ to $-\infty$;

As θ increases from $180°$ to $270°$, cosec θ increases from $-\infty$ to -1 ;

And as θ increases from $270°$ to $360°$, cosec θ diminishes from -1 to $-\infty$ and as θ passes through $360°$, cosec θ suddenly changes from $-\infty$ to $+\infty$.

126. We know that if an angle increases by any complete multiple of 2π (or $360°$), all its trigonometrical ratios remain the same (*i.e.*, unaltered). If the radius vector OP of angle θ revolves further and makes complete revolutions the trigonometrical ratios of θ will be repeated for each complete revolution. Since there is a recurrence of the values of trigonometrical functions (ratios), they are called *periodic functions* (they being repeated after each period of 2π).

Graphs of Trigonometrical Functions

127. The graphs of trigonometrical functions ($\sin x$, $\cos\theta$, etc) can be drawn just like those of algebraic functions.

Here also the straight lines XOX′ and YOY′ intersecting at right angles at O are taken as the axes of co-ordinates, O being

the origin. The positive and negative directions of the axes are the same as in algebraic graphs.

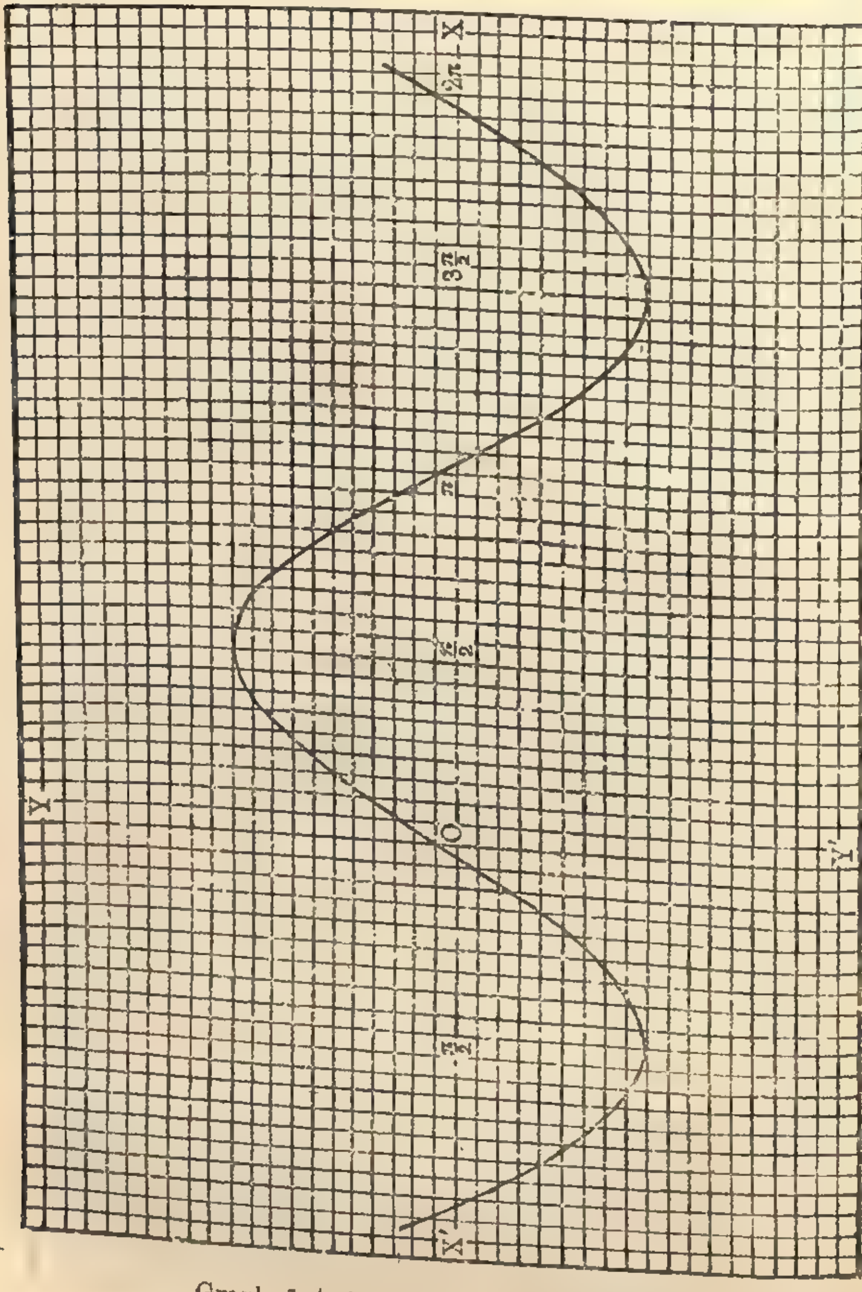
Here the values of the angles are represented by chosen lengths along the x-axis and the corresponding values of the trigonometrical ratios along the y-axis. Thus each pair of the corresponding values will be the co-ordinates of a point. Then a suitable scale being chosen, several points are plotted on the graph paper.

Now joining the plotted points free-hand we obtain the required graphs of the given trigonometrical function.

128. Sine graph or graph of sin *x*.

Let $y=\sin x$. The values of $\sin x$ (*i.e.*, y) corresponding to the different values of x can be found from the table of natural sines. Here the values of $\sin x$ (correct to 2 places of decimals) corresponding to the values of x, differing by 10°, are tabulated below.

x	-90°	-80°	-70°	-60°	-50°	-40°	30°	-20°
y or $\sin x$	-1	$-·98$	$-·94$	$-·87$	$-·77$	$-·64$	$-·50$	$-·34$
x	-10°	0°	10°	20°	30°	40°	50°	60°
y or $\sin x$	$-·17$	0	$·17$	$·34$	$·50$	$·64$	$·77$	$·87$
x	70°	80°	90°	100°	110°	120°	...	...
y or $\sin x$	$·94$	$·98$	1	$·98$	$·94$	$·87$	...	...



Graph 1 [sine graph ($-\pi$ to 2π)]

Let one small division along the x-axis represent 10° and 10 small divisions along the y-axis represent unity (*i.e.*, *1*). Now, plotting the points $(-90^\circ, -1)$, $(-80^\circ, -\cdot98)$, etc tabulated above and joining them free-hand, we get the required graph [See graph 1].

[*N.B.* The sines of angles from 0° to 90° are given in the table of natural sines. To find the sines of angles less than 0° and greater than 90°, we take the help of the formulas $\sin(-\theta) = -\sin\theta$, $\sin(180^\circ - \theta) = \sin\theta$, $\sin(180^\circ + \theta) = -\sin\theta$, $\sin(360^\circ - \theta) = -\sin\theta$. (2) The values of the angles may be taken at intervals of 10°, 5°, 15°, or at any other suitable intervals. (3) It is convenient to represent unity (*i. e.*, *1*) by 10 or 5 small divisions along y-axis. (4) This process will be followed in drawing other graphs.]

Some special features of sine graph

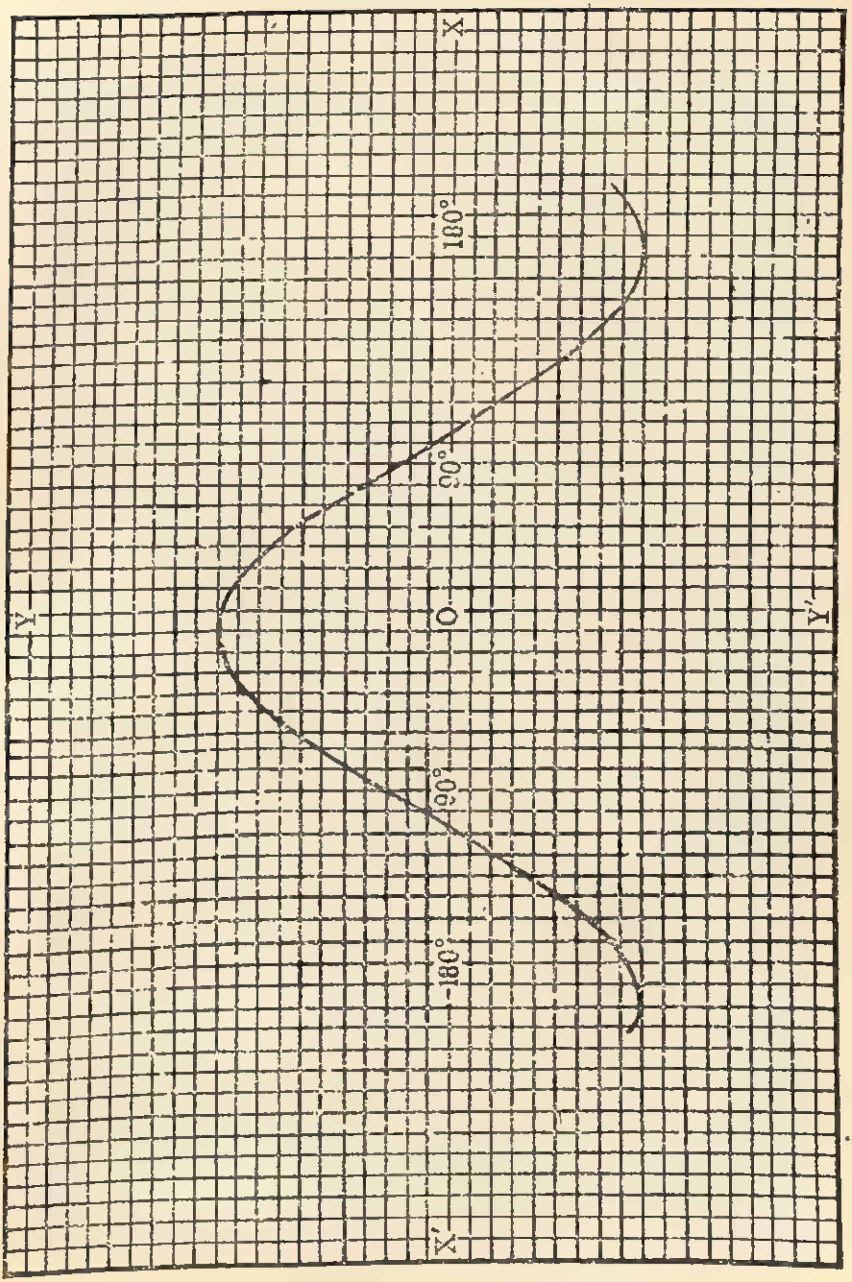
It appears from the graph that (i) it is a continuous graph extending on either side in symmetrical wave form. (ii) The ordinate of no point on the graph exceeds $+1$ or is less than -1, and so the maximum value of $\sin x$ is $+1$ and its minimum value is -1, and these values occur when the values of x are odd multiplies of 90°. (iii) At the origin and at points where x is an even multiple of 90°, the value of $\sin x$ is 0, for the graph cuts the x-axis at these points. (iv) $\because$ $\sin(2n\pi + x) = \sin x$, $\therefore$ the portion of the graph from 0° to 2π goes on being repeated on either side.

129. Cosine graph or graph of cos *x*.

Let $y = \cos x$. Here the values of x at intervals of 15° and the corresponding values of $\cos x$ (to 2 places of decimals), obtained from the natural cosine table, are tabulated below :

x	$-90°$	$-75°$	$-60°$	$-45°$	$-30°$	$-15°$	$0°$	$15°$
y or $\cos x$	0	·26	·5	·71	·87	·97	1	·97
x	30°	45°	60°	75°	90°	105°	120°	135°
y or $\cos x$	·87	·71	·5	·26	0	$-$·26	$-$·5	$-$·71
x	150°	165°	180°	195°	210°	225°	240°	...
y or $\cos x$	$-$·87	$-$·97	-1	$-$·97	$-$·87	$-$·71	$-$·5	...

Let one small division along the x-axis represent 10° and 10 small divisions along the y-axis represent 1. Now, plotting the above points and joining them free-hand, we obtain the required graph. [see graph 2]



Graph 2 [cosine graph (– 180° to 180°)]

Some special features of cosine graph

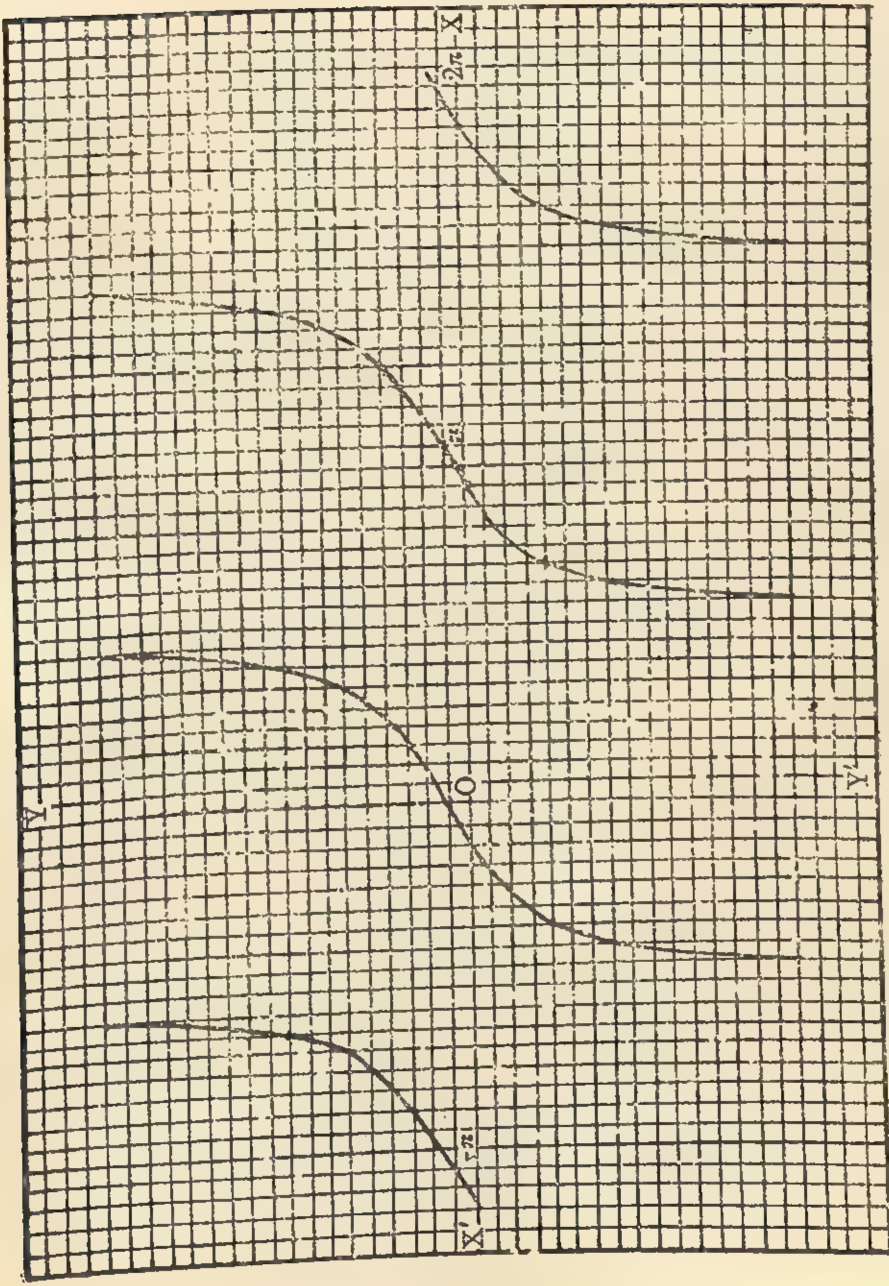
If appears from the graph that (i) the portion of the graph from -90° to 90° is symmetrical about the y-axis. This happens because $\cos(-x)=\cos x$. (ii) As $\cos(2n\pi+x)=\cos x$, the graph repeats itself at intervals of 360° (iii) The cosine graph becomes exactly the same as the sine graph, only if the sine graph is moved to the left through 90°-space, *i.e.*, if the origin O (in graph 1) be shifted 9 small divisions to the left and this new position of O be taken as the origin. This is so because $\cos x=\sin(90^\circ+x)$.

130. Tangent graph or graph of tan *x*.

Let $y=\tan x$. Here the values of x differing by 10° and the corresponding values of $\tan x$ (to 2 places of decimals) are tabulated below from the natural tangent table.

x	-120°	-110°	-100°	-90°	-80°	70°	-60°
y or $\tan x$	1·73	2·75	5·76	∞, $-\infty$	-5·67	-2·75	-1·73
x	-50°	-40°	-30°	-20°	-10°	0°	10°
y or $\tan x$	-1·19	$-$·84	$-$·58	$-$·36	$-$·18	0	·18
x	20°	30°	40°	50°	60°	70°	80°
y or $\tan x$	·36	·58	·84	1·19	1·73	2·75	5·67
x	90°	100°	110°	120°	...	...	...
y or $\tan x$	∞, $-\infty$	-5·67	-2·75	-1·73	...	...	...

Let 1 small division along the x-axis represent 10° and 3 small divisions along y-axis represent unity. Now plotting the above tabulated points and joining them free-hand, we obtain the graph of $\tan x$ [see graph 3].



Graph 3 [tanget graph (− π to 2π)]

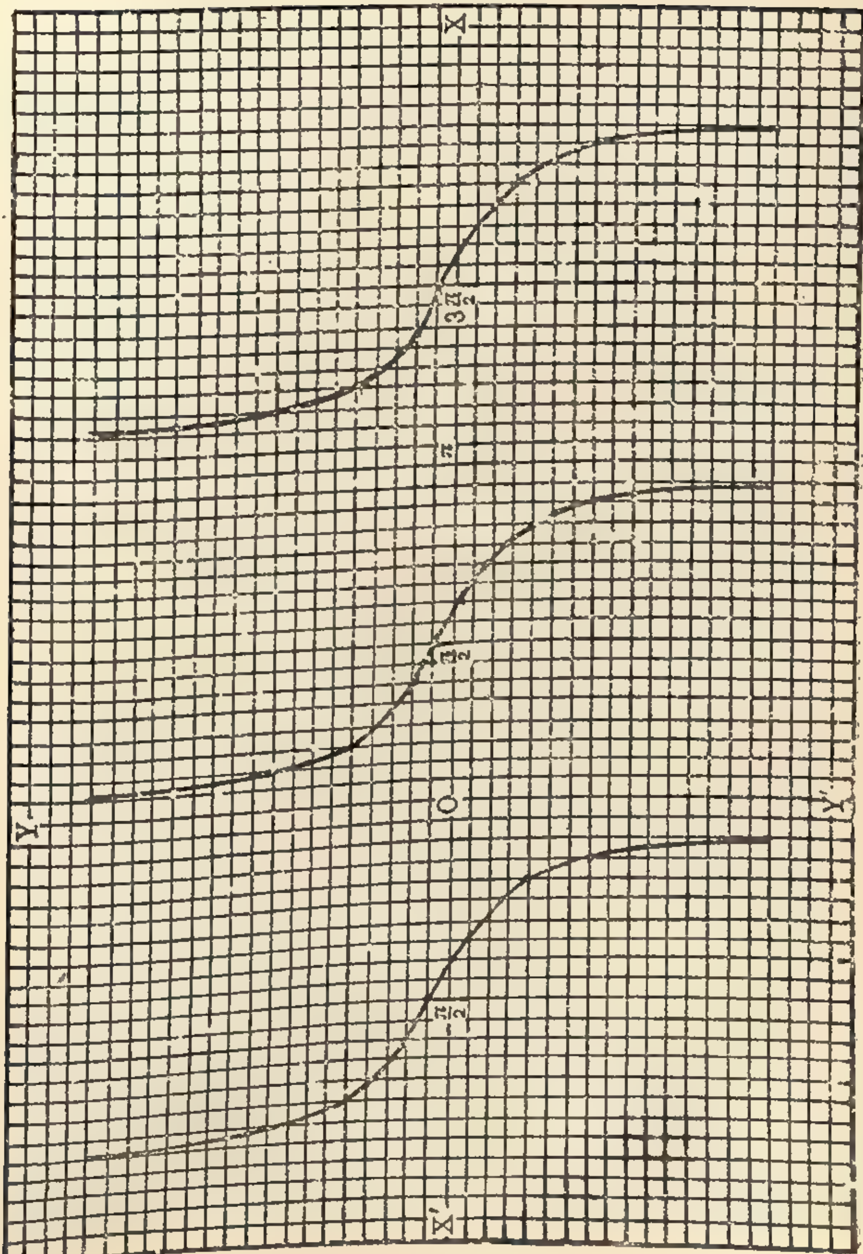
Some special features of tangent graph

It appears from the graph that (i) it is not a continuous curve, but consists of an infinite number of similar separate portions or branches, parallel to one another. The graph is disconnected at points where the value of x is an odd multiple of 90°. As x passes through these points from the left to the right, the value of tan x suddenly changes from ∞ to $-\infty$. (ii) The graph continually approaches the lines parallel to the y-axis at points corresponding to the odd multiples of 90° on either side of the x-axis. These lines are called the *asymptotes* to the curve of the graph. (iii) $\because$ $\tan(n.180°+x)=\tan x$ (where n is any integer), $\therefore$ each branch is repeated at an interval of 180°, the repetition being of the branch from $-90°$ to 90° on both sides (left and right).

131. Cotangent graph or graph of cot *x*.

Let $y=\cot x$. Here the values of x at intervals of 10° and the corresponding values of y (*i.e.*, of cot x) taken from the natural cotangent table are tabulated below.

x	$-120°$	$-110°$	$-100°$	$-90°$	$-80°$	$-70°$	$-60°$
y or cot x	·58	·36	·18	0	$-$·18	$-$·36	$-$·58
x	$-50°$	$-40°$	$-30°$	$-20°$	$-10°$	0°	10°
y or cot x	$-$·84	$-$1·19	$-$1·73	$-$2·75	$-$5·67	$-\infty$, ∞	5·67
x	20°	30°	40°	50°	60°	70°	80°
y or cot x	2·75	1·73	1·19	·84	·58	·36	·18
x	90°	100°	110°	120°	...	...	...
y or cot x	0	$-$·18	$-$·36	$-$·58	...	...	...



Graph 4 [cotangent graph $(-\pi$ to $2\pi)$]

Let 1 small division along x-axis represent $10°$ and 3 small divisions along y-axis represent unity or 1. Plotting the above points and joining them free-hand, we obtain the required graph [see graph 4].

Some special features of cotangent graph

(i) It is also a discontinuous graph. The continuity breaks at points where $x = 0°$ or any multiple of $180°$. (ii) The tangent graph becomes the cotangent graph, being shifted through $90°$ either to the left or to the right. (iii) $\because$ $\cot(n.\ 180° + x) = \cot x$, $\therefore$ the portion of the graph between $0°$ and $180°$ are repeated again and again on either side. (iv) The graph continually approaches the lines parallel to the y-axis on both sides of the x-axis at points where $x = 0°$ or any multiple of $180°$, but never actually meets them. These lines are called the asymptotes to the curve.

132. Cosecant graph or graph of cosec x.

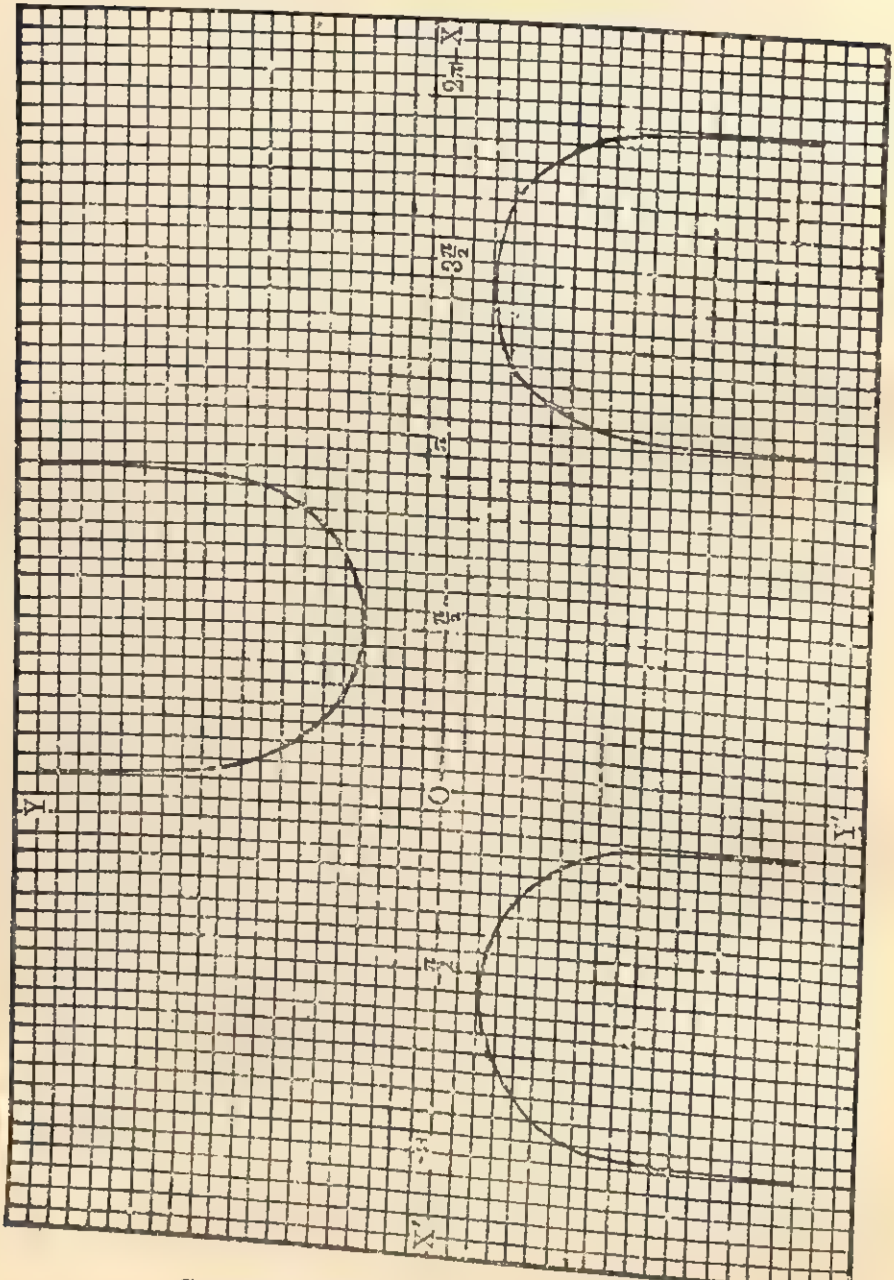
Let $y = \operatorname{cosec} x$. Here the values of x at intervals of $15°$ and the corresponding values of cosec x are tabulated below from the natural cosecant table [if this table be not available, the values of cosec x can be found from the natural sine table, as $\operatorname{cosec} x = \frac{1}{\sin x}$.]

x	$-105°$	$-90°$	$-75°$	$-60°$	$-45°$	$-30°$	$-15°$
y or cosec x	$-1·04$	-1	$-1·04$	$-1·15$	$-1·41$	-2	$-3·86$
x	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$
y or cosec x	3·86	2	1·41	1·15	1 04	1	1·04
x	$120°$	$135°$	$150°$	$165°$	$195°$	$210°$	$225°$
y or cosec x	1·15	1·41	2	3·86	$-3·86$	-2	$-1·41$

Let one small division along the x-axis denote $10°$ and 3 such divisions along the y-axis denote 1. Now, plotting the above points and joining them free-hand, we obtain the required graph. [see graph 5].

Some special features of cosecant graph

(i) This graph also is not continuous. It consists of an infinite number of detached branches. At angle $0°$ and at each multiple angle of $180°$, the graph is disconnected. At these points the lines parallel to the y-axis are asymptotes to the curve.



Graph 5 [cosecant graph ($-\pi$ to 2π)]

Ex. 6. Find the sum of $1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{4}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{8}+\ldots$to ∞.

The given series

$$=1-\frac{1}{2}\cdot\frac{1}{2}+\frac{\frac{1}{2}(\frac{1}{2}+1)}{\lfloor 2}\cdot\left(\frac{1}{2}\right)^2-\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{\lfloor 3}\cdot\left(\frac{1}{2}\right)^3+\ldots\ldots$$

$$=(1+\tfrac{1}{2})^{-\frac{1}{2}}=(\tfrac{3}{2})^{-\frac{1}{2}}=(\tfrac{2}{3})^{\frac{1}{2}}=\sqrt{\tfrac{2}{3}}.$$

Ex. 7. Find the binomial expression whose expansion is $1-\frac{1}{6}+\frac{1.3}{6.12}-\frac{1.3.5}{6.12.18}+\ldots\ldots$, and find its sum.

The given series

$$=1-\frac{1}{2}\cdot\frac{1}{3}+\frac{\frac{1}{2}(\frac{1}{2}+1)}{\lfloor 2}\left(\frac{1}{3}\right)^2-\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{\lfloor 3}\left(\frac{1}{3}\right)^3+\ldots\ldots$$

$=(1+\frac{1}{3})^{-\frac{1}{2}}$, which is a binomial expression.

$\therefore$ the required sum$=(1+\frac{1}{3})^{-\frac{1}{2}}=(\frac{4}{3})^{-\frac{1}{2}}=(\frac{3}{4})^{\frac{1}{2}}=\frac{\sqrt{3}}{2}$.

Ex. 8. Show that $\sqrt{3}=1+\frac{1.2}{2.3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\ldots$to ∞.

$$\sqrt{3}=3^{\frac{1}{2}}=(\tfrac{1}{3})^{-\frac{1}{2}}=(1-\tfrac{2}{3})^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}\cdot\frac{2}{3}+\frac{\frac{1}{2}(\frac{1}{2}+1)}{\lfloor 2}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{\lfloor 3}\cdot\left(\frac{2}{3}\right)^3+\ldots\ldots\text{to }\infty$$

$$=1+\frac{1.2}{2.3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\ldots\text{to }\infty.$$

Ex. 9. Show that $\frac{5}{3.6}+\frac{5.7}{3.6.9}+\frac{5.7.9}{3.6.9.12}+\ldots=\frac{1}{3}(3\sqrt{3}-2)$.

[Annamalai '41]

Multiplying each term on the L. H. S. by 3 and dividing the product by 3 we have

$$\text{the L. H. S.}=\frac{1}{3}\left\{\frac{3.5}{3.6}+\frac{3.5.7}{3.6.9}+\frac{3.5.7.9}{3.6.9.12}+\ldots\ldots\right\}$$

$$=\frac{1}{3}\left\{\frac{\frac{3}{2}\cdot\frac{5}{2}}{\lfloor 2}\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{\lfloor 3}\left(\frac{2}{3}\right)^3+\ldots\ldots\right\}$$

$$=\frac{1}{3}\left[\left\{1+\frac{3}{2}\left(\frac{2}{3}\right)+\frac{\frac{3}{2}\cdot\frac{5}{2}}{\lfloor 2}\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{\lfloor 3}\left(\frac{2}{3}\right)^3+\ldots\ldots\right\}-1-\frac{3}{2}\left(\frac{2}{3}\right)\right]$$

$$=\tfrac{1}{3}\left[\{(1-\tfrac{2}{3})^{-\frac{3}{2}}\}-2\right]=\tfrac{1}{3}\{(\tfrac{1}{3})^{-\frac{3}{2}}-2\}=\tfrac{1}{3}(3^{\frac{3}{2}}-2)=\tfrac{1}{3}(3\sqrt{3}-2).$$

Ex. 10. If x be so small that its cube and higher powers are negligible, show that $\frac{1}{(1+3x)^2}-\frac{1}{(1+2x)^3}=3x^2$.

$$\frac{1}{(1+3x)^2}-\frac{1}{(1+2x)^3}=(1+3x)^{-2}-(1+2x)^{-3}$$

$$=\left\{1-2.3x+\frac{2.3}{\lfloor 2}(3x)^2+\ldots\right\}-\left\{1-3.2x+\frac{3.4}{\lfloor 2}(2x)^2+\ldots\ldots\right\}$$

[x^3 and the higher powers of x being negligible, all other terms in the expansion are negligible.]

$$=(1-6x+27x^2+\ldots)-(1-6x+24x^2+\ldots)=3x^2.$$

Ex. 11. Find the sum of the coefficients of the first $(r+1)$ terms in the expansion of $(1-x)^{-4}$.

Let $(1-x)^{-4}=a_0+a_1x+a_2x^2+\ldots\ldots+a_rx^r+\ldots \quad \ldots(1)$

Hence, the sum will be obtained by finding the value of

$$a_0+a_1+a_2+\ldots\ldots+a_r.$$

We have $(1-x)^{-1}=1+x+x^2+\ldots+x^r+\ldots \qquad \ldots(2)$

$\therefore$ the reqd. sum $=a_0+a_1+a_2+\ldots+a_r$

$=$ the coefficient of x^r in the product of (1) and (2)

$=$ the coefficient of x^r in $(1-x)^{-4}\times(1-x)^{-1}$

$=$ the coefficient of x^r in $(1-x)^{-5}$

$$=\frac{5.6.7\ldots(5+r-1)}{\lfloor r}=\frac{5.6.7\ldots(4+r)}{\lfloor r}=\frac{1.2.3.4.5.6.7\ldots(r+4)}{1.2.3.4\lfloor r}$$

$$=\frac{\lfloor r.(r+1)(r+2)(r+3)(r+4)}{1.2.3.4\lfloor r}=\frac{(r+1)(r+2)(r+3)(r+4)}{24}.$$

[Recurring Decimal]

Ex. 12. Exhibit $\dot{.5}$ as an infinite series in G. P. and hence find its value as a vulgar fraction.

$$\dot{.5}=.555\ldots\ldots \text{to } \infty$$

$$=.5+.05+.005+\ldots\ldots \text{to } \infty$$

$=\frac{5}{10}+\frac{5}{10^2}+\frac{5}{10^3}+\ldots\ldots$to ∞, it is an infinite series in G. P. of which $\frac{5}{10}$ is the first term and $\frac{1}{10}$ is the common ratio.

$$\therefore \quad \dot{.5}=\frac{a}{1-r}=\frac{\frac{5}{10}}{1-\frac{1}{10}}=\frac{5}{9}.$$

Ex. 13. Show that $\cdot 1\dot{6}$ is equivalent to an infinite geometric progression. By assuming it find its value. [C. U. '11]

$\cdot 1\dot{6}=\cdot 1666\ldots\ldots$to $\infty=\cdot 1+\cdot 06+\cdot 006+\cdot 0006+\ldots\ldots$to ∞

$=\frac{1}{10}+\left(\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\ldots\ldots\text{to}\,\infty\right)$, the terms within brackets are in G. P., whose first term is $\frac{6}{10^2}$ and common ratio is $\frac{1}{10}$.

$$\therefore \quad \cdot 1\dot{6}=\frac{1}{10}+\frac{\frac{6}{10^2}}{1-\frac{1}{10}}=\frac{1}{10}+\frac{6}{90}=\frac{15}{90}=\frac{1}{6}.$$

[Approximate Value]

Ex. 14. Find the value of $(1{\cdot}02)^5$ correct to 3 places of decimals by the binomial theorem.

$$(1{\cdot}02)^5=\left(1+\frac{2}{100}\right)^5=\left(1+\frac{2}{10^2}\right)^5$$

$$=1+5.\frac{2}{10^2}+\frac{5.4}{\underline{|2}}.\frac{2^2}{10^4}+\frac{5.4.3}{\underline{|3}}.\frac{2^3}{10^6}+\ldots\ldots$$

$$=1+\frac{1}{10}+\frac{4}{10^3}+\frac{8}{10^5}+\ldots\ldots$$

$$=1+\cdot 1+\cdot 004+\cdot 00008+\ldots\ldots$$

$=1{\cdot}104$ (correct to 3 places of decimals).

[N. B. Here the other terms of the expansion are left out, as the value is required correct to 3 places of decimals only.]

Ex. 15. Find by the binomial theorem the cube root of 122 to 4 places of decimals.

$\because$ the nearest cube to 122 is 125 or 5^3,

$$\therefore \sqrt[3]{122}=(122)^{\frac{1}{3}}=(125-3)^{\frac{1}{3}}=(5^3-3)^{\frac{1}{3}}=\left\{5^3\left(1-\frac{3}{5^3}\right)\right\}^{\frac{1}{3}}$$

$$=5\left(1-\frac{3}{5^3}\right)^{\frac{1}{3}}=5\left\{1-\frac{1}{3}\cdot\frac{3}{5^3}-\frac{1}{9}\cdot\frac{3^2}{5^6}-\frac{5}{81}\cdot\frac{3^3}{5^9}-\ldots\ldots\right\}$$

$$=5-\frac{1}{3}\cdot\frac{3}{5^2}-\frac{1}{9}\cdot\frac{3^2}{5^5}-\frac{1}{81}\cdot\frac{3^3}{5^7}-\ldots\ldots$$

$$=5-\frac{1}{5^2}-\frac{1}{5^5}-\frac{1}{3}\cdot\frac{3}{5^7}-\ldots\ldots$$

$$=5-\frac{2^2}{10^2}-\frac{2^5}{10^5}-\frac{1}{3}\cdot\frac{2^7}{10^7}-\ldots\ldots$$

$$=5-{\cdot}04-{\cdot}00032-\frac{{\cdot}0000128}{3}-\ldots\ldots$$

$$=5-{\cdot}04-{\cdot}00032-{\cdot}0000042-\ldots\ldots$$

$=5-{\cdot}0403242=4{\cdot}9597$ (correct to 4 places of decimals).

Ex. 16. Find the approximate value of (i) $\frac{465}{10003}$ correct to 5 places of decimals and (ii) $\dfrac{(1{\cdot}0002)^3}{({\cdot}9993)^2}$ correct to 4 places of decimals.

(i) $$\frac{465}{10003}=\frac{465}{(10000+3)}=\frac{465}{10^4(1+{\cdot}0003)}=\frac{{\cdot}0465}{(1+{\cdot}0003)}$$

$$={\cdot}0465(1+{\cdot}0003)^{-1}={\cdot}0465(1-{\cdot}0003) \text{ [App.]}$$

$$={\cdot}0465-{\cdot}00001395={\cdot}04649 \text{ [App.]}$$

(ii) $(1{\cdot}0002)^3=(1+{\cdot}0002)^3=1+3\times{\cdot}0002=1+{\cdot}0006$ [App.]

$({\cdot}9993)^2=(1-{\cdot}0007)^2=1-2\times{\cdot}0007=1-{\cdot}0014$ [App.]

$$\therefore \quad \text{the quotient} = \frac{1+\cdot0006}{1-\cdot0014} = (1+\cdot0006)(1-\cdot0014)^{-1}$$
$$= (1+\cdot0006)(1+\cdot0014)$$
$$= 1+\cdot0006+\cdot0014 \text{ [App.]}$$
$$= 1\cdot0020 \text{ [correct to 4 places of decimals]}$$

[**N. B.** In solution (i) above, only the first two terms in the expansion of $(1+\cdot0003)^{-1}$ and in solution (ii) only the first two terms of the expansions of the numerator as well as the denominator have been taken into consideration, for the other terms can be neglected for approximats values [vide Art. 56]. Hence, in these cases $(1+x)^n = 1+nx$. Again, in the product of $(1+\cdot0006)(1+\cdot0014)$ in (ii), the product of $\cdot0006 \times \cdot0014$ has been neglected as it ($\cdot00000084$) begins with 6 ciphers and can be ignored for the value correct to 4 decimal places.]

Exercise 7

Find to 4 places of decimals the value of :—

1. $(1\cdot04)^5$ 2. $\sqrt[4]{621}$ 3. $(9996)^{\frac{1}{4}}$ 4. $\sqrt{99}$

5. $(47)^{-\frac{1}{2}}$ 6. $\sqrt[3]{\frac{501}{500}}$ 7. $\sqrt[3]{2}$ 8. $(630)^{-\frac{1}{4}}$

9. Evaluate $\sqrt{24}$ by means of the Binomial Theorem to 5 places of decimals. [C. U.]

Find approximately the values of :—

10. $1\cdot0036 \times \cdot996$ 11. (*a*) $\frac{638}{10005}$ 11. (*b*) $\frac{1}{\cdot9997}$

12. $\frac{(1\cdot00013)^3}{(\cdot9992)^2}$.

If a be so small that its square and higher powers may be neglected, find the value of :—

13. $\frac{\sqrt{1+3a}}{(1-2a)^{\frac{1}{5}}}$ 14. $\frac{\sqrt[3]{8+3a}-(1-a)^{\frac{1}{5}}}{(1+5a)^{\frac{3}{5}}}$

15. If x be so small that its cube and higher powers may be neglected, show that $\dfrac{1}{(1+5x)^3}-\dfrac{1}{(1+3x)^5}=15x^2$.

16. If x be so large that $\dfrac{1}{x^9}$ is negligible, show that $\sqrt[3]{x^3+1}-\sqrt[3]{x^3-1}$ is approximately equal to $\dfrac{2}{3x^2}$.

17. Find the square of

$$1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots\ldots\text{to }\infty.$$

Find the sum to infinity of the following series :—

18. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots\ldots$ 19. $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3.\sqrt{3}}+\ldots\ldots$

20. $\frac{1}{3}+\frac{1}{7}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{3^3}+\frac{1}{7^3}+\ldots\ldots$

21. $1+3x+5x^2+7x^3+\ldots\ldots$(when $-1<x<1$)

22. $1-5x+9x^2-13x^3+\ldots\ldots$(if $-1<x<1$)

23. $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots\ldots$ [C. U. '50]

24. $1-\frac{1}{4}+\frac{1.3}{4.8}-\frac{1.3.5}{4.8.12}+\ldots\ldots$

25. $1+\frac{1}{6}+\frac{1.3}{2.4}\cdot\frac{1}{3^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^3}+\ldots\ldots$

26. $1-\frac{1}{8}+\frac{1.5}{8.16}-\frac{1.5.9}{8.16.24}+\ldots\ldots$

27. $1+\frac{1}{6}+\frac{1.4}{3.6}\cdot\frac{1}{4}+\frac{1.4.7}{3.6.9}\cdot\frac{1}{8}+\ldots\ldots$

28. $1+\frac{3}{2.4}+\frac{1.3.3^2}{2^2.4^2\lfloor 2}+\frac{1.3.5.3^3}{2^3.4^3.\lfloor 3}+\ldots\ldots$

29. $1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\ldots\ldots$ [B. U. '52]

30. $1+\frac{5}{8}+\frac{5.8}{8.12}+\frac{5.8.11}{8.12.16}+\ldots\ldots$ [Annamalai '49]

31. $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\ldots\ldots$ [Gujrat '52]

32. $1+\frac{4}{6}+\frac{4.5}{6.9}+\frac{4.5.6}{6.9.12}+\ldots\ldots$ [Andhra '54]

33. $2+\frac{5}{2\,!\,3}+\frac{5.7}{3\,!\,3^2}+\frac{5.7.9}{4\,!\,3^3}+\ldots\ldots$ [A. U. '46]

34. $\frac{1}{2.4.6}+\frac{1.3}{2.4.6.8}+\frac{1.3.5}{2.4.6.8.10}+\ldots\ldots$ [B. U. '47]

35. Find the binomial expression whose expansion is $1-\frac{1}{4}+\frac{1.3}{2.4}.\frac{1}{2^2}-\frac{1.3.5}{2.4.6}.\frac{1}{2^3}+\ldots\ldots$ and find its sum.

36. Prove that $\sqrt{8}=1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots$ to ∞.

37. Show that $\frac{\sqrt{3}}{2}=1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}.\frac{1}{3^2}-\frac{1.3.5}{2.4.6}.\frac{1}{3^3}+\ldots\ldots$

Exhibit the following as series in G. P. continued to infinity and therefrom deduce their values :—

38. $\cdot\dot{0}\dot{9}$ 39. $\cdot 0\dot{5}$ 40. $\cdot 08\dot{1}$ 41. $1\cdot\dot{5}$

42. Show by the method of summation of a series in G. P. that $\sqrt{\cdot 444\ldots}=\cdot 666\ldots$

43. Find the sum of the coefficients of the first $(r+1)$ terms in the expansion of $(1-x)^{-3}$.

44. Find the sum of the coefficients of the first $(r+1)$ terms in the expansion of $(1+x)^{-p}$.

45. Show that $\frac{1}{2}n+\frac{n(n+1)}{1.2}\cdot\frac{1}{2^2}+\frac{n(n+1)(n+2)}{1.2.3}\cdot\frac{1}{2^3}+\ldots\ldots$

$=2^n-1.$

46. Show that the middle term of $\left(x+\frac{1}{x}\right)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-n-\frac{1}{2}}$. [P. U. '55]

47. If x is so small that x^3 and higher powers of x can be neglected, show that the nth root of $(1+x)$ is equal to $\frac{2n+(n+1)x}{2n+(n-1)x}$ nearly. [Travancore '53]

48. If $x>-\frac{1}{2}$, prove that

$$\frac{x}{\sqrt{(1+x)}}=\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3+\frac{1.3.5}{2.4.6}\left(\frac{x}{1+x}\right)^4+\cdots$$

[Karnatak '54]

49. If n is any positive integer, show that the integral part of $(3+\sqrt{7})^n$ is an odd number. [B. U. '48]

[Hints : From the expansion we have $(3+\sqrt{7})^n+(3-\sqrt{7})^n$ = an even number.]

50. Prove that $1=\frac{1}{4}+\frac{1.3}{4.6}+\frac{1.3.5}{4.6.8}+\ldots\ldots$ [Agra '41]

51. Identifying as a binomial expansion, show that $\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots\ldots=0{\cdot}4$ nearly. [Rajputana '50]

52. Prove that the coefficient of x^n in the expansion of $\frac{1}{1+x+x^2}$ is 1, 0 or -1 according as n is of the form $3m$, $3m-1$, or $3m+1$. [P. U. '53]

CHAPTER II

TRIGONOMETRY

SOME IMPORTANT TRIGONOMETRICAL FORMULAS

I. 1 radian $= 57^\circ 17' 44{\cdot}8''$ (app.), $1^\circ = {\cdot}01745$ radian (nearly),

2 right angles $= 180^\circ = 200^g = \pi$ radians.

$$\pi = \tfrac{22}{7} = 3{\cdot}1416 \text{ (app.)}$$

Circumference of a circle $= 2\pi r$; area of a circle $= \pi r^2$.

Radian measure of an angle $=$ Arc $\div$ radius.

II. $\left.\begin{array}{l} \sin^2\theta + \cos^2\theta = 1 \\ \sec^2\theta = 1 + \tan^2\theta \\ \operatorname{cosec}^2\theta = 1 + \cot^2\theta \end{array}\right\} \quad \begin{array}{l} \dfrac{\sin\theta}{\cos\theta} = \tan\theta \\ \dfrac{\cos\theta}{\sin\theta} = \cot\theta \end{array}$

III. $\sin 0^\circ = 0$; $\cos 0^\circ = 1$; $\tan 0^\circ = 0$.

$\sin 30^\circ = \frac{1}{2}$; $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$ $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$.

$\sin 45^\circ = \dfrac{1}{\sqrt{2}}$ $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$ $\tan 45^\circ = 1$.

$\sin 60^\circ = \dfrac{\sqrt{3}}{2}$ $\cos 60^\circ = \dfrac{1}{2}$; $\tan 60^\circ = \sqrt{3}$.

$\sin 90^\circ = 1$; $\cos 90^\circ = 0$; $\tan 90^\circ = \infty$.

$\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15^\circ = 2 - \sqrt{3}$;

$\sin 18^\circ = \frac{1}{4}(\sqrt{5}-1)$; $\cos 18^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}$;

$\sin 22^\circ\frac{1}{2} = \frac{1}{2}\sqrt{2-\sqrt{2}}$; $\cos 22^\circ\frac{1}{2} = \frac{1}{2}\sqrt{2+\sqrt{2}}$;

$\tan 22^\circ\frac{1}{2} = \sqrt{2} - 1$

$\sin 36^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$; $\cos 36^\circ = \frac{1}{4}(\sqrt{5}+1)$;

$\sin 54^\circ = \frac{1}{4}(\sqrt{5}+1)$; $\cos 54^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$;

$\sin 67^\circ\frac{1}{2} = \frac{1}{2}\sqrt{2+\sqrt{2}}$; $\cos 67^\circ\frac{1}{2} = \frac{1}{2}\sqrt{2-\sqrt{2}}$;

$\tan 67^\circ\frac{1}{2} = \sqrt{2} + 1$;

$\sin 72^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}$; $\cos 72^\circ = \frac{1}{4}(\sqrt{5}-1)$;

$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\tan 75^\circ = 2+\sqrt{3}$.

$\sin 120^\circ = \frac{\sqrt{3}}{2}$; $\cos 120^\circ = -\frac{1}{2}$; $\tan 120^\circ = -\sqrt{3}$.

$\sin 180^\circ = 0$; $\cos 180^\circ = -1$; $\tan 180^\circ = 0$.

$\sin 270^\circ = -1$; $\cos 270^\circ = 0$; $\tan 270^\circ = \infty$.

$\sin 360^\circ = 0$; $\cos 360^\circ = 1$; $\tan 360^\circ = 0$.

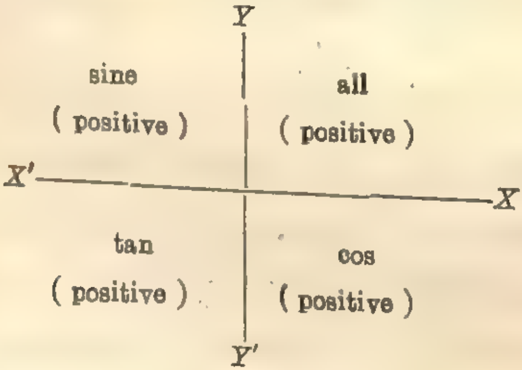
IV. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

$\sin(90^\circ \pm \theta) = \cos\theta$; $\cos(90^\circ \pm \theta) = \mp\sin\theta$;
$\tan(90^\circ \pm \theta) = \mp\cot\theta$.

$\sin(180^\circ \pm \theta) = \mp\sin\theta$; $\cos(180^\circ \pm \theta) = -\cos\theta$;
$\tan(180^\circ \pm \theta) = \pm\tan\theta$.

$\sin(270^\circ \pm \theta) = -\cos\theta$; $\cos(270^\circ \pm \theta) = \pm\sin\theta$;
$\tan(270^\circ \pm \theta) = \mp\cot\theta$.

$\sin(360^\circ \pm \theta) = \pm\sin\theta$; $\cos(360^\circ \pm \theta) = \cos\theta$;
$\tan(360^\circ \pm \theta) = \pm\tan\theta$.



V. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

$$\cot(A \pm B) = \frac{\cot B \cot A \mp 1}{\cot B \pm \cot A}.$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

VI. $2 \sin A \cos B = \sin (A+B) + \sin (A-B).$

$2 \cos A \sin B = \sin (A+B) - \sin (A-B).$

$2 \cos A \cos B = \cos (A+B) + \cos (A-B).$

$2 \sin A \sin B = \cos (A-B) - \cos (A+B).$

VII. $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}.$

$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}.$

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}.$

$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}.$

VIII. $\sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

$\cos (A+B) \cos (A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

IX. $\sin 2A = 2 \sin A \cos A.$

$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A.$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$; $\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$

$\left.\begin{array}{l} 1 + \cos 2A = 2 \cos^2 A \\ 1 - \cos 2A = 2 \sin^2 A \end{array}\right\}$; $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$

$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$ $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$

X. $\sin 3A = 3 \sin A - 4 \sin^3 A.$

$\cos 3A = 4 \cos^3 A - 3 \cos A.$

$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$; $\cot 3A = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$

XI. $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}.$

$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}.$

$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$; $\cot \theta = \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}}.$

$$\sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\ ;\qquad \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}.$$

$$\left.\begin{array}{l}1+\cos\theta=2\cos^2\frac{\theta}{2}\\ 1-\cos\theta=2\sin^2\frac{\theta}{2}\end{array}\right\}\ ;\qquad \tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}.$$

XII. When $A+B+C=\pi$, then

(a) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$

(b) $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

(c) $\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$

(d) $\cos A+\cos B+\cos C=1+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C.$

(e) $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$

(f) $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

(g) $\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C.$

(h) $\cos^2A+\cos^2B+\cos^2C=1-2\cos A\cos B\cos C.$

(i) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

(j) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}$

(k) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$

(l) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$

(m) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

XIII. When $\sin\theta=\sin\alpha$, then $\theta=n\pi+(-1)^n\alpha$

[where n is 0 or any integer, positive or negative]

" $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$, [" "]

" $\tan\theta=\tan\alpha$, then $\theta=n\pi+\alpha$, [" "]

" $\sin\theta=0$ or $\tan\theta=0$, then $\theta=n\pi$ [" "]

$\cos\theta=0$ or $\cot\theta=0$, then $\theta=(2n+1)\frac{\pi}{2}$ ["]

when $\sin\theta=1$, then $\theta=(4m+1)\frac{\pi}{2}$

[where m is 0 or any integer, positive or negative]

„ $\sin\theta=-1$, then $\theta=(4m-1)\frac{\pi}{2}$ [„ „]

„ $\cos\theta=1$, „ $\theta=2m\pi$, [„ „]

„ $\cos\theta=-1$, „ $\theta=(2m+1)\pi$ [„ „]

XIV. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

$\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$

$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\frac{xy-1}{y+x}$

$\cot^{-1}x-\cot^{-1}y=\cot^{-1}\frac{xy+1}{y-x}$

$2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

$2\cos^{-1}x=\cos^{-1}(2x^2-1)$

$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}$.

$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}$

$\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\}$

XV. $\log_a(m\times n)=\log_a m+\log_a n$.

$\log_a\frac{m}{n}=\log_a m-\log_a n$.

$\log_a(m)^n=n\log_a m$.

$\log_a m=\log_b m\times\log_a b$.

$\log_a 1=0$; $\log_a a=1$.

XVI. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R.$

$$\cos A=\frac{b^2+c^2-a^2}{2bc};\quad \cos B=\frac{c^2+a^2-b^2}{2ca};$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}.$$

$a=b\cos C+c\cos B$
$b=c\cos A+a\cos C$
$c=a\cos B+b\cos A.$

In $\triangle$ABC, where $2s=a+b+c$,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.\qquad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\cot\frac{A}{2}=\frac{s(s-a)}{\Delta}.$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$$

$$\sin B=\frac{2\Delta}{ca};\quad \sin C=\frac{2\Delta}{ab}.$$

$$\Delta=\tfrac{1}{2}bc\sin A=\tfrac{1}{2}ca\sin B=\tfrac{1}{2}ab\sin C$$
$$=\sqrt{s(s-a)(s-b)(s-c)}=\frac{abc}{4R}.$$

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}.$$

$$r=\frac{\Delta}{s}$$
$$=4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$
$$=(s-a)\tan\tfrac{1}{2}A=(s-b)\tan\tfrac{1}{2}B=(s-c)\tan\tfrac{1}{2}C$$

$$r_1=\frac{\Delta}{s-a}$$
$$=4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C=s\tan\tfrac{1}{2}A.$$

$$r_2=\frac{\Delta}{s-b}=4R\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C=s\tan\tfrac{1}{2}B.$$

$$r_3=\frac{\Delta}{s-c}=4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C=s\tan\tfrac{1}{2}C.$$

Some Important Results

$\sqrt{2}=1{\cdot}4142135\cdots$	$\pi=3{\cdot}14159265\cdots$
$\sqrt{3}=1{\cdot}7320508\cdots$	$\frac{1}{\pi}=0{\cdot}31830989\cdots$
$\sqrt{5}=2{\cdot}2360679\cdots$	
$\sqrt{6}=2{\cdot}4494897\cdots$	$\log 2={\cdot}30103$
$\sqrt{7}=2{\cdot}6457513\cdots$	$\log 3={\cdot}47712$
$\sqrt{8}=2{\cdot}8284271\cdots$	$\log 5={\cdot}69897$
$\sqrt{10}=3{\cdot}1622776\cdots$	$\log 7={\cdot}84510$

Trigonometrical Equations and General Values

69. You have seen before that there may be an infinite number of angles whose trigonometrical ratios have the same values. As for example, suppose $\sin\theta=\frac{1}{2}$. Here evidently one value of θ is $30°$, which is here the least positive value of θ, for $\sin 30°=\frac{1}{2}$. Now, since the trigonometrical ratios of supplementary angles are equal, we have $\sin 150°=\sin 30°=\frac{1}{2}$. Again, since angles, which differ from $30°$ or $150°$ by complete multiples of $360°$, have the same sine (and other ratios), so each of the angles $30°$, $150°$, $390°$ $(30°+360°)$, $-330°$ $(30°-360°)$, $-210°$, etc. has the same sine (*i.e.* $\frac{1}{2}$).

Similarly an infinite number of angles may have the same cosine or other trigonometrical ratios.

Hence, for the solution of trigonometrical equations, it is necessary to determine a single general expression that contains all the angles having the same given trigonometrical ratios.

70. *To find the general expression for angles whose any one of the trigonometrical ratios is zero.*

(*i*) Suppose $\sin\theta=0$. To find the general expression for the value of θ.

If in any angle a perpendicular be drawn from any point of one of its arms to the other, then $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$ is the sine of the

angle. Thus, the sine of an angle is zero, if the length of the perpendicular be zero and this again will be possible only when the two arms of the angle are in the same straight line. Hence such angles must be some odd or even multiples of π(or 180°).

$\therefore$ **If** $\sin\theta=0$, **then** $\theta=n\pi$, where n is zero or any integer (positive or negative).

(*ii*) Cosine of an angle$=\dfrac{\text{base}}{\text{hypotenuse}}$. So, the cosine of an angle is zero, when the length of the base is zero and this will be possible, if the arms of the angle be at right angles to each other. Hence, the angles must be $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or, any odd multiple of $\frac{\pi}{2}$.

$\therefore$ **If** $\cos\theta=0$, **then** $\theta=(2n+1)\frac{\pi}{2}$, where n is zero or any integer, positive or negative.

(*iii*) If the tangent, *i.e.*, $\dfrac{\sin}{\cos}$ be zero, then, evidently, the numerator, *i.e.*, the sine of the angle must be zero.

$\therefore$ **If** $\tan\theta=0$, **then** $\theta=n\pi$.

Similarly, if $\cot\theta=0\left(i.e., \dfrac{\cos\theta}{\sin\theta}=0\right)$, then $\cos\theta=0$.

$\therefore$ **If** $\cot\theta=0$, **then** $\theta=(2n+1)\frac{\pi}{2}$.

(*iv*) Since $\sec\theta$ and $\operatorname{cosec}\theta$ can never be numerically less then 1, so they can never be zero.

71. *General expression of angles having the same sine.*

Let α be the smallest angle whose sine is equal to a given value. Let θ be any other angle having the same sine. Then we have the equation $\sin\theta=\sin\alpha$. Here we have to find the general value of θ, which will satisfy the equation.

$$\because \sin\theta=\sin\alpha, \qquad \therefore \sin\theta-\sin\alpha=0,$$

$$\text{or, } 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0,$$

$\therefore$ either $\cos\dfrac{\theta+\alpha}{2}=0$, or, $\sin\dfrac{\theta-\alpha}{2}=0$.

Now, if $\cos\dfrac{\theta+\alpha}{2}=0$, then $\dfrac{\theta+\alpha}{2}$ = any *odd* multiple of $\dfrac{\pi}{2}$

[See Art. 70 (*ii*)]

$\therefore$ $\theta+\alpha$ = any odd multiple of π,

$\therefore$ θ = any odd multiple of $\pi-\alpha=(2m+1)\pi-\alpha \cdots(1)$

where n is zero or any integer, positive or negative.

Again, if $\sin\dfrac{\theta-\alpha}{2}=0$, then $\dfrac{\theta-\alpha}{2}$ = any multiple of π,

i.e., $\theta-\alpha$ = any *even* multiple of π.

$\therefore$ θ = any even multiple of $\pi+\alpha=2m\pi+\alpha\cdots(2)$

Hence, combining (1) and (2) we have the general expression $\theta=n\pi+(-1)^n\alpha\cdots(3)$, where n is zero or any positive or negative integer (odd or even).

[*N.B.* If in (3) n be odd, it will correspond to (1) and if n be even, it will be similar to (2).]

Corollary : If cosec θ = cosec α, then $\sin\theta=\sin\alpha$, Hence the general expression for all angles having the same *cosecant* is $\theta=n\pi+(-1)^n\alpha$.

[Geometrical Proof]

71. A. Let AOP be an angle whose sine is given and let it be denoted by α.

Construction. Draw a circle AP′A′ with centre O and with radius OP. Draw PM⊥OA and cut off OM′ = OM from MO produced. Draw. M′P′⊥OA′, then M′P′ cuts the circumference at P′. Join OP′.

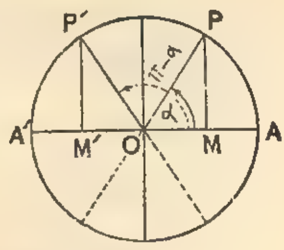


Fig. 1

Proof. Evidently Δ^s OPM and P′OM′ are congruent.

$\therefore$ $\angle$P′OM′ = $\angle$POM = α, $\therefore$ $\angle$AOP′ = $\pi-\alpha$,

Only when the radius vector OP (*i.e.*, the angle generating line OP or the boundary line of the angle) is in the position OP or OP′ (and in no other position), then the sine of the angle generated will be equal to the given sine.

Now, the radius vector occupies the position OP (*i.e.*, is co-terminal with OP), when it traces out the angle α after making some complete revolutions. Hence, the angle generated by it is $2m\pi+\alpha\cdots(1)$, for it makes angle 2π by one complete revolution. Here m is zero or any integer, positive or negative.

Similarly, when the radius vector is coterminal with OP′ it traces out an angle $(2m\pi+\text{AOP}')$, or angle $(2m\pi+\pi-\alpha)$, *i.e.*, angle $(2m+1)\pi-\alpha\cdots(2)$, where m is zero or any positive or negative integer.

Now, the expression $n\pi+(-1)^n\alpha\cdots(3)$ includes all the angles expressed by (1) and (2). For, if $n=2m$, then $(-1)^{2m}=1$ and then from (3) we have $2m\pi+\alpha$ which is expressed by (1). Again, if $n=2m+1$, then $(-1)^{2m+1}=-1$, and so from (3) we have $(2m+1)\pi-\alpha$ which is expressed by (2).

Hence the general value of the angles having the same given sine is $n\pi+(-1)^n\alpha$, where n is zero or any integer, positive or negative.

72. *To find the general value of angles having the same cosine.*

Let α be the smallest positive angle whose cosine is equal to the cosine of angle θ. So here the equation $\cos\theta=\cos\alpha$ is to be solved.

$$\because\quad \cos\theta=\cos\alpha, \qquad \therefore\quad \cos\theta-\cos\alpha=0,$$

$$\text{or,}\quad 2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0,$$

$$\therefore\quad \text{either } \sin\frac{\theta+\alpha}{2}=0, \text{ or, } \sin\frac{\theta-\alpha}{2}=0.$$

Now, if $\sin\dfrac{\theta+\alpha}{2}=0$, then $\dfrac{\theta+\alpha}{2}=$ any multiple of π.

or, $\theta+\alpha$ = any multiple of 2π.

$\therefore$ θ = any multiple of $2\pi-\alpha=2n\pi-\alpha$.

Again, if $\sin\dfrac{\theta-\alpha}{2}=0$, then $\dfrac{\theta-\alpha}{2}$ = any multiple of π,

or, $\theta-\alpha$ = any multiple of 2π.

$\therefore$ θ = any multiple of $2\pi+\alpha=2n\pi+\alpha$.

Hence, the general formula for angles having a given cosine is $\theta=2n\pi\pm\alpha\cdots(4)$, where n is zero or any integer, positive or negative.

Corollary. If $\sec\theta=\sec\alpha$, then $\cos\theta=\cos\alpha$. Hence the general value of all angles having the same *secant* is expressed by $\theta=2n\pi\pm\alpha$.

[Geometrical Proof]

72. A. Let AOP be an angle having a given cosine and let it be denoted by α.

Construction. Draw a circle APP′ with centre O and with radius OP. Draw PM⊥OA and let PM produced meet the circumference at P′. Join OP, OP′.

Proof. Evidently $\triangle$MOP and $\triangle$MOP′ are congruent.

$\therefore$ PM = P′M and $\angle$P′OM = $\angle$POM = α, so $\angle$AOP′ = $2\pi-\alpha$.

If the radius vector (*i.e.*, the boundary line of the angle) is in the position OP or OP′ (and in no other position), then the cosine of the angle thus generated is equal to the given cosine.

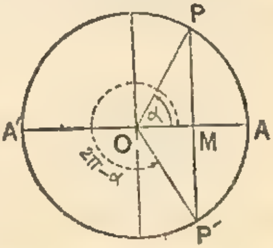


Fig. 2

Now, the radius vector is coterminal with OP, when it traces out an angle α after making some complete revolutions. So, the angle thus generated by it is $2n\pi+\alpha\cdots(1)$, where n is zero or any integer, positive or negative.

Again, the radius vector may occupy the position OP′ when

it traces out an angle $-\alpha$ after making some complete revolutions. Then the angle generated by it is $2n\pi-\alpha$ ...(2).

Now, $2n\pi\pm\alpha$ includes all the angles expressed by (1) and (2).

Hence the general value of the angles having the same cosine is $2n\pi\pm\alpha$, where n is zero or any integer, positive or negative.

73. *To find the general value of angles having the same tangent.*

Let α be the smallest angle whose tangent is equal to the tangent of angle θ.

Here, the equation $\tan\theta=\tan\alpha$ is to be solved.

$\because$ $\tan\theta=\tan\alpha$, $\therefore$ $\tan\theta-\tan\alpha=0$, or, $\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin\alpha}{\cos\alpha}=0$.

or, $\dfrac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha}=0$, or, $\dfrac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha}=0$,

or, $\sin(\theta-\alpha)\left(\dfrac{1}{\cos\theta\cos\alpha}\right)=0$,

$\therefore$ either $\sin(\theta-\alpha)=0$, or, $\dfrac{1}{\cos\theta\cos\alpha}=0$.

Now, since the cosine of any angle cannot be infinity,

$\therefore$ $\dfrac{1}{\cos\theta\cos\alpha}$ cannot be zero.

$\therefore$ Here $\sin(\theta-\alpha)=0$, $\therefore$ $\theta-\alpha=$ any multiple of $\pi=n\pi$,

$\therefore$ $\theta=n\pi+\alpha$.

Hence, the general expression for all angles having the same tangent is $\theta=n\pi+\alpha$ ...(5) where n is zero, or any integer, positive or negative.

Corollary. If $\cot\theta=\cot\alpha$, then $\tan\theta=\tan\alpha$, so $\theta=n\pi+\alpha$ is the general expression of all angles having the same *cotangent.*

[Geometrical Proof]

73. A. Let AOP be an angle having a given tangent and let it be denoted by α.

Construction. Draw a circle APA′ with centre O and with radius OP. Let PO produced cut the circumference at P′.

Draw PM and P'M' perpendicular to AA'. Then $\angle AOP' = \pi + \alpha$.

Proof. $\because$ $\triangle POM$ and $\triangle P'OM'$ are congruent,

$\therefore$ $PM = P'M'$, $OM = OM'$ and $\angle P'OM' = \angle POM = \alpha$.

$\therefore$ the tangents of $\angle AOP$ and $\angle AOP'$ are equal.

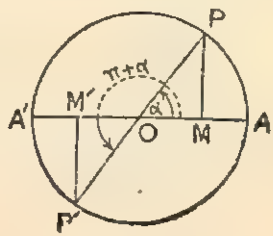


Fig. 3

Here, the radius vector will come to the position OP when it traces out an angle α after making some complete revolutions. The angle thus traced out by it is $2m\pi + \alpha$ ···(1) where m is zero or any integer, positive or negative.

Again, if the radius vector traces out an angle $\pi + \alpha$, after making some complete revolutions, it will be coterminal with OP' and then the angle generated is $2m\pi + \pi + \alpha$,

or, $(2m+1)\pi + \alpha$ ···(2) where m is zero or any integer, positive or negative.

All the angles represented by (1) and (2) are included in $n\pi + \alpha$ ···(3) For, if n is even, then $n = 2m$ and from (3) we have $2m\pi + \alpha$ which is expressed by (1), and if n is odd, then $n = 2m+1$ and from (3) we have $(2m+1)\pi + \alpha$ which is expressed by (2).

Hence the general value of the angles having the same tangent is $n\pi + \alpha$ where n is zero or any integer, positive or negative.

74. *The following general values are important and should be remembered :—*

If $n = 0$ or any positive or negative integer, we have ;

(1) If $\sin\theta = 0$, then $\theta = n\pi$.

If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$

If $\sin\theta = 1 = \sin\frac{\pi}{2}$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$.

If $\sin\theta = -1 = \sin\left(-\frac{\pi}{2}\right)$, then $\theta = 2n\pi - \frac{\pi}{2} = (4n-1)\frac{\pi}{2}$.

(2) If $\cos\theta=0$, then $\theta=(2n+1)\frac{\pi}{2}$.

If $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$.

If $\cos\theta=1$, then $\theta=2n\pi$.

If $\cos\theta=-1$, then $\theta=(2n+1)\pi$.

(3) If $\tan\theta=0$, then $\theta=n\pi$.

If $\tan\theta=\tan\alpha$, then $\theta=n\pi+\alpha$.

(4) If $\cot\theta=0$, then $\theta=(2n+1)\frac{\pi}{2}$.

If $\cot=\theta\cot\alpha$, then $\theta=n\pi+\alpha$.

Examples (8)

Ex. 1. Find the general expression for all angles whose consine is $-\frac{1}{2}$.

Of the angles whose cosine is $-\frac{1}{2}$, the least is 120° or $\frac{2\pi}{3}$.

$\therefore$ The general form of the angles, whose consine is $-\frac{1}{2}$, is $2n\pi\pm\frac{2\pi}{3}$ [Art. 72]

Ex. 2. Find the general solution of $\sin^2 x=\sin^2\theta$.

From the given equation we have
$$\left.\begin{aligned}\sin x&=\sin\theta \text{...(1)}\\ \text{or, } \sin x&=-\sin\theta=\sin(-\theta)\text{......(2)}\end{aligned}\right\}$$

Now, from (1) we have $x=n\pi+(-1)^n\theta$,

and from (2) „ „ $x=n\pi+(-1)^n(-\theta)$

$\therefore$ the required solution is $x=n\pi\pm\theta$.

Ex. 3. Find the general value of α which satisfies both the equations $\cos\alpha=-\frac{1}{\sqrt{2}}$ and $\tan\alpha=1$.

225° or $\left(\pi+\frac{\pi}{4}\right)$ is the value of the angle between 0° and 360°, whose cosine is $-\frac{1}{\sqrt{2}}$ and tangent is 1.

$\therefore$ the general value of $\alpha=2n\pi+\pi+\frac{\pi}{4}=(2n+1)\pi+\frac{\pi}{4}$.

Ex. 4. Solve the equation $\cot x - \tan x = 2$. [C. U. '37]

$\cot x - \tan x = 2$,

or, $\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x} = 2$,

or, $\cos^2 x - \sin^2 x = 2 \sin x \cos x$,

or, $\cos 2x = \sin 2x$, or, $\dfrac{\sin 2x}{\cos 2x} = 1$,

or, $\tan 2x = 1 = \tan \dfrac{\pi}{4}$,

$\therefore\ 2x = n\pi + \dfrac{\pi}{4}$, $\therefore\ x = \dfrac{n\pi}{2} + \dfrac{\pi}{8} = (4n+1)\dfrac{\pi}{8}$.

Ex. 5. Solve $\sin x + \cos x = \sqrt{2}$. [C. U. '35]

Dividing both sides of the equation by $\sqrt{1^2+1^2}$ or $\sqrt{2}$ we have

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = 1,$$

or, $\sin \dfrac{\pi}{4} \sin x + \cos \dfrac{\pi}{4} \cos x = 1$

$$\left(\because \sin \frac{\pi}{4} = \sin 45^\circ = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}\right)$$

or, $\cos\left(x - \dfrac{\pi}{4}\right) = 1$, $\therefore\ x - \dfrac{\pi}{4} = 2n\pi$ [see Art. 74 (ii)]

$\therefore\ x = 2n\pi + \dfrac{\pi}{4} = (8n+1)\dfrac{\pi}{4}$.

[*N. B.* (1) A trigonometrical equation may be solved in various ways and the results obtained, though different in forms, are included in the same series of angles.

(2) The Ex. 5 may also be worked out by squaring both sides of the equation, but this method is not free from flaws. For all the solutions thus obtained may not satisfy the given equation, *i.e.*, such solutions may include some *extraneous roots* besides the correct solutions. Let us show it from the equation in Ex. 5 above.

$\sin x+\cos x=\sqrt{2}$,

$\therefore$ $\sin^2 x+\cos^2 x+2\sin x\cos x=2$,

or, $1+2\sin x\cos x=2$,

or, $1+\sin 2x=2$, or, $\sin 2x=1=\sin\frac{\pi}{2}$.

$\therefore$ $2x=(4m+1)\frac{\pi}{2}$, $\therefore$ $x=(4m+1)\frac{\pi}{4}$. Here $m=0$ or any integer.

If m is even, suppose $m=2n$.

Then $x=(8n+1)\frac{\pi}{4}\cdots\cdots(1)$.

If m is odd, suppose $m=2n+1$.

Then $x=(8n+5)\frac{\pi}{4}\cdots\cdots(2)$.

Here we have an additional solution $x=(8n+5)\frac{\pi}{2}$, which is not included in the general values of x in the solution of Ex. 5. We find that this value of x does not satisfy the equation and is, therefore, an *extraneous root*. Hence, if any equation be solved by this method, the roots obtained must be verified to see if there is any extraneous root. If there be any, it should be stated as an extraneous root.

Ex. 6. Solve $\cos 2\theta=\cos\theta\sin\theta$.

$\cos 2\theta=\cos\theta\sin\theta$,

or, $2\cos 2\theta=2\cos\theta\sin\theta=\sin 2\theta$,

or, $\cot 2\theta=\frac{1}{2}=\cot\alpha$ (say)

$2\theta=n\pi+\alpha$. $\therefore$ $\theta=\frac{1}{2}(n\pi+\alpha)$, where $\cot\alpha=\frac{1}{2}$.

Ex. 7. Solve $\cos 9x=\cos 5x-\cos x$.

$\cos 9x=\cos 5x-\cos x$,

or, $\cos 9x+\cos x=\cos 5x$,

or, $2\cos 5x\cos 4x-\cos 5x=0$,

or, $\cos 5x(2\cos 4x-1)=0$,

$\therefore$ either $\cos 5x=0\cdots\cdots(1)$, or, $2\cos 4x-1=0\cdots(2)$

From (1) we have $5x=(2n+1)\frac{\pi}{2}$, $\therefore$ $x=(2n+1)\frac{\pi}{10}$;

and from (2) we have $\cos 4x=\frac{1}{2}=\cos\frac{\pi}{3}$, $\therefore$ $4x=2n\pi\pm\frac{\pi}{3}$,

$\therefore$ $x=(6n\pm1)\frac{\pi}{12}$. $\therefore$ $x=(2n+1)\frac{\pi}{10}$ or, $(6n\pm1)\frac{\pi}{12}$.

Ex. 8. Solve $a\cos x+b\sin x=c$, where $c \not> \sqrt{a^2+b^2}$.

Dividing the equation by $\sqrt{a^2+b^2}$ we have,

$$\frac{a}{\sqrt{a^2+b^2}}\cos x+\frac{b}{\sqrt{a^2+b^2}}\sin x=\frac{c}{\sqrt{a^2+b^2}} \quad \cdots\cdots(1)$$

Now suppose $\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha$ and $\frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$.

$\therefore$ From (1) we have $\cos\alpha\cos x+\sin\alpha\sin x=\frac{c}{\sqrt{a^2+b^2}}$,

or, $\cos(x-\alpha)=\frac{c}{\sqrt{a^2+b^2}}=\cos\beta$ (say)

$\therefore$ $x-\alpha=2n\pi\pm\beta$, $\therefore$ $x=2n\pi\pm\beta+\alpha$.

[*N.B.* (1) Here a, b and c being known, α and β are known.

(2) A new angle that is introduced to help trigonometrical calculations is called a *subsidiary* angle. Here α and β are subsidiary angles.]

(3) In solving such equations, we have to divide both sides by the square root of the sum of the squares of the coefficients of $\cos x$ and $\sin x$.]

Ex. 9. Solve the equation $\tan 3\theta=\cot 2\theta$.

Here $\tan 3\theta=\cot 2\theta=\tan\left(\frac{\pi}{2}-2\theta\right)$

The general value of the angles whose tangents are equal to the tangent of angle $\left(\frac{\pi}{2}-2\theta\right)$, is $n\pi+\left(\frac{\pi}{2}-2\theta\right)$, where n is any positive or negative integer.

Hence, here $3\theta = n\pi + \frac{\pi}{2} - 2\theta$, or, $5\theta = n\pi + \frac{\pi}{2}$,

$\therefore \quad \theta = \frac{1}{5}\left(n\pi + \frac{\pi}{2}\right)$, where n is any integer.

Ex. 10. Solve $\tan x + \tan 2x + \tan 3x = 0$.

[B. H. U. '46, B.U.E. '63]

Here $\tan x + \tan 2x + \tan (x + 2x) = 0$,

or, $(\tan x + \tan 2x) + \dfrac{\tan x + \tan 2x}{1 - \tan x . \tan 2x} = 0$,

or, $(\tan x + \tan 2x)\left(1 - \dfrac{1}{1 - \tan x \tan 2x}\right) = 0$,

$\therefore$ either $\tan x + \tan 2x = 0 \cdots\cdots(1)$.

or, $\left(1 + \dfrac{1}{1 - \tan x \tan 2x}\right) = 0 \cdots\cdots(2)$.

From (1) we have $\tan 2x = -\tan x = \tan(-x) = \tan(n\pi - x)$

$\therefore \quad 2x = n\pi - x$, or, $3x = n\pi$, $\therefore \; x = \dfrac{n\pi}{3}$.

Again, from (2) we have $1 = \dfrac{1}{\tan x \tan 2x - 1}$,

or, $\tan x \tan 2x - 1 = 1$, or, $\tan x \tan 2x = 2$,

or, $\tan x . \dfrac{2 \tan x}{1 - \tan^2 x} = 2$,

or, $2 \tan^2 x = 2 - 2 \tan^2 x$, or, $4 \tan^2 x = 2$,

or, $\tan^2 x = \frac{1}{2}$, $\therefore \; \tan x = \pm \dfrac{1}{\sqrt{2}} = \tan \alpha$ (say)

$\therefore \quad x = n\pi \pm \alpha$, where $\tan^2 \alpha = \frac{1}{2}$.

Ex. 11. Solve $\dfrac{\sin \theta}{\sin 2x} + \dfrac{\cos \theta}{\cos 2x} = 2$. [C. U. '51]

$$\frac{\sin \theta}{\sin 2x} + \frac{\cos \theta}{\cos 2x} = 2,$$

or, $\sin \theta \cos 2x + \cos \theta \sin 2x = 2 \sin 2x \cos 2x$,

or, $\sin(\theta + 2x) = \sin 4x$

$\therefore \quad 4x = m\pi + (-1)^m(\theta + 2x)$, where m is any integer.

Now, if m=an even number$=2n$ (say),

then $4x=2n\pi+\theta+2x$, or, $2x=2n\pi+\theta$, $\therefore\ x=n\pi+\frac{\theta}{2}$.

Again if m=an odd number$=2n+1$ (say),

then $4x=(2n+1)\pi-(\theta+2x)$, or, $6x=(2n+1)\pi-\theta$,

$$\therefore\quad x=(2n+1)\frac{\pi}{6}-\frac{\theta}{6}.$$

Ex. 12. Solve $7\cos\theta+3\sin\theta=3$, given $\tan 72^\circ 18'=2\frac{1}{3}$.

Suppose $72^\circ 18'=\alpha$;

Here, $7\cos\theta+3\sin\theta=3$, or, $\frac{7}{3}\cos\theta+\sin\theta=1$,

or, $\tan\alpha\cos\theta+\sin\theta=1$, $[\because\ \frac{7}{3}=\tan 72^\circ 18'=\tan\alpha]$

or, $\dfrac{\sin\alpha}{\cos\alpha}\cos\theta+\sin\theta=1$,

or, $\sin\alpha\cos\theta+\cos\alpha\sin\theta=\cos\alpha$,

or, $\sin(\alpha+\theta)=\cos\alpha=\sin\left(\frac{\pi}{2}-\alpha\right)$

$$\therefore\quad \alpha+\theta=n\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right)$$

$$=n\pi+(-1)^n.17^\circ 42'\ \left[\because\quad \frac{\pi}{2}-\alpha=90^\circ-72^\circ 18'=17^\circ 42'\right]$$

$$\therefore\quad \theta=n\pi+(-1)^n 17^\circ 42'-\alpha=n\pi+(-1)^n 17^\circ 42'-72^\circ 18'.$$

$$=n\pi+(-)^n\frac{59\pi}{600}-\frac{241\pi}{600}.$$

Ex. 13. Find all the values of θ between 0° and 1000° that satisfy the equation $2\sin^2\theta-\sin\theta=3$.

Here $2\sin^2\theta-\sin\theta-3=0$,

or, $2\sin^2\theta-3\sin\theta+2\sin\theta-3=0$,

or, $(\sin\theta+1)(2\sin\theta-3)=0$,

$\therefore$ either $\sin\theta+1=0$ or, $2\sin\theta-3=0$, $\therefore\sin\theta=-1$ or $\frac{3}{2}$;

but the numerical value of the sine of any angle cannot be $\frac{3}{2}$.

$$\therefore\quad \sin\theta=-1,\quad \therefore\quad \theta=(4n-1)\frac{\pi}{2}=(4n-1)90^\circ$$

Here, θ lies between 0° and 1000°, if $n=1$, 2 and 3.

$$\therefore \quad \theta=270^\circ, 630^\circ, 990^\circ.$$

Ex. 14. Solve $\sqrt{3}\cos A+\sin A=\sqrt{3}$ for $-2\pi<A<2\pi$.

Here, $(\sqrt{3})^2+(1)^2=3+1=4$; dividing both sides of the equation by $\sqrt{4}$ or 2 we have

$$\cos A.\frac{\sqrt{3}}{2}+\frac{1}{2}.\sin A=\frac{\sqrt{3}}{2},$$

$$\text{or,}\quad \cos A\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\sin A=\frac{\sqrt{3}}{2},$$

$$\text{or,}\quad \cos\left(A-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6},\quad \therefore\ A-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{6}$$

$$\therefore\quad A=2n\pi+\frac{\pi}{3}\cdots(1)\quad \text{or,}\quad A=2n\pi\cdots(2).$$

Now, from (1) we find that if $n=0$ and -1, then the value of A lies between -2π and 2π. $\therefore$ $A=\frac{\pi}{3}$, $-\frac{5\pi}{3}$.

Again, from (2) we find that only if $n=0$, A lies between the limits -2π to 2π. $\therefore$ $A=0$.

$$\text{Hence,}\quad A=-\frac{5\pi}{3},\ 0,\ \frac{\pi}{3}.$$

Exercise 8

1. Write down the general expression of all angles whose sine is equal to $\frac{\sqrt{3}}{2}$.

2. What are the general values of θ which satisfy the equation $\tan\theta=-1$?

3. What is the most general value of θ which satisfy both the equations $\sin\theta=-\frac{1}{2}$ and $\tan\theta=\frac{1}{\sqrt{3}}$?

Solve the following equations :—

4. $\sqrt{3}\cos\theta+\sin\theta=1$. [B. U. E. '64]
5. $\tan^2\theta=3\operatorname{cosec}^2\theta-1$. [C. U. '39]
6. $2\sin^2x+\sin^2 2x=2$. [C. U. '40]
7. $\tan x+\cot x=4$. [C. U. '13]
8. $\sin x+\cos x=\dfrac{1}{\sqrt{2}}$. [B. H. U. '48]
9. $\cos\theta+\sqrt{3}\sin\theta=2$.
10. $\sin x+\sqrt{3}\cos x=\sqrt{2}$. [B.H.U. '47]
11. $\operatorname{cosec}^2\theta+\cot^2\theta=3\cot\theta$.
12. $\sin m\theta+\sin n\theta=0$. [C.U.]
13. $\sin 4\theta=\sin\theta$. [B.H.U. '49]
14. $\tan 5\theta=\cot 2\theta$. [A. U. '43] 14. (*a*) $\tan px=\cot qx$. [C. U. '53]
15. $\sin 2\theta=\cos\theta$.
16. $\cos x+\cos 2x+\cos 3x=0$. [C. U. '46]
17. $\cos\theta+\sin\theta+\sqrt{2}=0$.
18. $\sec\theta-1=(\sqrt{2}-1)\tan\theta$.
19. $\sin 7\theta-\sin 3\theta-\sin\theta=0$. [P. U. '39]
20. $\sin 5x\cos 3x=\sin 9x.\cos 7x$. [B.H.U. '46]
21. $\cos 3\theta-\cos 5\theta=\sin\theta$. [A.U. '39]
22. $\tan x+\tan 3x=2\tan 2x$.
23. $\tan^2x-(1+\sqrt{3})\tan x+\sqrt{3}=0$.
24. $\tan x+\tan 2x+\tan x\tan 2x=1$. [C. U. '41, '45, '48]
25. $\tan\theta+\tan 2\theta=\tan 3\theta$.
26. $\tan^2x+\cot^2x=2$. 27. $\cos 2\alpha=\cos\alpha+\sin\alpha$.
28. $4\sin 4\theta+1=\sqrt{5}$. [C. U.] 29. $\tan^2 3x=\cot^2\alpha$.
30. $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)=4$. [C.U. '49, N. B. '64]
31. $\cos x+\sin x=\cos 2x+\sin 2x$. [C. U. '43]
32. $\tan x+\tan 2x+\tan 3x=0$. [A. I. '41]
33. $\cos x-\sin x=\cos\alpha+\sin\alpha$. [B.H.U. '38]
34. $\cos x+\cos 3x+\cos 5x+\cos 7x=0$. [C.U. '59]

35. $\dfrac{\cos\alpha}{\cos x}+\dfrac{\sin\alpha}{\sin x}=2.$

36. Solve $3\cos x+2\sin x=2$, having given $\tan 65°22'=1\frac{1}{2}$.

37. Solve $\sin\dfrac{n+1}{2}\theta=\sin\dfrac{n-1}{2}\theta+\sin\theta.$ [U.P.B. '53]

38. Find all the values of θ between $0°$ and $1000°$ that satisfy the equation $3\sin^2\theta-8\sin\theta+5=0$.

39. Solve $\tan x+\tan 2x+\tan 3x=\tan x\tan 2x\tan 3x$. [C. U. '56]

40. Solve $\sin\theta+2\cos\theta=1$. [C. U. '34]

41. If $\sin A=\sin B$, and $\cos A=\cos B$, prove that either A and B are equal or they differ by some multiple of four right angles. [C. U. '35]

42. Solve $4\sin\theta\cos\theta=1-2\sin\theta+2\cos\theta$ in the interval $0<\theta<\pi$. [C. U. '50]

43. Solve the equation $\sin 4\theta=\cos 3\theta+\sin 2\theta$ in the interval $0<\theta<\pi$. [C. U. '51]

44. Solve $\cos\theta-\sin\theta=\dfrac{1}{\sqrt{2}}$ in $-\pi<\theta<+\pi$. [C. U. '52]

Solve the following for θ when $0<\theta<2\pi$:—

45. $\cos\theta+\sqrt{3}\sin\theta=2$. [C. U. '36]

46. $\cot\theta-\tan\theta=2$. [C. U. '37]

47. $\sin\theta+\sqrt{3}\cos\theta=\sqrt{2}$. [C. U. '38]

48. Solve the equations (general solutions not required) $\tan x+\tan y=2$ and $2\cos x\cos y=1$. [C. U. '55]

49. Solve $2\cos x+5\sin x=5$, given $\tan 68°12'=2\frac{1}{2}$.

50. If $\cos\theta-\sin\theta=\cos\alpha-\sin\alpha$, prove that

$$\theta+\frac{\pi}{4}=2n\pi\pm\left(\alpha+\frac{\pi}{4}\right).$$ [B.H.U. '40]

Inverse Circular Functions

75. If $\sin\theta=k$, where k is a known quantity, the value of θ is not definitely known, but we only know that θ is any one of the series of angles whose sine is equal to k. Hence $\sin\theta=k$ signifies that θ is one of the angles whose sine is k. This statement can be inversely expressed as $\theta=\sin^{-1}k$. Thus the *symbol* $\sin^{-1}k$ denotes an angle whose sine is k. In the inverse notation $\theta=\sin^{-1}k$, θ stands on one side of the equation and is regarded as an angle whose value is known through the medium of its sine.

Similary, $\cos^{-1}x$ denotes any one of the angles whose cosine is x. Thus $\tan^{-1}\sqrt{3}$ indicates one of the angles whose tangent is $\sqrt{3}$. But we have seen that all such angles are included in the formula $n\pi+\frac{\pi}{3}$. Hence, $\theta=\tan^{-1}\sqrt{3}$ and $\theta=n\pi+\frac{\pi}{3}$ are equivalent, though different in forms.

It is to be noted here that *$\sin^{-1}k$ is an angle,* whereas *$\sin\theta$ is a number.* Sin $\theta=k$ and $\theta=\sin^{-1}k$ are identical in sense and if one of them is known, the other is also known. Again, since the sine of an angle cannot be greater than 1, so $\sin^{-1}x$ will be meaningless unless $-1\leqslant x\leqslant 1$.

Similarly, $-1\leqslant x\leqslant 1$ in the case of $\cos^{-1}x$; in both the cases, $\operatorname{cosec}^{-1}x$ and $\sec^{-1}x$, $-1\leqslant x$ and $x\geqslant 1$, and x may have any value in $\tan^{-1}x$ and $\cot^{-1}x$.

76. The symbol *$\sin^{-1}k$ is read as "sine inverse k"* or as *sine minus one k*". In some books, it is expressed as "arc sin k". $\cos^{-1}k$ is read as 'cos-inverse k'. The expressions of the form $\sin^{-1}x$, $\cos^{-1}a$, $\tan^{-1}b$ are called **Inverse Circular Functions.**

N. B. It is to be noted here that $\sin^{-1}k$ and $(\sin k)^{-1}$ are not same. $(\sin k)^{-1}=\dfrac{1}{\sin k}$, but in $\sin^{-1}k$, -1 is not an index.

77. If θ be an angle whose sine is k, then the sine of all the angles included in $n\pi+(-1)^n\theta$ will be k. So, $\sin^{-1}k$ has an infinite number of values *i.e.*, it is a multivalued function.

Hence, the general value of $\sin^{-1} k = n\pi + (-1)^n \sin^{-1} k$. Similarly, the general values of $\cos^{-1} k$ and $\tan^{-1} k$ are respectively $2n\pi \pm \cos^{-1} k$ and $n\pi + \tan^{-1} k$.

78. Principal Value : The smallest numerical value of θ, positive or negative, is called *principal value* of $\sin^{-1} k$.

If there be two numerically equal angles, one positive and the other negative, corresponding to the same ratio, then the positive angle is *generally* taken as the principal value.

In numerical instances the principal value is usually selected.

Examples. (1) The principal value of $\sin^{-1} \frac{1}{\sqrt{2}} = 45°$.

(2) The principal value of $\tan^{-1}(-1) = -45°$.

(3) We have $\cos 60° = \frac{1}{2}$ and $\cos(-60°) = \frac{1}{2}$; here the cosine of both the angles $60°$ and $-60°$ is $\frac{1}{2}$, but the principal value of $\cos^{-1} \frac{1}{2}$ is $60°$ and not $-60°$.

79. From the above relations we find that $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, and $\operatorname{cosec}^{-1} x$ always lie between $-90°$ and $90°$, but $\cos^{-1} x$ and $\sec^{-1} x$ always lie between $0°$ and $180°$.

If x *is positive*, $\sin^{-1} x$ lies between $0°$ and $90°$, but if x is negative $\sin^{-1} x$ lies between $-90°$ and $0°$.

If x *is positive*, $\cos^{-1} x$ lies between $0°$ and $90°$, but it lies between $90°$ and $180°$, if x is negative.

80. (*i*) If $\sin\theta = x$, then $\theta = \sin^{-1} x$, $\therefore$ $\theta = \sin^{-1} \sin\theta$.
Similarly, $\theta = \cos^{-1}\cos\theta$, $\theta = \tan^{-1}\tan\theta$, etc.

(*ii*) Again if $\theta = \sin^{-1} x$, then $\sin\theta = x$, $\therefore$ $\sin\sin^{-1} x = x$.
Similarly $\cos\cos^{-1} x = x$, $\tan\tan^{-1} x = x$, etc.

(*iii*) To prove that $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$, $\cot^{-1} x = \tan^{-1}\frac{1}{x}$

and $\sec^{-1} x = \cos^{-1}\frac{1}{x}$.

Proof. Let $\operatorname{cosec}^{-1} x=\theta$, then $\operatorname{cosec}\theta=x$.

$$\therefore \quad \sin\theta=\frac{1}{\operatorname{cosec}\theta}=\frac{1}{x}, \quad \therefore \sin^{-1}\frac{1}{x}=\theta.$$

$$\text{Hence, } \operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}.$$

The other two relations may be similarly proved.

[*N.B.* It may be similarly shown that $\operatorname{cosec}^{-1}\frac{1}{x}=\sin^{-1}x$,

$\cot^{-1}\frac{1}{x}=\tan^{-1}x$, $\sec^{-1}\frac{1}{x}=\cos^{-1}x$]

81. Since all the trigonometrical ratios can be expressed in terms of any one of them, we can express all the inverse trigonometrical ratios in terms of any one inverse ratio.

Suppose $\theta=\sin^{-1}x$, then $\sin\theta=x$.

$$\therefore \quad \cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}, \ \tan\theta=\frac{x}{\sqrt{1-x^2}},$$

$$\cot\theta=\frac{\sqrt{1-x^2}}{x}, \ \sec\theta=\frac{1}{\sqrt{1-x^2}}, \ \operatorname{cosec}\theta=\frac{1}{x}.$$

Hence, we have $\theta=\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x}.$$

N.B. From the above relations we have $\theta=\sin^{-1}x$ and $\theta=\cos^{-1}\sqrt{1-x^2}$, but we cannot conclude from this that the equation $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$ ...(1) is *identically* true.

For, in $\theta=\sin^{-1}x$ and $\theta=\cos^{-1}\sqrt{1-x^2}$, θ has an infinite number of values, but as the general values of sine and cosine are not the same, the equation-(1) is not always true.

As for example, let $x=\frac{1}{2}$, then $\sqrt{1-x^2}=\frac{\sqrt{3}}{2}$.

Now $\sin^{-1} x=\sin^{-1} \frac{1}{2}=30°, 150°, 390°, 510°$, etc.

and $\cos^{-1} \sqrt{1-x^2}=\cos^{-1} \frac{\sqrt{3}}{2}=30°, 330°, 390°, 690°$, etc.

Hence, evidently, $\sin^{-1} x$ and $\cos^{-1} \sqrt{1-x^2}$ are not always equal.

82. To prove that : (*i*) $\sin^{-1} x+\cos^{-1} x=\frac{\pi}{2}$.

(*ii*) $\tan^{-1} x+\cot^{-1} x=\frac{\pi}{2}$, (*iii*) $\sec^{-1} x+\operatorname{cosec}^{-1} x=\frac{\pi}{2}$.

Proof. (i) Let $\sin^{-1} x=\theta$, then $\sin \theta=x$.

Now, $\because$ $\sin \theta=\cos (90°-\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$,

$\therefore$ $\cos\left(\frac{\pi}{2}-\theta\right)=x$, $\therefore$ $\cos^{-1} x=\frac{\pi}{2}-\theta$.

$\therefore$ $\sin^{-1}x+\cos^{-1}x=\theta+\frac{\pi}{2}-\theta=\frac{\pi}{2}$.

(ii) Let $\tan^{-1} x=\theta$, then $\tan \theta=x$.

Again, $\tan \theta=\cot\left(\frac{\pi}{2}-\theta\right)$, $\therefore$ $\cot\left(\frac{\pi}{2}-\theta\right)=x$.

$\therefore$ $\cot^{-1} x=\frac{\pi}{2}-\theta$. $\therefore$ $\tan^{-1}x+\cot^{-1}x=\theta+\frac{\pi}{2}-\theta=\frac{\pi}{2}$.

(iii) Suppose $\sec^{-1}x=\theta$, then $\sec \theta=x$.

Now, $\sec \theta=\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)$

$\therefore$ $\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)=x$, $\therefore$ $\operatorname{cosec}^{-1}x=\frac{\pi}{2}-\theta$.

$\therefore$ $\sec^{-1}x+\operatorname{cosec}^{-1}x=\theta+\frac{\pi}{2}-\theta=\frac{\pi}{2}$.

83. To prove that :

(i) $\tan^{-1}\mathbf{x}+\tan^{-1}\mathbf{y}=\tan^{-1} \dfrac{\mathbf{x}+\mathbf{y}}{1-\mathbf{xy}}$

and (ii) $\tan^{-1}\mathbf{x}-\tan^{-1}\mathbf{y}=\tan^{-1} \dfrac{\mathbf{x}-\mathbf{y}}{1+\mathbf{xy}}$.

Proof. (i) Let $\tan^{-1}x=\alpha$ and $\tan^{-1}y=\beta$.

$\therefore$ $\tan\alpha=x$ and $\tan\beta=y$.

Now, $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+y}{1-xy}$.

$\therefore$ $\alpha+\beta=\tan^{-1}\dfrac{x+y}{1-xy}$.

Hence, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$.

(iii) Suppose $\tan^{-1}x=\alpha$ and $\tan^{-1}y=\beta$;

so, $\tan\alpha=x$ and $\tan\beta=y$.

Now, $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{x-y}{1+xy}$,

$\therefore$ $\alpha-\beta=\tan^{-1}\dfrac{x-y}{1+xy}$.

Hence, $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}$.

84. Prove that :

(*i*) $\cot^{-1}x+\cot^{-1}y=\cot^{-1}\dfrac{xy-1}{y+x}$,

(*ii*) $\cot^{-1}x-\cot^{-1}y=\cot^{-1}\dfrac{xy+1}{y-x}$.

Proof. (i) Let $\cot^{-1}x=\alpha$ and $\cot^{-1}y=\beta$,

then $\cot\alpha=x$ and $\cot\beta=y$.

Now, $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=\dfrac{xy-1}{y+x}$

$\therefore$ $\alpha+\beta=\cot^{-1}\dfrac{xy-1}{y+x}$, $\therefore$ $\cot^{-1}x+\cot^{-1}y=\cot^{-1}\dfrac{xy-1}{y+x}$.

(ii) Similarly, $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\dfrac{xy+1}{y-x}$,

$\therefore$ $\alpha-\beta=\cot^{-1}\dfrac{xy+1}{y-x}$, $\therefore$ $\cot^{-1}x-\cot^{-1}y=\cot^{-1}\dfrac{xy+1}{y-x}$.

85. To prove that :

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{x+y}{1-xy}+\tan^{-1}z$$

$$=\tan^{-1}\frac{\frac{x+y}{1-xy}+z}{1-\frac{x+y}{1-xy}\times z}=\tan^{-1}\frac{\frac{x+y+z-xyz}{1-xy}}{\frac{1-xy-yz-zx}{1-xy}}$$

$$=\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}.$$

[**Alternative proof**] :

Let $\tan^{-1}x=A$, $\tan^{-1}y=B$, $\tan^{-1}z=C$.

$\therefore$ $\tan A=x$, $\tan B=y$ and $\tan C=z$.

Now, $\tan(A+B+C)=\tan\{(A+B)+C\}$

$$=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}=\frac{\frac{\tan A+\tan B}{1-\tan A.\tan B}+\tan C}{1-\frac{\tan A+\tan B}{1-\tan A\tan B}\times\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

$$\therefore A+B+C=\tan^{-1}\left\{\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}\right\}$$

Now, substituting the values of A, B, C and tan A, tan B, tan C, we have

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{x+y+z-xyz}{1-xy-yz-zx}.$$

86. To prove that :

(i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{(1-x)^2(1-y^2)}\}$

(iv) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{(1-x)^2(1-y^2)}\}$.

Proof. (i) Suppose $\sin^{-1}x=\alpha$, and $\sin^{-1}y=\beta$,

then $\sin\alpha=x$, and $\sin\beta=y$.

$\therefore\ \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-x^2}$,

and $\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-y^2}$.

Now, $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$\therefore\ \alpha+\beta=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$

$\therefore\ \sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$

(ii) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=x\sqrt{1-y^2}-y\sqrt{1-x^2} \qquad \text{[Vide (i)]}$$

$\therefore\ \alpha-\beta=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$

$\therefore\ \sin^{-1}x-\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$

(iii) Let $\cos^{-1}x=\alpha$ and $\cos^{-1}y=\beta$,

then $\cos\alpha=x$ and $\cos\beta=y$.

$\therefore\ \sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-x^2}$

and $\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-y^2}$.

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=xy-(\sqrt{1-x^2})(\sqrt{1-y^2})$$

$\therefore\ \alpha+\beta=\cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}$

$\therefore\ \cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}$.

(iv) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$=xy+\sqrt{(1-x^2)(1-y^2)} \qquad \text{[Vide (iii)]}$$

$\therefore\ \alpha-\beta=\cos^{-1}\{xy+\sqrt{(1-x^2)(1-y^2)}\}$

$\therefore\ \cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{(1-x^2)(1-y^2)}\}$.

87. To prove that :

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$

(iii) $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

Proof. (i) Suppose $\sin^{-1}x=\theta$, then $\sin\theta=x$,

$\therefore\quad \cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$.

Now, $\sin 2\theta=2\sin\theta\cos\theta=2x\sqrt{1-x^2}$.

$\therefore\quad 2\theta=\sin^{-1}(2x\sqrt{1-x^2})$,

$\therefore\quad 2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$.

(ii) Let $\cos^{-1}x=\theta$, then $\cos\theta=x$.

$\therefore\quad \cos 2\theta=2\cos^2\theta-1=2x^2-1$, $\quad\therefore\quad 2\theta=\cos^{-1}(2x^2-1)$

$\therefore\quad 2\cos^{-1}x=\cos^{-1}(2x^2-1)$.

(iii) Suppose $\tan^{-1}x=\theta$, then $\tan\theta=x$.

Now, $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2x}{1-x^2}$,

$\therefore\quad 2\theta=\tan^{-1}\dfrac{2x}{1-x^2}$, $\therefore\quad 2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

88. To prove that :

(i) $\mathbf{3\sin^{-1}x=\sin^{-1}(3x-4x^3)}$

(ii) $\mathbf{3\cos^{-1}x=\cos^{-1}(4x^3-3x)}$

(iii) $\mathbf{3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}}$.

Proof. (i) Let $\sin^{-1}x=\alpha$, then $\sin\alpha=x$.

Now, $\sin 3\alpha=3\sin\alpha-4\sin^3\alpha=3x-4x^3$

$\therefore\quad 3\alpha=\sin^{-1}(3x-4x^3)$.

$\therefore\quad 3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

(ii) Let $\cos^{-1}x=\alpha$, then $\cos\alpha=x$.

Now, $\cos 3\alpha=4\cos^3\alpha-3\cos\alpha=4x^3-3x$,

$\therefore\quad 3\alpha=\cos^{-1}(4x^3-3x)$,

$\therefore\quad 3\cos^{-1}x=\cos^{-1}(4x^3-3x)$.

(iii) Let $\tan^{-1}x=\alpha$, then $\tan\alpha=x$.

Now $\because\quad \tan 3\alpha=\dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\dfrac{3x-x^3}{1-3x^2}$,

$\therefore\quad 3\alpha=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$, $\therefore\quad 3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$.

39. To prove that :

$$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}.$$

Proof. Let $\tan^{-1}x=\theta$, then $\tan\theta=x$.

Now $\because$ $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2x}{1+x^2}$,

$\therefore$ $2\theta=\sin^{-1}\dfrac{2x}{1+x^2}$, $\therefore$ $2\tan^{-1}x=\sin^{-1}\dfrac{2x}{1+x^2}$.

Again, $\because$ $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-x^2}{1+x^2}$,

$\therefore$ $2\theta=\cos^{-1}\dfrac{1-x^2}{1+x^2}$, $\therefore$ $2\tan^{-1}x=\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

Again, $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2x}{1-x^2}$

$\therefore$ $2\theta=\tan^{-1}\dfrac{2x}{1-x^2}$, $\therefore$ $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

Hence, $2\tan^{-1}x=\sin^{-1}\dfrac{2x}{1+x^2}=\cos^{-1}\dfrac{1-x^2}{1+x^2}=\tan^{-1}\dfrac{2x}{1-x^2}$.

[*N. B.* The above relations are very important and should be remembered.]

Examples (9)

Ex. 1. Find the value of $\sin^{-1}\dfrac{\sqrt{3}}{2}$.

Suppose $\theta=\sin^{-1}\dfrac{\sqrt{3}}{2}$, then $\sin\theta=\dfrac{\sqrt{3}}{2}=\sin 60^\circ=\sin\dfrac{\pi}{3}$.

$\therefore$ the required value $=\dfrac{\pi}{3}$.

Ex. 2. Express $\sec^{-1}x$ in terms of other inverse functions.

Let $\sec^{-1}x=\theta$, then $\sec\theta=x$.

$\therefore$ $\dfrac{1}{\cos\theta}=x$, or, $\cos\theta=\dfrac{1}{x}$, $\therefore$ $\theta=\cos^{-1}\dfrac{1}{x}$.

Again, $\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{1}{x^2}}=\frac{\sqrt{x^2-1}}{\pm x}$

$\therefore\quad \theta=\sin^{-1}\frac{\sqrt{x^2-1}}{\pm x}.$

Now, $\tan\theta=\frac{\sin\theta}{\cos\theta}=\pm\sqrt{x^2-1},$

$\therefore\quad \theta=\tan^{-1}(\pm\sqrt{x^2-1}),$

$\cot\theta=\pm\frac{1}{\sqrt{x^2-1}},\quad\therefore\quad\theta=\cot^{-1}\frac{\pm1}{\sqrt{x^2-1}}$

and $\operatorname{cosec}\theta=\frac{1}{\sin\theta}=\frac{\pm x}{\sqrt{x^2-1}},\quad\therefore\quad\theta=\operatorname{cosec}^{-1}\frac{\pm x}{\sqrt{x^2-1}}.$

Hence, $\sec^{-1}x=\sin^{-1}\frac{\sqrt{x^2-1}}{\pm x}=\cos^{-1}\frac{1}{x}=\tan^{-1}(\pm\sqrt{x^2-1})$

$$=\cot^{-1}\frac{\pm1}{\sqrt{x^2-1}}=\operatorname{cosec}^{-1}\frac{\pm x}{\sqrt{x^2-1}}.$$

Ex. 3. Show that $\sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}.$

Let $\sin^{-1}\frac{12}{13}=\theta$, then $\sin\theta=\frac{12}{13}.$

$\therefore\quad \cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{144}{169}}=\sqrt{\frac{25}{169}}=\frac{5}{13}.$

$\therefore\quad \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{12}{13}\times\frac{13}{5}=\frac{12}{5},\quad\therefore\quad\theta=\tan^{-1}\frac{12}{5}$

Hence, $\sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}.$

Ex. 4. Show that $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{16}{65}.$

Let $\sin^{-1}\frac{4}{5}=\alpha$ and $\cos^{-1}\frac{12}{13}=\beta$,
then $\sin\alpha=\frac{4}{5}$, and $\cos\beta=\frac{12}{13}.$

$\therefore$ the L.H.S.$=\alpha+\beta$, this is to be expressed in terms of inverse cosine, for the R.H.S. is in the terms of inverse cosine.

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Putting the values of the functions on the right side as obtained from fig. 4, we have

$$\cos(\alpha+\beta)=\tfrac{3}{5}\times\tfrac{12}{13}-\tfrac{4}{5}\times\tfrac{5}{13}$$
$$=\tfrac{36}{65}-\tfrac{4}{13}=\tfrac{16}{65}$$

$\therefore\quad \alpha+\beta=\cos^{-1}\frac{16}{65}.$

Hence,

$$\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{16}{65}.$$

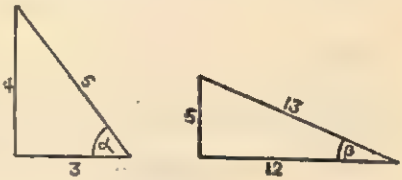


Fig. 4

[*N.B.* (i) In the above example, the value of cos α could be found from sin $\alpha=\frac{4}{5}$, and the value of sin β could be found from cos $\beta=\frac{12}{13}$, but it is easier to find those values by drawing diagrams as shown above. Here sin $\alpha=\frac{4}{5}$, hence in the first right-angled triangle (fig. 4), we take the hypotenuse = 5 units and the perpendicular = 4 units, consequently the base = 3 units.

Similarly in the second triangle, the hypotenuse = 13 units and the base = 12 units, and hence the perpendicular = 5 units.

(ii) In some cases it is easier to solve the problems by expressing the ratios as tangents or cotangents.]

Ex. 5. Show that $2\tan^{-1}\dfrac{1}{5}+\cos^{-1}\dfrac{63}{65}=\sin^{-1}\dfrac{3}{5}.$

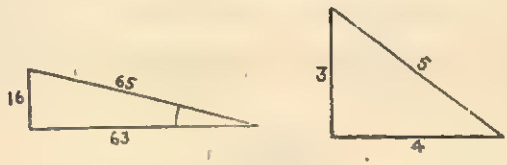


Fig. 5

The L.H.S. $=\tan^{-1}\dfrac{2\times\frac{1}{5}}{1-(\frac{1}{5})^2}+\tan^{-1}\frac{16}{63}$

[$\because$ from fig. 5 we have $\cos^{-1}\frac{63}{65}=\tan^{-1}\frac{16}{63}$]

$$=\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}+\tan^{-1}\frac{16}{63}=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{16}{63}$$

$$=\tan^{-1}\frac{\frac{5}{12}+\frac{16}{63}}{1-\frac{5}{12}\times\frac{16}{63}}=\tan^{-1}\frac{\frac{507}{12\times63}}{\frac{169}{189}}=\tan^{-1}\frac{3}{4}=\sin^{-1}\frac{3}{5}.$$

[*N.B.* Here $\cos^{-1}\frac{63}{65}$ being given, we take the hypotenuse $=65$ units and the base$=63$ units in the first triangle (fig. 5), so the perpendicular$=16$ units. Now, the tangent of the angle, whose cosine is $\frac{63}{65}$, is $\frac{16}{63}$. $\therefore$ we have $\cos^{-1}\frac{63}{65}=\tan^{-1}\frac{16}{63}$.

Again, the left hand side has been reduced to $\tan^{-1}\frac{3}{4}$, so in the second triangle we take the perpendicular$=3$ and the base$=4$ units and consequently the hypotenuse$=5$ units. Now, the angle, whose tangent is $\frac{3}{4}$, has its sine$=\frac{3}{5}$, $\therefore$ we have $\tan^{-1}\frac{3}{4}=\sin^{-1}\frac{3}{5}$]

Ex. 6. Prove that $\cos\tan^{-1}\cot\sin^{-1}x=x$.

Let $\sin^{-1}x=\theta$, then $\sin\theta=x$, $\therefore$ $\cos\theta=\sqrt{1-x^2}$.

$$\therefore \quad \cot\sin^{-1}x=\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{\sqrt{1-x^2}}{x}=\tan\left(\frac{\pi}{2}-\theta\right).$$

$$\therefore \quad \tan^{-1}\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\pi}{2}-\theta$$

$$\therefore \quad \text{the L.H.S.}=\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=x.$$

Ex. 7. Prove that $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

[C. U. '60]

$$\because \quad \cot^{-1}(x+1)=\tan^{-1}\frac{1}{x+1},$$

$$\therefore \quad \text{the L.H.S.}=\tan^{-1}x+\tan^{-1}\frac{1}{x+1}=\tan^{-1}\frac{x+\dfrac{1}{x+1}}{1-x.\dfrac{1}{x+1}}$$

$$=\tan^{-1}\frac{\dfrac{x^2+x+1}{x+1}}{\dfrac{1}{x+1}}=\tan^{-1}(x^2+x+1).$$

Ex. 8. Prove that $\sin(2\sin^{-1}x)=2x\sqrt{1-x^2}$.

Let $\sin^{-1}x=\theta$, $\therefore$ $\sin\theta=x$.

$\therefore$ $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$.

Now, $\sin 2\theta=2\sin\theta\cos\theta=2x\sqrt{1-x^2}$,

$$\therefore \quad 2\theta=\sin^{-1}(2x\sqrt{1-x^2})$$

or, $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

$\therefore \quad \sin(2\sin^{-1}x)=\sin\{\sin^{-1}(2x\sqrt{1-x^2})\}=2x\sqrt{1-x^2}.$

Ex. 9. Show that $\tan^{-1}\dfrac{x-y}{1+xy}+\tan^{-1}\dfrac{y-z}{1+yz}$

$$+\tan^{-1}\frac{z-x}{1+zx}=0.$$

$$\tan^{-1}\frac{x-y}{1+xy}=\tan^{-1}x-\tan^{-1}y\cdots(1)$$

$$\tan^{-1}\frac{y-z}{1+yz}=\tan^{-1}y-\tan^{-1}z\cdots(2)$$

and $\tan^{-1}\dfrac{z-x}{1+zx}=\tan^{-1}z-\tan^{-1}x\ldots(3)$

Now, adding (1), (2) and (3) we have

$$\tan^{-1}\frac{x-y}{1+xy}+\tan^{-1}\frac{y-z}{1+yz}=\tan^{-1}\frac{z-x}{1+zx}=0.$$

Ex. 10. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$,

show that $xy+yz+zx=1$.

Let $\tan^{-1}x=A$, $\tan^{-1}y=B$, $\tan^{-1}z=C$.

$\therefore \quad \tan A=x$, $\tan B=y$, $\tan C=z$.

Now, from ihe given condition we have

$A+B+C=\dfrac{\pi}{2}$, or, $A+B=\dfrac{\pi}{2}-C$.

$$\therefore \quad \tan(A+B)=\tan\left(\frac{\pi}{2}-C\right),$$

or, $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\cot C=\dfrac{1}{\tan C}$,

$\therefore \quad \tan A\tan C+\tan B\tan C=1-\tan A\tan B,$

or, $\tan A\tan B+\tan B\tan C+\tan C\tan A=1.$

$\therefore \quad xy+yz+zx=1$ [Putting the values of $\tan A$, $\tan B$, $\tan C$]

Ex. 11. Solve $\sin^{-1} x=\cos^{-1} x$.

Let $\cos^{-1} x=\theta$, $\therefore$ $\cos\theta=x$,

$\therefore$ $\sin\theta=\sqrt{1-x^2}$, $\therefore$ $\sin^{-1}\sqrt{1-x^2}=\theta$.

Now, we have $\sin^{-1} x=\cos^{-1} x=\sin^{-1}\sqrt{1-x^2}$,

or, $x=\sqrt{1-x^2}$ or, $x^2=1-x^2$, or, $2x^2=1$,

or, $x^2=\frac{1}{2}$, $\therefore$ $x=\pm\frac{1}{\sqrt{2}}$.

Ex. 12. Solve $\tan^{-1} 2x+\tan^{-1} 3x=\frac{\pi}{4}$.

$$\tan^{-1} 2x+\tan^{-1} 3x=\frac{\pi}{4},$$

or, $\tan^{-1}\dfrac{2x+3x}{1-2x\times 3x}=\dfrac{\pi}{4}$, or, $\tan^{-1}\dfrac{5x}{1-6x^2}=\dfrac{\pi}{4}$

$\therefore$ $\dfrac{5x}{1-6x^2}=\tan\dfrac{\pi}{4}=1$, or, $5x=1-6x^2$,

$\therefore$ $6x^2+5x-1=0$, or, $(6x-1)(x+1)=0$,

$\therefore$ either $6x-1=0$, or, $x+1=0$, $\therefore$ $x=\frac{1}{6}$ or -1.

$\because$ $x=-1$ does not satisfy the equation,

$\therefore$ it is an extraneous root. $\therefore$ the required solution is $x=\frac{1}{6}$.

Ex. 13. Solve $\tan^{-1}\dfrac{1}{1+2x}+\tan^{-1}\dfrac{1}{4x+1}=\tan^{-1}\dfrac{2}{x^2}$.

[Agra '47]

$\because$ $\tan^{-1} a+\tan^{-1} b=\tan^{-1}\dfrac{a+b}{1-ab}$,

$$\therefore\quad \tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{\dfrac{1}{1+2x}+\dfrac{1}{1+4x}}{1-\dfrac{1}{(1+2x)(1+4x)}}$$

$$=\tan^{-1}\frac{2(1+3x)}{2x(3+4x)}=\tan^{-1}\frac{1+3x}{x(3+4x)}.$$

Hence, $\tan^{-1}\dfrac{1+3x}{x(3+4x)}=\tan^{-1}\dfrac{2}{x^2}$,

$$\therefore \quad \frac{1+3x}{x(3+4x)}=\frac{2}{x^2}, \text{ or, } \frac{1+3x}{3+4x}=\frac{2}{x},$$

or, $\quad x+3x^2=6+8x,\quad$ or, $\quad 3x^2-7x-6=0,$

$\therefore \quad (3x+2)(x-3)=0,$

$\therefore \quad x=-\frac{2}{3}\quad$ or $\quad 3.$

Ex. 14. Solve $\sin^{-1} x - \cos^{-1} x=\sin^{-1}(3x-2)$.

Let $\sin^{-1} x=\alpha$ and $\cos^{-1} x=\beta$.

$\therefore \quad \sin\alpha=x$ and $\cos\beta=x$;

then $\cos\alpha=\sqrt{1-x^2}$ and $\sin\beta=\sqrt{1-x^2}$.

Now from the given equation we have

$$\alpha-\beta=\sin^{-1}(3x-2).$$

$$\begin{aligned}\therefore \quad 3x-2&=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &=x\times x-\sqrt{1-x^2}\times\sqrt{1-x^2}\\ &=x^2-(1-x^2)=2x^2-1,\end{aligned}$$

or, $\quad 2x^2-3x+1=0,\quad$ or, $\quad (2x-1)(x-1)=0,$

$\therefore \quad x=\frac{1}{2}\quad$ or $\quad 1.$

Exercise 9

Find the value of :—

1. $\sin^{-1}\dfrac{1}{\sqrt{2}}$ 2. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

3. $\tan^{-1}\dfrac{1}{\sqrt{3}}$ 4. $\cos^{-1} 0$ 5. $\sin^{-1}(\frac{4}{5})+\operatorname{cosec}^{-1}(\frac{5}{3})$

6. $\sin(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2})$ [C. U. '17]

7. Express the following in terms of other inverse functions :—

(i) $\sin^{-1} x$ (ii) $\tan^{-1} x$.

Show that :—

8. $\tan^{-1}\frac{8}{15}=\operatorname{cosec}^{-1}\frac{17}{8}$.

9. $\sin^{-1}\frac{77}{85}=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$.

10. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$. [B. H. U. '48]

11. $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\frac{\pi}{4}$. [C. U. '39]

12. $\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{1}{2}$.

13. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$. [C. U. '41]

14. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

15. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$.

16. $4(\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5})=\pi$.

17. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$. [U. P. B.]

18. $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$. [C. U. '43]

19. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}$. [H. S. '65 ; C. U. '51]

20. $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$. [C. U. '37]

21. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45°$.

[B. H. U. '52]

22. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

[C.U.'52, '55 ; N. B. '64]

23. Given $\sec^{-1}x=\operatorname{cosec}^{-1}y$, prove by general reasoning that $\frac{1}{x^2}+\frac{1}{y^2}=1$. [C. U. '50]

Prove that :—

24. $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}c=\tan^{-1}a$.

[C. U. '55]

25. $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}$

$$=\tan^{-1}\frac{a^2-b^2}{1+a^2b^2}+\tan^{-1}\frac{b^2-c^2}{1+b^2c^2}+\tan^{-1}\frac{c^2-a^2}{1+c^2a^2}.$$

[P. U. '31]

26. $\tan^{-1}(\cot x)+\cot^{-1}(\tan x)=\pi-2x.$

27. $\sin(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2})=1.$

28. $\sin\cot^{-1}\tan\cos^{-1}p=p.$

29. $\tan^{-1}(\frac{1}{2}\tan 2A)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0.$ [C. U. '48]

30. $\tan^{-1}r+\cot^{-1}(r+1)=\tan^{-1}(r^2+r+1).$

31. $\cot^{-1}\dfrac{xy+1}{x-y}+\cot^{-1}\dfrac{yz+1}{y-z}+\cot^{-1}\dfrac{zx+1}{z-x}=0.$

32. $\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)$
$=\cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z).$ [C. U. B. Sc. '32]

Solve for x :—

33. $\tan^{-1}x=\cot^{-1}x.$ 34. $\sin^{-1}x+\sin^{-1}2x=\dfrac{\pi}{3}.$

35. $\tan^{-1}2x+\tan^{-1}3x=n\pi+\dfrac{3\pi}{4}.$

36. $\cot^{-1}x+\sin^{-1}\dfrac{1}{\sqrt{5}}=\dfrac{\pi}{4}.$

37. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x.$

38. $\sin^{-1}\dfrac{5}{x}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}.$ [C. U. '14]

39. $\tan^{-1}(x+1)-\cot^{-1}\dfrac{1}{x-1}=\tan^{-1}2.$

40. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x.$ [C. U. B. Sc.]

41. $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\dfrac{1}{x}=\frac{1}{4}\pi.$ [Agra.'42]

42. $2\cot^{-1}2+\cos^{-1}\frac{3}{5}=\operatorname{cosec}^{-1}x.$

43. $\tan^{-1}\dfrac{x-1}{x-2}+\cot^{-1}\dfrac{x+2}{x+1}=\dfrac{\pi}{4}.$

44. $\sin^{-1}\dfrac{2a}{1+a^2}+\sin^{-1}\dfrac{2b}{1+b^2}=2\tan^{-1}x.$ [C. U. '47]

45. $\tan(\cos^{-1}x)=\sin(\tan^{-1}2).$

46. $3\tan^{-1}\dfrac{1}{2+\sqrt{3}}-\tan^{-1}\dfrac{1}{x}=\tan^{-1}\dfrac{1}{3}.$ [Agra '43]

47. Solve : $\left.\begin{array}{l}\sin^{-1}x+\sin^{-1}y=\frac{2}{3}\pi\\ \cos^{-1}x-\cos^{-1}y=\frac{1}{3}\pi\end{array}\right]$ [H.S. '68 ; C.U. '40]

48. Find if there is any value of x which strictly satisfies the equation $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\tan^{-1}(-7).$

49. Find the value of $\tan(\tan^{-1}x+\cot^{-1}x).$

50. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz.$ [B. U. E '63 ; C. U. '54]

51. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that $x^2+y^2+z^2+2xyz=1.$ [C. U. B. Sc. '41]

52. If $A+B+C=\pi$ and if $A=\tan^{-1}2$, $B=\tan^{-1}3$, show that $C=\dfrac{\pi}{4}.$ [C. U. '51]

53. Show that $\tan^{-1}\dfrac{1}{a-b}+\tan^{-1}\dfrac{b}{a^2+ab+1}=\tan^{-1}\dfrac{1}{a}.$ [C. U. '49]

54. Prove that $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15.$ [C. U. '56]

55. Prove that $\cos\tan^{-1}\sin\cot^{-1}x=\left(\dfrac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}.$ [A.U. '47]

56. If $\sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$, show that $\alpha\sqrt{1-\alpha^2}+\beta\sqrt{1-\beta^2}+\gamma\sqrt{1-\gamma^2}=2\alpha\beta\gamma.$ [B.H.U. '44]

Properties of Triangles

Section (A). [Relations between the sides and the trigonometrical ratios of the angles.]

90. You know from geometry that each triangle has six parts, viz., the three sides and the three angles. There are six kinds of triangles, viz., the acute-angled, the obtuse-angled and the right-angled triangles with respect to the angles, and the scalene, the equilateral and the isosceles triangles with respect to the sides.

The angles of the $\triangle$ABC are usually denoted by A, B and C and the sides opposite to these angles are denoted by a, b and c respectively. The area of a triangle is generally denoted by $\triangle$ or S. Half of its perimeter is denoted by s, so $s=\frac{1}{2}(a+b+c)$.

In sections (A) and (B) we shall determine the formulas relating to the sides, angles, area, circum-radius, in-radius and ex-radius of a triangle.

91. *In any triangle* ABC *prove that* $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

[Sine Rule]

Or, In any triangle the sines of angles are proportional to the opposite sides.

Or, Prove that in $\triangle$ABC, $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

Let the $\triangle$ABC be acute-angled in fig. 6 (i), obtuse-angled in fig. (ii) and right-angled in fig. (iii).

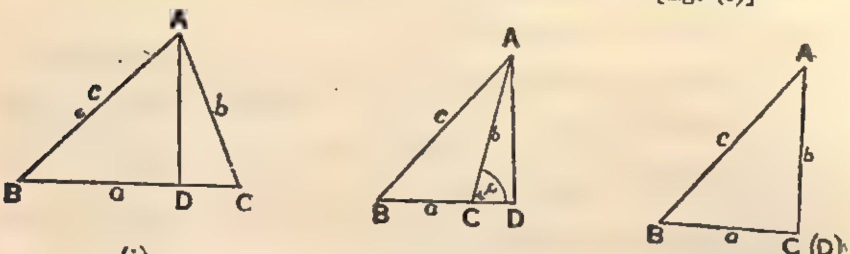
To prove that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

Proof. From A draw AD perpendicular to BC or BC produced.

In fig. (iii) the perpendicular AD coincides with AC.

Now, from $\triangle$ABD, AD = AB sin ABD = c sin B,

and from $\triangle$ACD, AD = AC sin ACD = b sin C [fig. (i)]



(i) (ii) (iii) Fig. 6

or, from $\triangle$ACD, AD = b sin (π − C) [in fig. (ii)] = b sin C.

$$\therefore \quad b \sin C = c \sin B, \quad \therefore \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, drawing a perpendicular from B to AC, it can be proved that $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

As $\angle$C is a right angle in fig. (iii),

$$\sin A = \frac{a}{c}, \ \sin B = \frac{b}{c} \text{ and } \sin C = \sin 90^\circ = 1,$$

$$\therefore \quad \frac{a}{\sin A} = c, \ \frac{b}{\sin B} = c \text{ and } c = \frac{c}{\sin C} \quad [\because \sin C = 1].$$

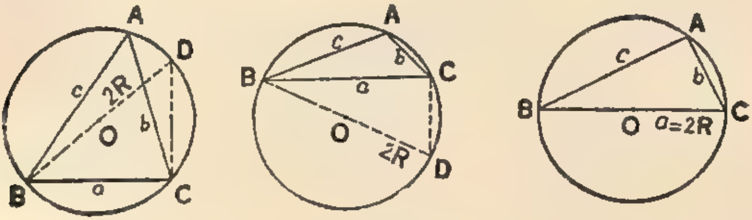
$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence, in any triangle $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

[**Alternative proof**]

Let O be the centre and R the radius of the circum-circle of $\triangle$ABC.

If A be an acute or obtuse angle (fig. IV and fig. V), join BO and produce it to cut the circumference at D. Join DC.



(iv) (v) (vi) Fig. 7

Now, BO=DO=R, $\therefore$ BD=2R.

If $\angle$A is acute (fig. IV),

the $\angle$BCD=1 right angle, being an angle in the semi-circle.

$\therefore$ from ΔBCD we have $\sin BDC=\frac{BC}{BD}=\frac{a}{2R}$.

Again, $\angle$BDC= $\angle$A (being in the same segment)

$$\therefore \quad \frac{a}{2R}=\sin A, \quad \text{or,} \quad \frac{a}{\sin A}=2R.$$

If $\angle$A is obtuse (fig. V),

The $\angle BDC=180^\circ-\angle A$, ABDC being a cyclic quadrilateral.

$\therefore$ $\sin A=\sin(180^\circ-A)=\sin BDC=\frac{a}{2R}$, $\therefore$ $\frac{a}{\sin A}=2R$.

Again, if $\angle$A is a right angle (fig. VI),

$$\sin A=\sin 90^\circ=1=\frac{a}{a}=\frac{a}{2R}, \quad \therefore \frac{a}{\sin A}=2R.$$

Hence, in any triangle, $\frac{a}{\sin A}=2R$.

Similarly, join AO and produce it to cut the circumference at E. Now by joining BE and CE, it can be proved that

$$\frac{b}{\sin B}=2R \text{ and } \frac{c}{\sin C}=2R.$$

Hence, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ [$\because$ each $=2R$]

Corollary. $\because$ $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$,

(*i*) $\therefore$ $a=2R \sin A$, $b=2R \sin B$ and $c=2R \sin C$.

(*ii*) $\sin A=\dfrac{a}{2R}$, $\sin B=\dfrac{b}{2R}$, $\sin C=\dfrac{c}{2R}$.

(*iii*) $\therefore$ $\dfrac{a}{b}=\dfrac{\sin A}{\sin B}$, etc.

[*N. B.* (*i*) The above relation is known as *Sine Rule.*

(*ii*) The point of intersection of the perpendicular bisectors of any two sides of a triangle is equidistant from its vertices. The circle, drawn with this point as centre and with radius equal to its distance from any vertex, is the circum-circle of the triangle. In a right-angled triangle, the middle point of the hypotenuse is the centre of the circum-circle and the hypotenuse is a diameter.]

92. Cosines of angles of a triangle in terms of sides. (Cosine Rule.)

Or, In any triangle to prove that

$$a^2=b^2+c^2-2bc \cos A, \text{ or, } \cos A=\frac{b^2+c^2-a^2}{2bc},$$

$$b^2=c^2+a^2-2ca \cos B, \text{ or, } \cos B=\frac{c^2+a^2-b^2}{2ca},$$

$$c^2=a^2+b^2-2ab \cos C, \text{ or, } \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Let ABC be any triangle. The angle C is acute in Fig. (i), obtuse in Fig (ii) and a right angle in Fig (iii).

From A, AD is drawn perpendicular to BC (in fig. i) or BC produced (in fig. ii). Now, $BC = a$, $AC = b$ and $AB = c$.

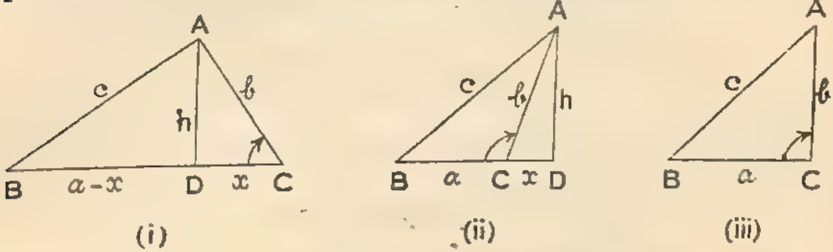


Fig. 8

Supoose $AD = h$ and $CD = x$.

In both the figs. (i) and (ii), $h^2 = b^2 - x^2 \cdots (1)$

Case I. When $\angle C$ is acute (fig. i), $BD = BC - DC = a - x$ and from $\triangle ACD$ we have $\cos C = \dfrac{CD}{AC} = \dfrac{x}{b}$, $\therefore \quad x = b \cos C \cdots (2)$.

Now, $c^2 = AB^2 = AD^2 + BD^2$ [$\because$ $\angle ADB$ is a right angle]
$= h^2 + (a - x)^2 = b^2 - x^2 + a^2 + x^2 - 2ax$ [from (1)]
$= a^2 + b^2 - 2ax$
$= a^2 + b^2 - 2ab \cos C$ [from (2)].

Case II. When $\angle C$ is obtuse (Fig. ii), $BD = BC + CD = a + x$, and $\angle ACD = 180° - C$. Then from $\triangle ACD$ we have

$\cos \angle ACD = \cos(180° - C) = \dfrac{x}{b}$. or, $\quad -\cos C = \dfrac{x}{b}$,

$$\therefore \quad x = -b \cos C \cdots\cdots (3).$$

Now, $c^2 = AB^2 = BD^2 + AD^2$ [$\because$ $\angle ADB = 90°$] $= (a + x)^2 + h^2$
$= a^2 + x^2 + 2ax + b^2 - x^2$ [from (1)] $= a^2 + b^2 + 2ax$
$= a^2 + b^2 + 2a(-b \cos C)$ [from (3)] $= a^2 + b^2 - 2ab \cos C$.

Case III. When C is a right angle (fig. iii),

$\because \quad C = 90°, \quad \therefore \quad \cos C = 0.$

Now, from the right-angled $\triangle ABC$, $AB^2 = BC^2 + AC^2$.

$\therefore \quad c^2 = a^2 + b^2 = a^2 + b^2 - 2ab \times 0$
$\qquad = a^2 + b^2 - 2ab \cos C$ ($\because$ $\cos C = 0$).

Hence in any triangle ABC, $c^2 = a^2 + b^2 - 2ab \cos C$,

i.e., $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

The other two relations can be similarly proved.

93. From the Art. 92 and the corollary of Art. 91, we have

$$\sin A=\frac{a}{2R} \text{ and } \cos A=\frac{b^2+c^2-a^2}{2bc}.$$

$$\therefore \tan A=\frac{\sin A}{\cos A}=\frac{a}{2R}\cdot\frac{2bc}{b^2+c^2-a^2}=\frac{abc}{R}\cdot\frac{1}{b^2+c^2-a^2}.$$

Similarly $\tan B=\frac{abc}{R}\cdot\frac{1}{c^2+a^2-b^2}$ and

$$\tan C=\frac{abc}{R}\cdot\frac{1}{a^2+b^2-c^2}.$$

94. *In any* $\triangle ABC$, *to prove that,*

$$a=b\cos C+c\cos B$$
$$b=c\cos A+a\cos C$$
$$c=a\cos B+b\cos A.$$

If in fig. (i) under Art. 91, the angle C *of* $\triangle ABC$ *be acute,* then $BC=BD+CD=AB\cos ABD+AC\cos ACD$

$$\therefore a=c\cos B+b\cos C.$$

If $\angle C$ *be obtuse* [in fig. (ii)],

then $BC=BD-CD=AB\cos ABD-AC\cos ACD$

$$=c\cos B-b\cos(180^\circ-C)$$
$$=c\cos B+b\cos C \quad [\because \cos(180^\circ-C)=-\cos C]$$

Again, if $\angle C$ *be a right angle* [in fig. (iii)],

then $BC=AB\cos B$,

$$\therefore a=c\cos B+b\cos 90^\circ \quad [\because \cos 90^\circ=0]$$
$$=c\cos B+b\cos C \quad [\because C=90^\circ]$$

Hence, in all cases $a=b\cos C+c\cos B$.

The other two relations can be similarly proved.

[*N. B.* Here we find that *any side of a triangle is equal to the sum of the projections of the other two sides upon it.*]

95. *To find the sines of half the angles in terms of the sides.*

It has been proved before that in any triangle,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

Also, $\cos A=1-2\sin^2\frac{A}{2}$.

$$\therefore \quad 2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a+b-c)(a-b+c)}{2bc}\cdots(1)$$

Let s be the semi-perimeter of the triangle,

then, $2s=a+b+c.$

Now, $a+b-c=a+b+c-2c=2s-2c=2(s-c),$

and $a-b+c=a+b+c-2b=2s-2b=2(s-b).$

Hence, $2\sin^2\frac{A}{2}=\frac{2(s-b).2(s-c)}{2bc}=\frac{2(s-b)(s-c)}{bc}.$

$$\therefore \quad \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}.$$

$$\therefore \quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Here only the positive value of the square root of $(s-b)(s-c)$ should be taken, for $A<180^\circ$ being an angle of a triangle, and hence $\frac{A}{2}<90^\circ$ and, therefore, $\sin\frac{A}{2}$ is always positive.

Similarly it can be proved that

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}} \text{ and } \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

96. *To find the cosines of half the angles in terms of the sides.*

It has been already proved that $\cos A=\frac{b^2+c^2-a^2}{2bc},$

and we know $\cos A=2\cos^2\frac{A}{2}-1.$

$$\therefore 2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$=\frac{2s(2s-2a)}{2bc} \quad [\text{taking } 2s=a+b+c]$$

$$=\frac{4s(s-a)}{2bc}=\frac{2s(s-a)}{bc},$$

$\therefore \quad \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}, \quad \therefore \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$

Here only the positive value of the square root must be taken, for A being an angle of a triangle $\frac{A}{2} < 90°$, and consequently $\cos \frac{A}{2}$ is always positive.

Similarly it can be proved that

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} \text{ and } \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

97. *To find the tangents of half the angles in terms of the sides.*

$$\text{We have } \tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}.$$

Hence from Arts. 95 and 96 we have

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly it can be proved that

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \text{ and } \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Here $A < 180°$, being an angle of a triangle, so $\frac{A}{2} < 90°$, $\therefore \tan \frac{A}{2}$ is always positive. Hence only the positive value of the square root must be taken here.

98. *To find the area of a triangle*

[Draw the fig. 6(i) here]

Let ABC be a triangle and let Δ be the symbol for the area of the ΔABC.

Draw AD⊥BC. Then AD is the altitude of the triangle.

Now from ΔACD we have

$AD = AC \sin C = b \sin C.$

Again, the area of a triangle $= \frac{1}{2} \times$ base $\times$ altitude

$\therefore \quad \Delta = \frac{1}{2} BC.AD = \frac{1}{2} ab \sin C.$

Similarly, by drawing perpendiculars from B and C to the opposite sides, it can be proved that $\Delta=\frac{1}{2}\,bc\sin A=\frac{1}{2}\,ca\sin B$.

$$\therefore\quad \Delta=\tfrac{1}{2}\,\text{ab}\sin C$$
$$=\tfrac{1}{2}\,\text{ac}\sin B$$
$$=\tfrac{1}{2}\,\text{bc}\sin A$$

$\therefore\quad \Delta=\frac{1}{2}$ (*product of two sides*) $\times$ *sine of the included angle*......(1)

[**Otherwise**]

$$\Delta=\tfrac{1}{2}\,bc\sin A=\frac{1}{2}\,bc\times 2\sin\frac{A}{2}\cos\frac{A}{2}=bc\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=bc\sqrt{\frac{(s-b)(s-c)}{bc}}\times\sqrt{\frac{s(s-a)}{bc}}=bc\times\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

$$=\sqrt{s(s-a)(s-b)(s-c)}\ \ldots\ldots\text{(ii)}$$

Again, $\because\quad s=\frac{1}{2}(a+b+c)$,

$\therefore\quad a+b+c=2s,\ a+b-c=2(s-c),\ b+c-a=2(s-a),$
$c+a-b=2(s-b).$

$$\therefore\quad \Delta=\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\sqrt{\tfrac{1}{2}(a+b+c).\tfrac{1}{2}(b+c-a).\tfrac{1}{2}(c+a-b).\tfrac{1}{2}(a+b-c)}$$
$$=\tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$
$$=\tfrac{1}{4}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}\ \cdots\cdots\text{(iii)}$$

Again, $\because\ \Delta=\frac{1}{2}bc\sin A=\frac{1}{2}bc.\frac{a}{2R}$ [R = circum-radius]

$$\therefore\quad \Delta=\frac{abc}{4R},\ \text{or, } R=\frac{abc}{4\Delta}.$$

Corollary. $\sin A=\dfrac{2\Delta}{bc},\ \sin B=\dfrac{2\Delta}{ca},\ \sin C=\dfrac{2\Delta}{ab}.$

99. *Sine of an angle of a triangle in terms of the sides.*

$$\sin A = 2 \sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, $\sin B = \dfrac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$,

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

Again, $\sin A = \dfrac{2\Delta}{bc} = \dfrac{2 \times \frac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}}{bc}$

$$= \frac{\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}}{2bc},$$

$\sin B$ and $\sin C$ can be similarly found.

[Here $A < 180°$, hence only the positive square root has been taken into account.]

100. From Art. 97 we have

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)} \cdot \frac{(s-b)(s-c)}{(s-b)(s-c)}}$$

$$= \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta}.$$

Similarly, $\tan\dfrac{B}{2} = \dfrac{(s-c)(s-a)}{\Delta}$ and $\tan\dfrac{C}{2} = \dfrac{(s-a)(s-b)}{\Delta}$.

Again, $\cot\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}} = \sqrt{\dfrac{s(s-a)}{(s-b)(s-c)} \cdot \dfrac{s(s-a)}{s(s-a)}}$

$$= \frac{s(s-a)}{\Delta}.$$

Similarly, $\cot\dfrac{B}{2} = \dfrac{s(s-b)}{\Delta}$ and $\cot\dfrac{C}{2} = \dfrac{s(s-c)}{\Delta}$.

101. To prove that in any $\triangle$ABC,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}\qquad[\text{ Tangent Rule }]$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

Here, $A+B+C=180°$, $\therefore\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$.

Since in any $\triangle$ABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},\quad\therefore\ \frac{b}{c}=\frac{\sin B}{\sin C},$$

$$\therefore\quad\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2\cos\frac{B+C}{2}.\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}$$

$$=\cot\frac{B+C}{2}\tan\frac{B-C}{2}=\tan\frac{A}{2}\tan\frac{B-C}{2}$$

$$\left[\because\ \frac{A}{2}+\frac{B+C}{2}=90°,\ \therefore\ \cot\frac{B+C}{2}=\cot\left(90°-\frac{A}{2}\right)=\tan\frac{A}{2}\right]$$

$$\therefore\quad\tan\frac{B-C}{2}=\frac{b-c}{b+c}.\frac{1}{\tan\frac{A}{2}}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

Similarly the other two relations can be proved.

Corollary : $\dfrac{a-b}{a+b}=\dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)},\quad\dfrac{b-c}{b+c}=\dfrac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$

and $\dfrac{c-a}{c+a}=\dfrac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}$.

Examples (10)

Ex. 1. If in a triangle ABC, $a=7$, $b=3$, $c=5$, find A.

$\because \cos A=\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{3^2+5^2-7^2}{2.3.5}=\dfrac{-15}{30}=-\dfrac{1}{2}=\cos 120°$,

$\therefore$ A$=120°$.

Ex. 2. If B$=60°$, $c=2\sqrt{3}$, $b=3\sqrt{2}$, find A.

We have $\sin C=\dfrac{c \sin B}{b}=\dfrac{2\sqrt{3}}{3\sqrt{2}}.\sin 60°$

$$=\frac{2\sqrt{3}}{3\sqrt{2}}\times\frac{\sqrt{3}}{2}=\frac{1}{\sqrt{2}}=\sin 45° \text{ or } \sin 135°.$$

$\therefore$ C$=45°$, or, $135°$, but here C cannot be equal to $135°$ for in that case B$+$C will be greater than $180°$.

$\therefore$ C$=45°$. $\therefore$ A$=180°-(45°+60°)=75°$.

Ex. 3. If $a=25$, $b=52$ and $c=63$, find $\tan \dfrac{B}{2}$.

Here $s=\dfrac{25+52+63}{2}=70$.

$\therefore s-a=70-25=45$, $s-b=70-52=18$, $s-c=70-63=7$.

$\therefore \tan\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}=\sqrt{\dfrac{7\times 45}{70\times 18}}=\sqrt{\dfrac{1}{4}}=\dfrac{1}{2}$.

[Here the negative square root is inadmissible.]

Ex. 4. If $a^4+b^4+c^4=2c^2(a^2+b^2)$, find C.

From the given condition $a^4+b^4+c^4-2a^2c^2-2b^2c^2=0$

or, $a^4+b^4+c^4-2a^2c^2-2b^2c^2+2a^2b^2=2a^2b^2$,

or, $(a^2+b^2-c^2)^2=2a^2b^2$,

or, $\left(\dfrac{a^2+b^2-c^2}{2ab}\right)^2=\dfrac{1}{2}$ [dividing both sides by $4a^2b^2$].

$\therefore \cos^2 C=\dfrac{1}{2}$ $\left[\because \cos C=\dfrac{a^2+b^2-c^2}{2ab}\right]$

$\therefore \cos C=\pm\dfrac{1}{\sqrt{2}}$, $\therefore$ C$=45°$ or $135°$.

Ex. 5. If $c=2a$ and $C=3A$, find the angles of the triangle ABC.

$$\because \quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

$$\therefore \quad \frac{\sin C}{\sin A}=\frac{c}{a}, \quad \text{or,} \quad \frac{\sin 3A}{\sin A}=\frac{2a}{a}=2,$$

or, $\dfrac{3 \sin A-4 \sin^3 A}{\sin A}=2$, or, $3 \sin A-4 \sin^3 A=2 \sin A$,

or, $4 \sin^3 A=\sin A$, or, $\sin^2 A=\frac{1}{4}$,

$\therefore$ $\sin A=\frac{1}{2}=\sin 30°$, $\therefore$ $A=30°$. $\therefore$ $C=3A=90°$,

and $B=180°-(A+C)=60°$. $\therefore$ $A=30°, B=60°, C=90°$.

Ex. 6. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Determine the value of the greatest angle. [C. U. '10]

Let θ be the greatest angle.

As the angle opposite to the greatest side is the greatest, we have to ascertain which of the given sides is the greatest.

$\because$ the side of a triangle cannot be negative,

$\therefore$ x^2-1 is positive, $\therefore$ x must be greater than 1.

If $x>1$, then x^2+x+1 will be the greatest side.

$$\text{Now, } \cos\theta=\frac{(x^2-1)^2+(2x+1)^2-(x^2+x+1)^2}{2(x^2-1)(2x+1)}$$

$$=\frac{-2x^3-x^2+2x+1}{2(2x^3+x^2-2x-1)}$$

$$=\frac{-(2x^3+x^2-2x-1)}{2(2x^3+x^2-2x-1)}=-\frac{1}{2},$$

$\therefore$ $\cos\theta=-\frac{1}{2}=\cos 120°$, $\therefore$ $\theta=120°$.

Ex. 7. In $\triangle ABC$, show that $a(b \cos C-c \cos B)=b^2-c^2$.

The L. H. S. $=ab \cos C-ac \cos B$

$$=ab\times\frac{a^2+b^2-c^2}{2ab}-ac\times\frac{c^2+a^2-b^2}{2ca}$$

$$=\frac{a^2+b^2-c^2}{2}-\frac{c^2+a^2-b^2}{2}$$
$$=\tfrac{1}{2}(a^2+b^2-c^2-c^2-a^2+b^2)$$
$$=\tfrac{1}{2}(2b^2-2c^2)=b^2-c^2.$$

Ex. 8. In any triangle ABC, prove that

$$a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)=0.$$

The L. H. S. $=(a\sin B-b\sin A)+(b\sin C-c\sin B)$
$+(c\sin A-a\sin C)$.

Now, $\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ (in any triangle)

$\therefore\ a\sin B=b\sin A,\quad \therefore\ a\sin B-b\sin A=0.$

Similarly, $b\sin C-c\sin B=0$ and $c\sin A-a\sin C=0.$

$\therefore$ the L. H. S. $=0+0+0=0.$

Ex. 9. In $\triangle ABC$, show that

$$\frac{b-c}{a}\cos^2\frac{A}{2}+\frac{c-a}{b}\cos^2\frac{B}{2}+\frac{a-b}{c}\cos^2\frac{C}{2}=0.$$

$\because\ \cos^2\dfrac{A}{2}=\dfrac{s(s-a)}{bc},\ \cos^2\dfrac{B}{2}=\dfrac{s(s-b)}{ca},\ \cos^2\dfrac{C}{2}=\dfrac{s(s-c)}{ab}$

$\therefore$ the L. H. S. $=\dfrac{b-c}{a}.\dfrac{s(s-a)}{bc}+\dfrac{c-a}{b}.\dfrac{s(s-b)}{ca}+\dfrac{a-b}{c}.\dfrac{s(s-c)}{ab}$

$$=\frac{s}{abc}(s-a)(b-c)+\frac{s}{abc}(s-b)(c-a)+\frac{s}{abc}(s-c)(a-b)$$
$$=\frac{s}{abc}[s(b-c+c-a+a-b)-\{a(b-c)+b(c-a)+c(a-b)\}]$$
$$=\frac{s}{abc}\times 0=0.$$

Ex. 10. Prove that in $\triangle ABC$,

$$\sin 2A+\sin 2B+\sin 2C=\frac{32\Delta^3}{a^2b^2c^2}.$$

$\because\ A+B+C=180°,$

$\therefore\ \sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

[Vide trigonometrical identities in part II]

Again, $\sin A=\frac{2\Delta}{bc}$, $\sin B=\frac{2\Delta}{ca}$ and $\sin C=\frac{2\Delta}{ab}$.

$\therefore$ the L. H. S. $=4\sin A\sin B\sin C$

$$=4\times\frac{2\Delta}{bc}\times\frac{2\Delta}{ca}\times\frac{2\Delta}{ab}=\frac{32\Delta^3}{a^2b^2c^2}.$$

Ex. 11. In any triangle ABC prove that

$$(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}.$$

$\because$ $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$,

$\therefore$ $b-c=2R(\sin B-\sin C)$,

$$\therefore\quad \frac{b-c}{a}=\frac{2R(\sin B-\sin C)}{2R\sin A}=\frac{\sin B-\sin C}{\sin A}$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{2\sin\frac{A}{2}\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\left[\because\ \cos\left(\frac{B+C}{2}\right)=\cos\left(90^\circ-\frac{A}{2}\right)=\sin\frac{A}{2}\right]$$

$$=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$\therefore$ by cross-multiplication, $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$.

Ex. 12. If the cosines of the angles are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled. [C. U. '23]

From the given condition we have $\frac{\cos A}{\cos B}=\frac{b}{a}$; also $\frac{b}{a}=\frac{\sin B}{\sin A}$.

$$\therefore\quad \frac{\cos A}{\cos B}=\frac{\sin B}{\sin A},$$

or, $\sin A \cos A = \sin B \cos B$,

or, $2 \sin A \cos A = 2 \sin B \cos B$, or, $\sin 2A = \sin 2B$

$\therefore$ $2A = 2B$ or $(180° - 2B)$

If $2A = 2B$, then $A = B$,

$\therefore$ The triangle is isosceles.

Again, if $2A = 180° - 2B$, then $A + B = 90°$,

$\therefore$ The remaining angle $C = 90°$.

$\therefore$ The triangle is right-angled.

Hence, the triangle is either isosceles or right-angled.

Ex. 13. In any triangle prove that cot A, cot B, cot C are in A. P., if a^2, b^2, c^2 are in A. P.

[A. U. '43]

$\because$ $\sin A = \dfrac{a}{2R}$ and $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$,

$\therefore$ $\cot A = \dfrac{\cos A}{\sin A} = \dfrac{R(b^2+c^2-a^2)}{abc}$.

Similarly, $\cot B = \dfrac{R(c^2+a^2-b^2)}{abc}$ and $\cot C = \dfrac{R(a^2+b^2-c^2)}{abc}$.

Now, cot A, cot B, cot C will be in A. P.,

if $\dfrac{R(b^2+c^2-a^2)}{abc}$, $\dfrac{R(c^2+a^2-b^2)}{abc}$, $\dfrac{R(a^2+b^2-c^2)}{abc}$ are in A. P.,

or, if $b^2+c^2-a^2$, $c^2+a^2-b^2$, $a^2+b^2-c^2$ are in A. P.,

or, if $(c^2+a^2-b^2)-(b^2+c^2-a^2)$

$$= (a^2+b^2-c^2)-(c^2+a^2-b^2),$$

or, if $2a^2-2b^2 = 2b^2-2c^2$, or, if $a^2-b^2 = b^2-c^2$,

or, if a^2, b^2, c^2 are in A. P.

Exercise 10

1. If $a=5$, $b=5\sqrt{3}$, $c=5$, find the angles of the triangle ABC.

2. If $a=3$, $b=3\sqrt{3}$ and $A=30^\circ$, find B. [C. U. '21]

3. If $a=48$, $b=35$, $C=60^\circ$, find c.

4. If $a=13$, $b=14$, $c=15$, find $\tan\frac{B}{2}$.

5. Find $\sin B$, if $a=18$, $b=24$, $c=30$.

6. If $(a+b+c)(b+c-a)=3bc$, find A.

Find the area of the triangle ABC :—

7. If $a=13$, $b=14$, $c=15$. 8. If $a=6$, $b=8$ and $s=12$.

9. If the sides are as 3 : 4 : 5 and $s=216$ ft.

10. If $b=2a$ and $B=3A$, find the angles of the triangle.

11. Find the greatest angle of the triangle whose sides are 5, $5\sqrt{3}$, 5.

In any triangle ABC prove the following :—

12. $a\sin A-b\sin B=c\sin(A-B)$ [C. U. '13]

13. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.

14. $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$.

15. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$. [P.U. '37]

16. $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$.

17. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$.

18. $\dfrac{\sin(B-C)}{\sin(B+C)}=\dfrac{b^2-c^2}{a^2}$.

19. $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$.

20. $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$.

21. $(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$.

22. $a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C$.

[A. U. '44]

23. $b^2\sin 2C+c^2\sin 2B=4\Delta$.

24. $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$.

25. $a^2\cot A+b^2\cot B+c^2\cot C=4\Delta$.

26. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2$.

27. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$.

28. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.

[C. U. '12]

29. $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$.

[B. H. U. '44]

30. If $\cos B=\frac{\sin A}{2\sin C}$, show that the triangle is isosceles.

[B. H. U. '45]

31. If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$, show that $C=60°$. [P. U.]

32. If $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin(A+B)}{\sin(A-B)}$, show that the triangle is either isosceles or right-angled.

33. In a triangle if a, b, c are in A. P., show that the cotangents of the semi-angles are also in A. P. [C. U. '54]

34. If the cosines of two angles of a triangle be proportional to the opposite sides, show that the triangle is isosceles.

[C. U. '24]

35. If the tangents of the semi-angles of a triangle are in A.P., show that the cosines of the angles are in A. P. [C. U. '54]

36. If in a triangle the angles be to one another as $1:2:3$, prove that the corresponding sides are as $1:\sqrt{3}:2$.

37. Prove geometrically that in any triangle,

$a=b \cos C+c \cos B$, discussing separately the two cases (i) when B and C are both acute and (ii) when B is acute and C is obtuse.

38. If $a \cos^2 \frac{C}{2}+c \cos^2 \frac{A}{2}=\frac{3b}{2}$, show that the sides of the triangle are in A. P. [P. U. '41]

Section (B) [Properties of triangles]

102. **Circum-circle.** The circle drawn through the vertices of a triangle is called its circum-circle or circumscribed circle. The point of intersection of the perpendicular bisectors of any two sides of the triangle is the centre and the distance of this point from any vertex is the radius of this circum-circle. This centre is said to be the **circum-centre** and is denoted by S. This radius is known as the circum-radius and is denoted by R. A triangle has only one circum-circle.

102. (a) To find the circum-radius of a triangle.

(i) Let DS and ES be the perpendicular bisectors of the sides BC and AB of the $\triangle ABC$ and let them meet at S. Then S is the circum-centre and AS or BS or CS is the circum-radius R of the triangle. Join AS, BS, CS.

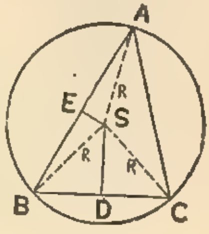


Fig. 9

$\because$ $\triangle BSD$ and $\triangle CSD$ are congruent,

$\therefore$ $\angle BSD = \angle CSC = A$ ($\because$ The angle at the circumference is half the angle at the centre, standing on the same arc.)

Now, in $\triangle BSD$, $BD=\frac{1}{2}BC=\frac{1}{2}a$, $\angle BSD=A$ and $\angle BDS=1$ right angle.

$$\therefore \quad \sin A=\sin BSD=\frac{BD}{BS}=\frac{\frac{1}{2}a}{R}=\frac{a}{2R}, \quad \therefore \quad R=\frac{a}{2\sin A}.$$

Similarly, $R=\dfrac{b}{2\sin B}=\dfrac{c}{2\sin C}$.

Hence, $$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}\ldots\ldots(i)$$

(ii) Again, since the area of the triangle $\Delta=\frac{1}{2}bc\sin A$

$$=\frac{1}{2}bc.\frac{a}{2R}=\frac{abc}{4R},$$

$$\therefore \quad R=\frac{abc}{4\Delta}\ldots\ldots(2).$$

103. In-circle : The circle drawn in a triangle touching its sides is called the inscribed circle or in-circle of the triangle.

The point of intersection of the bisectors of any two angles of the triangle is the centre of this circle. It is called the in-centre of the triangle and is denoted by I. The distance of this centre from any side of the triangle is the radius of the in-circle. It is called the in-radius and is denoted by r.

103. (a) To find the in-radius of a triangle.

(i) Let BI and CI bisect the angles B and C of the $\triangle ABC$ and meet at I and let ID, IE and IF be the perpendiculars to the sides of the triangle. They are equal. The circle, drawn with centre I and radius ID will touch the sides of $\triangle ABC$ at D, E and F. So the circle DEF is the in-circle of $\triangle ABC$ and ID (or, r) is the in-radius.

Then $ID=IE=IF=r$. Join IA.

Now, $\triangle ABC=\triangle IBC+\triangle IAB+\triangle IAC$

$$=\tfrac{1}{2}BC.ID+\tfrac{1}{2}AB.IF+\tfrac{1}{2}AC.IE$$

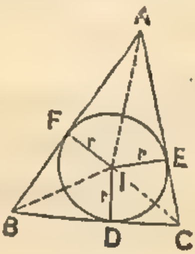


Fig. 10

$=\frac{1}{2}ar+\frac{1}{2}cr+\frac{1}{2}br=\frac{1}{2}r(a+b+c)$

$=rs$ [s=semi-perimeter of the triangle]

$$\therefore\ r=\frac{\text{Area of the triangle}}{s}=\frac{\Delta}{s}\quad \cdots\cdots(1)$$

(ii) Again, from $\triangle IBD$ we have $BD=r\cot\frac{B}{2}$

and from $\triangle ICD$, we have $CD=r\cot\frac{C}{2}$,

$$\therefore\quad a=BC=BD+CD=r\cot\frac{B}{2}+r\cot\frac{C}{2}$$

$$=r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=r\left(\frac{\cos\frac{1}{2}B}{\sin\frac{1}{2}B}+\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}\right)$$

$$=r\left(\frac{\cos\frac{1}{2}B\sin\frac{1}{2}C+\sin\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}B\sin\frac{1}{2}C}\right)$$

$$=r\times\frac{\sin\left(\frac{1}{2}B+\frac{1}{2}C\right)}{\sin\frac{1}{2}B\sin\frac{1}{2}C}=r\times\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}$$

[$\because\ \sin\left(\frac{1}{2}B+\frac{1}{2}C\right)=\sin\left(90^\circ-\frac{1}{2}A\right)=\cos\frac{1}{2}A$]

$$\therefore\quad r=\frac{a\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}.$$

Now, $\because\ a=2R\sin A=2R.\,2\sin\frac{1}{2}A\cos\frac{1}{2}A$

$=4R\sin\frac{1}{2}A\cos\frac{1}{2}A$,

$\therefore\ r=4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ [putting the value of a]$\cdots(2)$

(iii) Again, $\because$ $AF=AE$

$BD=BF$

$CD=CE$

$\therefore\ AF+BD+CD=$semi-perimeter$=s$.

$\therefore\ AF+BC=s$, or, $AF+a=s$, $\therefore\ AF=s-a$.

Similarly, $BF=s-b$ and $CE=s-c$.

Now, from ΔAIF we have, $IF=AF\tan IAF=AF\tan\frac{1}{2}A$,

$$\left.\begin{array}{ll}\text{or,} & r=(s-a)\tan\tfrac{1}{2}A \\ \text{Similarly,} & r=(s-b)\tan\tfrac{1}{2}B \\ \text{and} & r=(s-c)\tan\tfrac{1}{2}C\end{array}\right\}\cdots\cdots(3)$$

104. *The distances of the in-centre from the vertices of a triangle.*

In the above figure, IA, IB and IC are these distances.

From $\triangle AIF$, we have $IA = IF \operatorname{cosec} IAF$,

$\therefore \quad IA = r \operatorname{cosec} \tfrac{1}{2}A.$

Similarly, $IB = r \operatorname{cosec} \tfrac{1}{2}B$ and $IC = r \operatorname{cosec} \tfrac{1}{2}C$.

105. Ex-circle : The circle that touches one side of a triangle and the other two sides produced is called the escribed circle or the ex-circle of the triangle. Each triangle may have three ex-circles. The circle that touches the side BC of $\triangle ABC$ and the sides AB, AC produced is the escribed circle opposite to the angle A. Similarly there may be other two escribed circles opposite to the angles B and C.

The point of intersection of the internal bisector of any angle and the external bisector of any other angle (or, the bisectors of any two exterior angles) of the triangle is the centre of the escribed circle and is called the **ex-centre**. The perpendiculars drawn from this point to the sides of the triangle are equal and each is the **ex-radius** of the circle. The centres of the escribed circles opposite to A, B and C are respectively denoted by I_1, I_2, I_3 and the corresponding radii are denoted by r_1, r_2 and r_3 respectively.

105. (A) To find the ex-radii of a triangle.

Let the internal bisector of the angle A and the external bisector of $\angle B$ of the $\triangle ABC$ meet at I_1. Then I_1 is the centre of the ex-circle opposite to $\angle A$. From I_1 draw I_1D perpendicular to BC and I_1F, I_1E perpendiculars to AB and AC produced.

We know from geometry that I_1D, I_1F and I_1E are ex-radii and $I_1D=I_1E=I_1F=r_1$. The circle drawn with centre I_1 and radius I_1D (*i.e.*, r_1) will touch BC at D and AB, AC produced at F, E respectively.

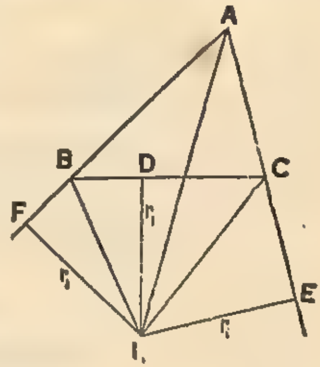


Fig. 11

Similarly, suppose r_2 and r_3 are respectively the radii of the escribed circles opposite to angles B and C.

(*i*) Join CI_1.

Now, $\triangle ABC=\triangle ABI_1+\triangle ACI_1-\triangle BCI_1$

$$=\tfrac{1}{2}AB.FI_1+\tfrac{1}{2}AC.EI_1-\tfrac{1}{2}BC.DI_1$$
$$=\tfrac{1}{2}cr_1+\tfrac{1}{2}br_1-\tfrac{1}{2}ar_1$$
$$=\tfrac{1}{2}r_1(b+c-a)=\tfrac{1}{2}r_1(b+c+a-2a)$$
$$=\tfrac{1}{2}r_1(2s-2a)=r_1(s-a),$$

$\therefore$ $\triangle$ (area of the triangle)$=r_1(s-a)$,

$$\left.\begin{aligned}\therefore\quad r_1&=\frac{\triangle}{s-a}\\ \text{Similarly, } r_2&=\frac{\triangle}{s-b}\\ \text{and } r_3&=\frac{\triangle}{s-c}\end{aligned}\right\}\cdots\cdots(1)$$

(ii) Again, $\angle DBI_1=\tfrac{1}{2}\angle FBD=\tfrac{1}{2}(180^\circ-\angle B)=90^\circ-\tfrac{1}{2}B.$

Now from $\triangle BDI_1$ we have

$$BD=I_1D.\frac{BD}{I_1D}=r_1\cot DBI_1=r_1\cot(90^\circ-\tfrac{1}{2}B).$$

Similarly from $\triangle CDI_1$ we have $CD=r_1\cot CDI_1$

$$=r_1\cot(90^\circ-\tfrac{1}{2}C)$$

$\therefore \quad a = BC = BD + CD = r_1 \cot (90° - \frac{1}{2}B) + r_1 \cot (90° - \frac{1}{2}C)$

$= r_1 \tan \frac{1}{2}B + r_1 \tan \frac{1}{2}C = r_1 (\tan \frac{1}{2}B + \tan \frac{1}{2}C)$

$$= r_1 \left(\frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B} + \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}C} \right)$$

$$= r_1 \left(\frac{\sin \frac{1}{2}B \cos \frac{1}{2}C + \sin \frac{1}{2}C \cos \frac{1}{2}B}{\cos \frac{1}{2}B \cos \frac{1}{2}C} \right)$$

$$= r_1 \frac{\sin (\frac{1}{2}B + \frac{1}{2}C)}{\cos \frac{1}{2}B \cos \frac{1}{2}C} = r_1 \frac{\cos \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C}$$

[$\because \quad \sin (\frac{1}{2}B + \frac{1}{2}C) = \sin (90° - \frac{1}{2}A) = \cos \frac{1}{2}A$]

$$\therefore \quad r_1 = \frac{a \cdot \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A},$$

But $a = 2R \cdot \sin A = 2R \cdot 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$

$= 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$

$\therefore \quad r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$ [putting the value of a]

$$\left.\begin{aligned} \text{Similarly, } r_2 &= 4R \sin \tfrac{1}{2}B \cos \tfrac{1}{2}A \cos \tfrac{1}{2}C \\ \text{and } \quad r_3 &= 4R \sin \tfrac{1}{2}C \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \end{aligned}\right\} \cdots\cdots(2)$$

(iii) Again, $AE = AC + CE = AC + CD = b + CD$

and $\quad AF = AB + FD = AB + BD = c + BD$;

But $AE = AF \quad (\because \quad \triangle AFI_1, \triangle AEI_1$ are congruent)

$\therefore \quad 2AE = AE + AF = b + CD + c + BD = b + c + (BD + CD)$

$= b + c + a = 2s \quad [\because \quad 2s =$ perimeter]

$\therefore \quad AE = s.$

Hence, from $\triangle AEI_1$ we have $EI_1 = AE \tan AEI_1$,

$$\left.\begin{aligned} \therefore \quad r_1 &= s \tan \tfrac{1}{2}A \\ \text{Similarly, } \quad r_2 &= s \tan \tfrac{1}{2}B \\ \text{and } r_3 &= s \tan \tfrac{1}{2}C \end{aligned}\right\} \cdots\cdots(3)$$

106. *To find the distances of the ex-centre from the angular points of the triangle.*

From $\triangle AFI_1$, we have $I_1A = I_1F \operatorname{cosec} I_1AF$

$\therefore \quad AI_1 = r_1 \operatorname{cosec} \frac{1}{2}A.$

From $\triangle I_1BD$ we have $I_1B = r_1 \operatorname{cosec} I_1BD = r_1 \operatorname{cosec} (90^\circ - \frac{1}{2}B)$,

$\therefore$ $BI_1 = r_1 \sec \frac{1}{2}B$, similarly $CI_1 = r_1 \sec \frac{1}{2}C$.

Similarly it can be shown that

$I_2B = r_2 \operatorname{cosec} \frac{1}{2}B$, $I_2A = r_2 \sec \frac{1}{2}A$, $I_2C = r_2 \sec \frac{1}{2}C$

and $I_3C = r_3 \operatorname{cosec} \frac{1}{2}C$, $I_3A = r_3 \sec \frac{1}{2}A$, $I_3B = r_3 \sec \frac{1}{2}B$.

Again, $AI_1 = r_1 \operatorname{cosec} \frac{1}{2}A$,

But $r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$

$$\therefore \left.\begin{aligned} AI_1 &= 4R \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C, \\ BI_1 &= 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}C, \\ CI_1 &= 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B. \end{aligned}\right\}$$

Similarly I_2A, I_2B, etc. can be found.

Examples (11)

Ex. 1. The sides of a triangle are 13, 14 and 15 ft., find R.

$$R = \frac{abc}{4\Delta}.$$

Here, $s = \frac{1}{2}(13+14+15)$ ft. $= 21$ ft.

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{21(21-13)(21-14)(21-15)} \text{ sq. ft.}$$
$$= \sqrt{21 \times 8 \times 7 \times 6} \text{ sq. ft.} = 84 \text{ sq. ft.}$$

$$\therefore \quad R = \frac{13 \times 14 \times 15 \text{ cu. ft.}}{4 \times 84 \text{ sq. ft.}} = \frac{65}{8} \text{ ft.} = 8\tfrac{1}{8} \text{ ft.}$$

Ex. 2. Show that $2R^2 \sin A \sin B \sin C = \Delta$.

$$\because \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

$\therefore$ $2R \sin A = a$, $2R \sin B = b$

$$\therefore \quad 2R^2 \sin A \sin B \sin C = \tfrac{1}{2} \times 2R \sin A \times 2R \sin B \sin C$$

$$= \tfrac{1}{2}a.b. \sin C = \tfrac{1}{2}.\ a.\ b. \frac{c}{2R} \left[\because \quad \sin C = \frac{c}{2R} \right]$$

$$= \frac{abc}{4R} = \Delta.$$

Ex. 3. Show that in $\triangle ABC$, $4Rrs = abc$.

$\because \quad R = \dfrac{abc}{4\Delta}$ and $r = \dfrac{\Delta}{s}$,

$\therefore \quad 4Rrs = 4 \times \dfrac{abc}{4\Delta} \times \dfrac{\Delta}{s} \times s = abc.$

Ex. 4. In a triangle prove that $r\, r_1\, r_2\, r_3 = \Delta^2$.

$\because$ In any triangle $r = \dfrac{\Delta}{s}$, $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$,

$$\therefore \quad r\, r_1\, r_2\, r_3 = \frac{\Delta}{s} \times \frac{\Delta}{s-a} \times \frac{\Delta}{s-b} \times \frac{\Delta}{s-c}$$

$$= \frac{\Delta^2 \times \Delta^2}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2 \times \Delta^2}{\Delta^2} = \Delta^2.$$

Ex. 5. Prove that $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$. [C. U.; B. H. U.]

$\because \quad r_1 = \dfrac{\Delta}{s-a}, \quad r_2 = \dfrac{\Delta}{s-b}, \quad r_3 = \dfrac{\Delta}{s-c}$,

$$\therefore \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{3s-(a+b+c)}{\Delta}$$

$$= \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r} \quad \left[\because \quad r = \frac{\Delta}{s} \right]$$

Ex. 6. Prove that $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

$$\text{L. H. S.} = \left(\frac{\Delta}{s-a} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{s-b} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right)$$

$$= \Delta^3 \left(\frac{1}{s-a} - \frac{1}{s}\right)\left(\frac{1}{s-b} - \frac{1}{s}\right)\left(\frac{1}{s-c} - \frac{1}{s}\right)$$

$$= \Delta^3 \times \frac{a}{s(s-a)} \times \frac{b}{s(s-b)} \times \frac{c}{s(s-c)}$$

$$= \Delta^3 \times \frac{abc}{s^2 \times \Delta^2} = \Delta \times \frac{abc}{s^2}$$

$$=\Delta\times\frac{4\Delta R}{s^2}\quad\left[\because\ R=\frac{abc}{4\Delta}\right].$$

$$=4R\times\frac{\Delta^2}{s^2}=4Rr^2\quad\left[\because\ r=\frac{\Delta}{s}\right].$$

Ex. 7. Show that $\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\cdot\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$

$\because\quad \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$ [See Ex. 5]

$$\therefore\quad\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\left(\frac{1}{r}+\frac{1}{r}\right)^2=\left(\frac{2}{r}\right)^2=\frac{4}{r^2}$$

$$=\frac{4}{r}\cdot\frac{1}{r}=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right).$$

Ex. 8. If I be the in-centre of the $\triangle ABC$, prove that

$$IA.IB.IC=abc\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

$$\because\quad IA=r\operatorname{cosec}\tfrac{1}{2}A=\frac{r}{\sin\frac{A}{2}},\quad IB=\frac{r}{\sin\frac{B}{2}},\quad IC=\frac{r}{\sin\frac{C}{2}},$$

$$\therefore\quad IA.IB.IC=\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}.$$

Now, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{(s-a)(s-b)(s-c)}{abc}$.

and $r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$.

$$\therefore\quad IA.IB.IC=\frac{(s-a)(s-b)(s-c)\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{\frac{(s-a)(s-b)(s-c)}{abc}}$$

$$=abc\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

Ex. 9. Prove that $\cos A+\cos B+\cos C=1+\frac{r}{R}$. [C. U'58]

$\because \quad A+B+C=180°$,

$\therefore \quad \cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdots(1)$

Again, $\because \quad r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$,

$\therefore \quad \frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdots(2)$

$\therefore$ From (1) and (2) we have $\cos A+\cos B+\cos C=1+\frac{r}{R}$.

Ex. 10. If $r_1=R$ in $\triangle ABC$, show that $\cos B+\cos C=\cos A$.

$\because \quad r_1=4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$,

$\therefore \quad 4\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=\frac{r_1}{R}=1 \quad [\because \quad r_1=R \text{ here}]$

Now, $\cos B+\cos C-\cos A$

$=2\cos\frac{B+C}{2}\cos\frac{B-C}{2}-\left(1-2\sin^2\frac{A}{2}\right)$

$=2\sin\frac{A}{2}\cos\frac{B-C}{2}+2\sin^2\frac{A}{2}-1 \quad [\because \frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=90°]$

$=2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}+\sin\frac{A}{2}\right)-1$

$=2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}+\cos\frac{B+C}{2}\right)-1 \left[\because \quad \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°\right]$

$=2\sin\frac{A}{2}.\ 2\cos\frac{B}{2}\cos\frac{C}{2}-1=4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-1$

$=1-1=0.$

$\therefore \quad \cos B+\cos C=\cos A.$

Ex. 11. Prove that $a\cot A+b\cot B+c\cot C=2(R+r)$.

The given L. H. S.$=a.\frac{\cos A}{\sin A}+b.\frac{\cos B}{\sin B}+c.\frac{\cos C}{\sin C}$

$=\frac{a}{\sin A}.\cos A+\frac{b}{\sin B}.\cos B+\frac{c}{\sin C}.\cos C$

$=2R\cos A+2R\cos B+2R\cos C$

$$\left[\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R\right]$$

$=2R(\cos A+\cos B+\cos C)=2R\left(1+\frac{r}{R}\right)$ [see Ex. 9]

$=2(R+r)$.

Ex. 12. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_2}{r_3}\right)=2$, show that the triangle is right-angled.

$\because \quad r_1=\frac{\Delta}{s-a}, \quad r_2=\frac{\Delta}{s-b}$ and $r_3=\frac{\Delta}{s-c}$,

$\therefore \quad \frac{r_1}{r_2}=\frac{\Delta}{s-a}\div\frac{\Delta}{s-b}=\frac{s-b}{s-a}$, and $\frac{r_1}{r_3}=\frac{s-c}{s-a}$.

Now, from the given condition we have

$\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)=2$, or, $\frac{b-a}{s-a}\times\frac{c-a}{s-a}=2$,

or, $(b-a)(c-a)=2(s-a)^2$

or, $(b-a)(c-a)=2(\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}a)^2$ $[\because s=\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c]$

or, $(b-a)(c-a)=\frac{1}{2}(b+c-a)^2$

or, $2bc-2ac-2ab+2a^2=b^2+c^2+a^2+2bc-2ab-2ac$,

or, $a^2=b^2+c^2$, $\quad\therefore$ A is a right angle.

Hence the triangle is right-angled.

Ex. 13. The perpendiculars from the angles of a triangle on the opposite sides meet at O and OA$=x$, OB$=y$, OC$=z$.

Show that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$. [A. U. '40]

Let S be the circum-centre of the triangle (draw a fig.) and SP be perpendicular to BC. Join SB and SC.

Now, $\angle BSC=2\angle A$, $\quad\therefore\quad \angle BSP=\angle A$.

$\because$ The distance of the circum-centre from any side of a triangle is half the distance of the ortho-centre from the opposite vertex, $\therefore$ $\mathrm{SP}=\frac{1}{2}\mathrm{AO}=\frac{1}{2}x$.

Again, $\mathrm{SP}=\mathrm{BP}\cot \mathrm{BSP}=\frac{1}{2}a\cot \mathrm{A}$,

i.e. $\frac{1}{2}x=\frac{1}{2}a\cot \mathrm{A}$, or, $x=a\cot \mathrm{A}$,

$$\therefore\quad \frac{a}{x}=\frac{1}{\cot \mathrm{A}}=\tan \mathrm{A}.$$

Similarly, $\frac{b}{y}=\tan \mathrm{B}$ and $\frac{c}{z}=\tan \mathrm{C}$.

$$\therefore\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\tan \mathrm{A}+\tan \mathrm{B}+\tan \mathrm{C}$$

Now, $\because$ $\mathrm{A}+\mathrm{B}+\mathrm{C}=180°$,

$$\therefore\quad \tan \mathrm{A}+\tan \mathrm{B}+\tan \mathrm{C}=\tan \mathrm{A}\tan \mathrm{B}\tan \mathrm{C}=\frac{a}{x}\cdot\frac{b}{y}\cdot\frac{c}{z}.$$

$$\therefore\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}.$$

Exercise 11

1. In a triangle ABC if $a=13$, $b=14$, and $c=15$, find r and r_1.

2. Express the circum-radius of a triangle in a form not involving the angles.

In a ΔABC, *prove the following* :—

3. $\sin \mathrm{A}+\sin \mathrm{B}+\sin \mathrm{C}=\frac{s}{\mathrm{R}}$.

4. $\Delta=\sqrt{r_1 r_2 r_3 r}$. 5. $\frac{r r_1}{r_2 r_3}=\tan^2\frac{\mathrm{A}}{2}$. [A. U. '47]

6. $r_1+r_2=c\cot \frac{1}{2}\mathrm{C}$. [C. U. '47]

7. $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2\mathrm{R}r}$. [B. H. U. '56]

8. $4\cos\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}=\frac{s}{\mathrm{R}}$.

9. $\frac{1}{2}\mathrm{S}=\mathrm{R}^2\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}$.

10. $\dfrac{b-c}{r_1}+\dfrac{c-a}{r_2}+\dfrac{a-b}{r_3}=0.$

11. $\Delta=r^2 \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}.$

12. $\dfrac{a}{\tan A}+\dfrac{b}{\tan B}+\dfrac{c}{\tan C}=2(R+r).$

13. $r_1+r_2+r_3-r=4R.$

14. $r_1r_2+rr_3=ab.$ 15. $(r_2+r_3)\sqrt{\dfrac{rr_1}{r_2r_3}}=a.$

16. $(r_1-r)(r_2-r)(r_3-r)=4r^2R.$ [A. U. '49]

17. $a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\dfrac{S}{R}.$

18. $\dfrac{bc-r_2r_3}{r_1}=\dfrac{ca-r_3r_1}{r_2}=\dfrac{ab-r_1r_2}{r_3}.$

19. $r_1r_2+r_2r_3+r_3r_1=s^2.$

20. $\left(\dfrac{1}{r}-\dfrac{1}{r_1}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_2}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_3}\right)=\dfrac{4R}{r^2s^2}=\dfrac{16R}{r^2(a+b+c)^2}.$

21. With the usual notation establish that
$\sin A+\sin B+\sin C=\dfrac{s}{R}.$ [C. U. '51. '55]

22. If $8R^2=a^2+b^2+c^2$, prove that the triangle is right-angled.

23. The sides of a triangle are as 3 : 7 : 8, find the ratio $R : r$.

24. If $r_1=r+r_2+r_3$, show that the triangle is right-angled.

25. The sides of a triangle are 5 ft., 8 ft. and 5 ft. Prove that two of its escribed circles are equal. [C. U. '18]

26. If $R=2r$, show that the triangle is equilateral.

27. If the lengths of the perpendiculars from the circum-centre on the sides BC, CA, AB of the $\triangle ABC$ are x, y, z respectively, prove that $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=\dfrac{abc}{4xyz}.$

28. If the altitudes of a triangle be h_1, h_2, h_3, prove that
$\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}.$

29. In any triangle, prove that the area of the in-circle is to the area of the triangle as $\pi : \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$.

30. If IA, IB, IC are respectively equal to x, y, z, show that $\frac{abc}{xyz}=\frac{s}{r}$.

31. If P be the area of the in-circle and P_1, P_2, P_3 the areas of the escribed circles of a triangle, prove that

$$\frac{1}{\sqrt{P}}=\frac{1}{\sqrt{P_1}}+\frac{1}{\sqrt{P_2}}+\frac{1}{\sqrt{P_3}}.$$

Use of Logarithmic and Trigonometric Tables

107. *Use of ordinary Logarithm Tables.*

You have already learnt the use of the common log tables and anti-log tables in class X. You will find some tables at the end of this book.

In Table I, only the mantissæ of the common logarithms of all the numbers from 10 to 10000 (*i.e.*, numbers consisting of 4 digits or less) are given. The characteristics are omitted, as they can be written by inspection.

Table II is the anti-log table, from which the number corresponding to a given logarithm can be found. The use of this table also is known to you.

Table III [Natural sine and cosine Table.]

Ordinary sines, cosines, etc. are usually stated as *natural sines, cosines, etc.* In table III, both the ordinary sines and cosines of all angles from 0° to 90° at intervals of 1′ (one minute) are given. This table consists of four portions. In the first portion (on the left side) 0°, 1°, 2°,……up to 90° are written from top vertically downwards, and in the third portion 89°, 88°,…up to 0° are noted in the same way. On top of the second portion, 0′, 10′, 20′, 30′, 40′, 50′, 60′ (60′=1 degree) are written horizontally and the

corresponding sines are given under each. The last or the fourth portion is the *mean difference* portion. Here 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′, 9′ are written horizontally on the top and the sines are given under each. For sines of an angle this table is read just like the log tables of ordinary numbers.

This table gives not only the sines but also the cosines of any angle. The *natural sines* are written from the left of the top towards the right and downwards, and the natural *cosines* are written from the bottom upwards from the right side to the left. It is to be noted that this table is so arranged that the sine of any angle and the cosine of its complementary angle coincide.

(*i*) *To find the natural sines*: Suppose sin 42°36′ is to be determined from this table. We first come to the line where 42° is written in the vertical column of the first portion and then proceeding to the right in the same row we come to the number ·67559 written under 30′ (written on the top). In the same row further to the right in the *difference portion* (*i.e.*, fourth portion) we come to the number 129 written under 6′ of the top. As the table is prepared correct to 5 places of decimals, 129 signifies ·00129 (*i.e.*, the requisite number of zeroes are supplied before 129).

It is to be noted that in the case of positive acute angles, as the angle increases, its sine increases. Hence *in determining the sine, the difference for the increased number of minutes is to be added.*

Hence, sin 42°36′ = ·67559 + ·00129 (diff.) = ·67688.

The process of finding the sine of an angle given up to the second will be discussed later on.

(*ii*) *To find the natural cosines*: It has been stated before that to find the cosines of a given angle from Table III, we have to proceed upwards from the bottom of the third column and come to the angle given and then move from the right side to the left in the row.

At the bottom of the table in the second portion are written $0', 10', 20', \cdots, 60'$ from the right side to the left, and at the bottom of the difference portion (i.e., the fourth portion) $1', 2', 3', \cdots, 9'$ are written from the left side to the right.

Now, suppose cosine of $46°4'3$ is to be found. In the third portion we first come to the row where $46°$ is written. Then we move to the left in the same row and come to the number ·68624 written in the column at the bottom of which $40'$ is noted ($\because$ we have to go upwards from the bottom to find cosines). Now, for $3'$ more, moving to the right in the same row we come to the difference portion (i.e. the fourth portion) and find the number 63 written in the column of $3'$ from the bottom. Here from 63 we have ·00063 (to make it a number correct to 5 decimal places).

$\therefore \quad \cos 46°43' = {}^{\cdot}68624 - {}^{\cdot}00063 = {}^{\cdot}68561.$

[*N.B.* In the case of positive acute angles, as the angle increases, its cosine decreases. Hence *in finding the cosine, the mean difference for the increased number of minutes* (given in the fourth portion) *is to be subtracted.*

Table IV [Natural tangent and cotangent Table]

In this table both the tangents and the cotangents of the angles from $0°$ to $90°$ at intervals of $1'$ are given. This table also is prepared in the same way as Table III. In determining the tangent of an angle we have to read from the top downwards and from the left to the right side in a row and in finding the cotangent we have to read upwards from the bottom and from the right side to the left in a row.

The tangent of an acute angle increases and the cotangent decreases as the angle increases. Hence the *mean difference* (made up into 5 places of decimals) for the increased number of minutes *is to be added in the case of tangents* and *subtracted in the case of contangents.*

Example. tan $30°23' = \cdot58513 + \cdot00118 = \cdot58631$, and

cot $45°32' = \cdot98270 - \cdot00114 = \cdot98156$.

Table V [Logarithmic sine and cosine Table]

You know that the sines and cosines of all angles are *numerically* less than unity (*i.e* 1) and the tangents of angles between 0° and 45° and the cotangents of angles between 45° and 90° are less than 1. Hence, their logarithms are negative. As it is inconvenient to tabulate negative values printing bars over the characteristics (such as $\bar{2}$, $\bar{3}$, etc), this table has been prepared *adding 10* to the *logarithms of trigonometrical ratios* to avoid this inconvenience. These are called *logarithmic sines, cosines,* etc.

They are written as $L \sin\theta$, $L\cos\theta$, $L\tan\theta$, etc.

Thus $L\sin\theta = 10 + \log\sin\theta$, $L\cos\theta = 10 + \log\cos\theta$, etc.

In this table the values of $L\sin\theta$ and $L\cos\theta$ are tabulated from 0° to 90° at intervals of 1′.

This table also is to be read as the table of natural sines and cosines to find the values of $L\sin\theta$ and $L\cos\theta$.

Suppose $L \sin 36°23'$ is to be determined. In the first column of this table we first come to the row of 36° and then moving to the right find the number ·77268 (*i.e.*, 9·77268) written in the column below 20′ of the top. So we have $L\sin 36°20' = 9\cdot77268$. Now, further to the right in the same row we find 51 written in the column under 3′ in the difference portion.

Hence, $L\sin 36°23' = 9\cdot77268 + \cdot00051 = 9\cdot77319$.

Similarly reading from below upwards and from the right side to the left we find $L\cos 45°32' = 9\cdot84566 - \cdot00026 = 9\cdot84540$ [the diff. being subtracted in the case of cosine.]

Table VI [Logarithmic tangent and cotangent table.]

The arrangement in table VI is similar to that in table V. The table VI gives $L\tan\theta$ and $L\cot\theta$ of all angles from 0° to 90° at intervals of 1′.

The mean difference is to be added in finding $L \tan \theta$ and is to be subtracted in finding $L \cot \theta$.

Thus, $L \tan 52°36' = 10{\cdot}11502 + {\cdot}00157 = 10{\cdot}11659$,

$$L \cot 65°45' = 9{\cdot}65535 - {\cdot}00168 = 9{\cdot}65367.$$

108. Principle of proportional parts

The principle of proportional parts states that (i) for a very small change in a number, the corresponding small change in the logarithm of the number is proportional to the change in the number itself. (ii) So, for a very small change in an angle, the corresponding small change in the trigonometrical ratio is proportional to the change in the angle itself.

Generally, when the increase is small in comparison with the number, the increase in the lagarithm is very nearly proportional to the increase in the number.

This principle is known as the *rule of proportional parts.*

The proof of this principle is beyond our scope. We shall, however, apply it where necessary. Notice the application of this rule in the following examples.

Examples (12)

Ex. 1. Given log $74583 = 4{\cdot}8726398$ and log $74584 = 4{\cdot}8726457$, find (i) log $74{\cdot}5836$ (ii) the number whose logarithm is $\bar{2}{\cdot}8726412$.

(i) Here evidently the mantissa of log $74583{\cdot}6$ lies between those of log 74583 and log 74584, and $74583{\cdot}6$ is greater than 74583 by ${\cdot}6$.

Here, log $74584 = 4{\cdot}8726457$
and log $74583 = 4{\cdot}8726398$

$\therefore$ increase for $1 = {\cdot}0000059$,

$\therefore$ increase for ${\cdot}6 = {\cdot}0000059 \times {\cdot}6 = {\cdot}00000354 = {\cdot}0000035$

[By the principle of proportional parts up to 7 places of decimals]

$\therefore$ log 74583·6 = 4·8726398 + ·0000035 = 4·8726433.

$\therefore$ the required log 74·5836 = 1·8726433.

(ii) 4·8726412 lies between 4·8726398 and 4·8726457, and its difference from the first is ·0000014.

Hence, the number whose logarithm is 4·8726412 must be between 74583 and 74584. Let the number be $74583 + x$.

Now, the difference for the increase of 1 is ·0000059 (*i.e.* diff. for 1 is 59)

$\therefore$ the diff. for $x =$ ·0000014 (*i.e.* briefly 14)

$\therefore$ From the Principle of proportional parts we have

$$59 : 14 :: 1 : x,$$

or, $x = \frac{14}{59} = {}^{\cdot}23\cdots$

$\therefore$ log 74583·23 ··· = 4·8726412.

Now, since the mantissa of the given logarithm $\bar{2}$·8726412 is the same as that of log 74583·23···(above), the required number must be formed of the same digits (as in 74583·23···) arranged in the same order. Its characteristic being $\bar{2}$, *i.e.*, -2, the required number = ·07458323···.

Ex. 2. Given $\sin 36°41' = 0{\cdot}59739$ and $\sin 36°42' = 0{\cdot}59763$, find $\sin 36°41'32''$.

$$\sin 36°42' = {}^{\cdot}59763$$
$$\sin 36°41' = {}^{\cdot}59739$$

$\therefore$ diff. for $1' = 24$ (briefly)

$\because$ $1' = 60''$, $\therefore$ diff. for $60'' = 24$

$\therefore$ diff. for $32'' = \frac{24 \times 32}{60} = 12{\cdot}8$ (*i.e.*, ·000128)

$\therefore$ $\sin 36°41'32'' = {}^{\cdot}59739 + {}^{\cdot}000128 = {}^{\cdot}597518$.

Ex. 3. Given cos $46°24' = 0·68962$ and difference for $1' = 21$, find cos $46°24'40''$.

Here, the difference between $46°24'40''$ and $46°24'$ is $40''$ and diff. for $1'$ or $60''$ is 21 (*i.e.* ·00021)

$\therefore$ diff for $40'' = \frac{21 \times 40}{60} = 14$ (*i.e.*, ·00014)

$\because$ for increasing angle, the cosine diminishes,

$\therefore$ cos $46°24'40'' = ·68962 - ·00014 = ·68948$.

Ex. 4. Given L sin $44°17' = 9·8439842$ and L sin $44°18' = 9·8441137$, find L sin $44°17'33''$ and deduce the value of L cosec $44°17'33''$.

(i) L sin $40°18' = 9·8441137$

L sin $40°17' = 9·8439842$

$\therefore$ diff. for $1' = 1295$ (*i.e.* ·0001295)

i.e., diff. for $60'' = 1295$

$\therefore$ diff. for $33'' = \frac{1295 \times 33}{60} = 712·25$, *i.e.*, = ·000071225

$\therefore$ L sin $40°17'33'' = 9·8439842 + ·0000712 = 9·8440554$.

(ii) Let $44°17'33'' = \theta$.

Here log sin $\theta = L$ sin $\theta - 10$.

$$\text{Now log cosec } \theta = \log \frac{1}{\sin \theta} = -\log \sin \theta \;[\because \log 1 = 0]$$

$$= -(L \sin \theta - 10) = 10 - L \sin \theta$$

$$= 10 - 9·8440554 = ·1559446$$

$\therefore$ L cosec $44°17'33'' = \log \text{cosec } \theta + 10 = 10·1559446$.

Ex. 5. Given cosec $13°8' = 4·4010616$ and cosec $13°9 = 4·3955817$, find the value of cosec $13°8'19''$.

Here for an increase of $1'$ or $60''$ in the angle, the cosecant decreases by $(4·4010616 - 4·3955217)$ or ·0054799.

$\therefore$ for an increase of 19″ in the angle, the cosecant diminishes by $\frac{19}{60}\times{}^{\cdot}0054799$ or $^{\cdot}0017353$.

$\therefore$ cosec $13°8'19'' = 4{\cdot}4010616 - {}^{\cdot}0017353 = 4{\cdot}3993263$.

Ex. 6. Given L cot $62°26' = 9{\cdot}5257779$ and L cot $62°27' = 9{\cdot}5253589$, find the value of L cot $62°26'47''$ and solve the equation L cot $\theta = 9{\cdot}5254782$.

(i) L cot $62°26' = 9{\cdot}5257779$

L cot $62°27' = 9{\cdot}5253589$

$\therefore$ diff. for $1'$ or $60'' = {}^{\cdot}0004190$

i.e., for an increase of 60″ in the angle, the L cot diminishes by $^{\cdot}0004190$,

$$\therefore \quad \text{diff. for } 47'' = \frac{47\times{}^{\cdot}0004190}{60} = {}^{\cdot}0003282.$$

$\therefore$ L cot $62°26'47'' = 9{\cdot}5257779 - {}^{\cdot}0003282 = 9{\cdot}5254497$.

(ii) Now, the equation L cot $\theta = 9{\cdot}5254782$ is to be solved.

Here L cot $62°26' = 9{\cdot}5257779$

and L cot $\theta = 9{\cdot}5254782$

$\therefore$ diff. $= 2997$ (briefly)

But from the given data, diff. for $1'$ or $60'' = 4190$ (briefly),

$\because$ L cot diminishes by 4190 for an increment of 60″ of the angle,

$\therefore$ L cot diminishes by 2997 for an increment of $\dfrac{60''\times 2997}{4190}$ or $42{\cdot}9''$ in the angle.

$\therefore$ $\theta = 62°26'43''$ (App.).

Ex. 7. Find by interpolation the angle whose L tan is $9{\cdot}732235$.

From the table of Logarithmic tangent (table VI) we have L tan $28°20' = 9{\cdot}73175$ and diff. for $1' = {\cdot}00030$.

$\therefore$ L tan $28°21' = 9{\cdot}73205$, and L tan $28°22' = 9{\cdot}73235$.

$\because$ the given L tan lies between $9{\cdot}73205$ and $9{\cdot}73235$,

$\therefore$ the angle lies between $28°21'$ and $28°22'$.

$9{\cdot}732235 - 9{\cdot}73205 = {\cdot}000185$

$\because$ ${\cdot}00030$ is the diff. for an increase of $1'$ or $60''$,

$\therefore$ ${\cdot}000185$ " " " " " " $\dfrac{60'' \times {\cdot}000185}{{\cdot}00030}$ or $37''$.

$\therefore$ the required angle $= 28°21'37''$.

Ex. 8. Find from the tables the value of

$$\frac{\tan 82°6' \times \sin 34°17'}{\sec 12°37'}.$$

Let the given exp. $= x$.

Then $x = \tan 82°6' \times \sin 34°17' \times \cos 12°37'$

$\therefore$ $\log x = \log \tan 82°6' + \log \sin 34°17' + \log \cos 12°37'$

Now, from the tables we have

$$\begin{aligned}
\log \tan 82°6' &= L \tan 82°6' - 10 = {\cdot}85806 \\
\log \sin 34°17' &= L \sin 34°17' - 10 = \bar{1}{\cdot}75072 \\
\log \cos 12°37' &= L \cos 12°37' - 10 = \bar{1}{\cdot}98938
\end{aligned}$$

$\therefore$ (Adding) $\log x = {\cdot}59816$

$\because$ Antilog ${\cdot}59816 = 3{\cdot}9642$, $\therefore$ $x = 3{\cdot}9642$

$\therefore$ The required value $= 3{\cdot}9642$.

Ex. 9. Solve $\sin x = {\cdot}7$, given $\log 7 = {\cdot}84510$,
$L \sin 44°25' = 9{\cdot}84501$ and diff. for $1' = 13$.

$\because$ $\sin x = {\cdot}7$

$\therefore$ $\log \sin x = \log {\cdot}7 = \bar{1}{\cdot}84510$

$\therefore$ $L \sin x = 9{\cdot}84510$

and $L \sin 44°25' = 9{\cdot}84501$

$\therefore$ diff. $=$ 9 (briefly)

Now, $13 = \text{diff. for } 1'$ or $60''$,

$\therefore \quad 9 = \text{diff. for } \dfrac{60''}{13} \times 9$ or $41{\cdot}5''$.

$\therefore \quad L \sin x = L \sin 40°25'41{\cdot}5''$, $\quad \therefore \quad x = 40°25'41{\cdot}5''$.

Exercise 12

Find from the Tables the values of :—

1. $\sin 44°\ 58'$
2. $\cos 25°\ 12'$
3. $\tan 35°\ 42'$
4. $\cot 38°\ 25'$
5. $\sec 36°\ 48'$
6. $\operatorname{cosec} 50°\ 25'$

Evaluate—

7. $L \cos 45°\ 15'$
8. $L \tan 22°\ 27'$
9. $L \sin 41°\ 15'$
10. $L \cot 27°\ 34'$
11. If $L \cos \theta = 9{\cdot}55533$, find θ to the nearest minute.
12. Given $\log 4827 = 3{\cdot}68367$ and $\log 4828 = 3{\cdot}68376$, find $\log 4827{\cdot}5$.
13. Given $\log 3534 = 3{\cdot}54826$ and $\log 3535 = 3{\cdot}54838$, find the number whose logarithm is $\bar{2}{\cdot}54831$.
14. Find the seventh root of $\cdot 034574$, having given $\log 34574 = 4{\cdot}5387496$, $\log 61837 = 4{\cdot}7912484$ and difference for $\cdot 00001 = \cdot 0000071$.
15. If $\log 256{\cdot}12 = 2{\cdot}4084435$, $\log 30{\cdot}317 = 1{\cdot}4816862$ and $\log 3{\cdot}0318 = \cdot 4817005$, find the fifth root of $\cdot 0025612$.
16. Given $\log 4376 = 3{\cdot}64108$ and $\log 4377 = 3{\cdot}64118$, find by interpolation the logarithm of $437{\cdot}66$.
17. If $\sin 35°24' = \cdot 57952$ and $\sin 35°25' = \cdot 57965$, find by interpolation the angle whose sine is $\cdot 57960$.
18. If $\cos 48°16' = \cdot 66566$ and difference for $1' = 22$, find $\cos 48°16'36''$.
19. If $\cos 58°18' = \cdot 5254716$ and $\cos 58°19' = \cdot 5252241$, find the angle whose cosine is $\cdot 5254221$.

20. Given $\tan 76°21' = 4{\cdot}1177784$ and $\tan 76°22' = 4{\cdot}1230079$, find the angle whose tangent is $4{\cdot}1203060$.

21. Find the value of $L \tan 79°\ 41'\ 24''$ from the table.

22. Given $L \sin 37°43'40'' = 9{\cdot}7867152$ and $L \sin 37°43'50'' = 9{\cdot}7867424$, find $L \sin 37°\ 43'56''$. [C. U. '10]

23. Given $L \tan 79°51'40'' = 10{\cdot}7475657$ and $L \tan 79°51'50'' = 10{\cdot}7476872$, find the angle whose L tan is $10{\cdot}7476532$. [C. U. '21]

24. If $L \sec 27°39' = 10{\cdot}0526648$ and difference for $10'' = 110$, find θ when $L \sec \theta = 10{\cdot}0527253$.

25. Given $\log 2 = {\cdot}30103$, $\log 6684 = 3{\cdot}82504$ and diff. for $1 = 7$, find $({\cdot}04)^{\frac{1}{8}}$.

26. Given $L \sin 14°6' = 9{\cdot}386704$, find $L \operatorname{cosec} 14°6'$.

27. Given $L \sin 35°20' = 9{\cdot}7621775$ and $L \cos 35°\ 20' = 9{\cdot}9115844$, find $L \tan 35°\ 20'$.

28. Prove that $L \sin \theta + L \operatorname{cosec} \theta = L \tan \theta + L \cot \theta = 20$, where θ is an acute angle.

Evaluate :—

29. $\sin 25°\ 12' \times \cos 45°\ 15'$. **30.** $\dfrac{\sin 47°\ 13'}{\tan 22°\ 27'}$.

31. Find the value of $\dfrac{\cot 27°12' \times \sin 34°\ 17'}{\sec 77°\ 23'}$,

given $L \cos 55°\ 43' = 9{\cdot}7507$, $L \tan 62°48' = 10{\cdot}2891$, $L \cos 77°\ 23' = 9{\cdot}3393$ and $\log 239{\cdot}4 = 2{\cdot}3791$.

32. Find θ, given $\sin \theta = {\cdot}6$, $\log 6 = {\cdot}77814$, $L \sin 36°\ 52' = 9{\cdot}77812$ and diff. for $1' = 17$.

32. Solve $\tan x = {\cdot}3$, given $\log 3 = {\cdot}4771213$, $L \tan 16°41' = 9{\cdot}4770875$ and diff. for $1' = 1352$.

Solution of Triangles

109. In any triangle the three sides and the three angles are the six parts of it. The solution of a triangle means the determination of these parts. The triangle is then completely known. These parts are not independent, but are restricted by the relations between the sides and angles of the triangle.

When any three parts of a triangle are given, *provided that one at least of these is a side,* the numerical values of the unknown parts can be determined. If, however, *only the three angles* are given, the triangle cannot be definitely known ; for, innumerable triangles having their angles equal to the given angles may be constructed.

The three parts of the triangle may be given in the following ways :

Case I. Three sides.

Case II. Three angles.

Case III. Any two sides and the included angle.

Case IV. Two angles and one side.

Case V. Two sides and an angle opposite to one of them.

Now, we shall deal with these cases one by one. It is to be noted here that the solution of a triangle is not possible, if the given data be inappropriate. As for example, two sides cannot be together equal to or less than the third, or two angles given cannot be obtuse or right angles.

110. Case I. Three sides given.

Let the three sides a, b, c of a triangle ABC be given. It is required to solve the triangle, *i.e.*, its three angles are to be determined.

From the formula $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, the value of $\cos A$ can be determined, a, b and c being known. Now, from the cosine

table, the angle corresponding to the obtained value of the cosine (cos A) can be found. Thus we obtain the value of A.

As the angle, being an angle of a triangle, lies between 0° and π and as within this limit an angle corresponding to a given cosine has only one value, the angle is definitely known.

Similarly B and C can be determined. If only two angles be thus determined, the third angle can be easily obtained.

Approximate value. The cosine table gives only the approximate values. It has been proved in higher mathematics that more accurate results, *i.e.* the *nearest approximate* can be obtained using the logarithmic tangent table.

Hence to determine A, we should apply the formula

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\text{ where } s=\tfrac{1}{2}(a+b+c).$$

Then we have $L\tan\frac{A}{2}=10+\log\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}^{\frac{1}{2}}$

$$=10+\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}.$$

The other two angles can be similarly determined.

[*N.B.* An angle of a triangle may be obtained using the sine formula, but in this case there may be ambiguity, for the sines of supplementary angles are equal in magnitude and are of the same sign. As for example, if $\sin A=\frac{1}{2}$, then $A=30^\circ$ or 150°. In such cases we shall have to verify which of the values of the angle satisfy the other conditions.

But there will be no such inconvenience, if the angle is obtained through the cosine or the tangent formula.

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

We have already stated that it is preferable to apply the tangent formula.]

111. Case II. Three angles given.

It has already been said that in this case the solution of the triangle is not possible, for an infinite number of triangles can be drawn with the same three given angles. These triangles will be equiangular and hence similar. So only the ratios of the lengths of the sides can be found in such cases but not their actual lengths.

Thus $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$, or, $a:b:c=\sin A:\sin B:\sin C$.

Examples (13)

Ex. 1. The sides of a triangle are 7, 8 and 9. Determine all the angles having given $\log 2=\cdot3010300$, $L\tan 24°5'40''=9\cdot6505069$, $L\tan 24°5'50''=9\cdot6505634$, $L\tan 20°12'20''=9\cdot7474183$ and $L\tan 29°12'30''=9\cdot7474677$.

[C. U. '38 ; B. H. U. '38]

Here $a=7$, $b=8$, $c=9$, hence $s=\frac{1}{2}(7+8+9)=12$.

$$\therefore\quad \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{(12-7)(12-9)}{12(12-8)}}$$

$$=\sqrt{\frac{5\times3}{12\times4}}=\sqrt{\frac{5}{16}}=\sqrt{\frac{10}{32}}$$

$$\therefore\quad L\tan\frac{B}{2}=10+\log\left(\frac{10}{32}\right)^{\frac{1}{2}}=10+\tfrac{1}{2}\log 10-\tfrac{1}{2}\log 32$$

$$=10+\tfrac{1}{2}\log 10-\tfrac{1}{2}\log 2^5=10+\tfrac{1}{2}\log 10-\tfrac{5}{2}\log 2$$

$$=10+\tfrac{1}{2}\times1-\tfrac{5}{2}\times\cdot3010300=10+\cdot5-\cdot7525750$$

$$=9\cdot7474250.$$

Now, $L\tan 29°12'30''=9\cdot7474677$

$L\tan 29°12'20''=9\cdot7474183$

$\therefore$ diff. for $10''=\cdot0000494$

Let $\frac{B}{2} = 29°12'20'' + x''$

$\therefore$ diff. for $x'' = 9{\cdot}7474250 - 9{\cdot}7474183 = {\cdot}0000067$

$\therefore \frac{x''}{10''} = \frac{{\cdot}0000067}{{\cdot}0000494} = \frac{67}{494}, \quad \therefore x = \frac{10'' \times 67}{494} = 1{\cdot}35''$

$\therefore \frac{B}{2} = 29°12'20'' + 1{\cdot}35'' = 29°12'21{\cdot}35''$

$\therefore B = 58°24'42{\cdot}7''$ (app.).

Again, $\tan \frac{A}{2} = \sqrt{\frac{(12-8)(12-9)}{12(12-7)}} = \sqrt{\frac{4 \times 3}{12 \times 5}} = \sqrt{\frac{1}{5}} = \sqrt{\frac{2}{10}}.$

$\therefore L \tan \frac{A}{2} = 10 + \frac{1}{2} \log 2 - \frac{1}{2} \log 10 = 10 + {\cdot}1505150 - {\cdot}5$
$= 9{\cdot}6505150.$

Now, diff. for $10'' = 9{\cdot}6505634 - 9{\cdot}6505069 = {\cdot}000565,$
and $9{\cdot}6505150 - 9{\cdot}6505069 = {\cdot}0000081.$

Let $\frac{A}{2} = 24°5'40'' + x''$

$\therefore 565 : 81 :: 10'' : x,$

$\therefore x = \frac{81 \times 10''}{565} = 1{\cdot}43''$ (app.),

$\therefore \frac{A}{2} = 24°5'40'' + 1{\cdot}43'' = 24°5'41{\cdot}43''$ (app.),

$\therefore A = 48°11'22{\cdot}86''$ (app.).

Hence, $C = 180° - A - B = 180° - 106°36'5{\cdot}56''$
$= 73°23'54{\cdot}44''$ (app.).

Ex. 2. The sides of a triangle are 32, 40, 66; find by the help of logarithmic tables the greatest angle. [C. U. '45]

Here the angle opposite to the greatest side 66 is the greatest angle. Let it be angle A.

Now, $\cos A = \frac{32^2 + 40^2 - 66^2}{2 \times 32 \times 40} = -\frac{1732}{2560} = -\frac{433}{640} = -{\cdot}67656\cdots$

From the tables we have $\cos 47°20' = \cdot 67773$

$$\begin{array}{rl} \text{and } \cos 47°30' &= \cdot 67559 \\ \hline \therefore \quad \text{diff. for } 10' &= \cdot 00214 \end{array}$$

$$\cdot 67773 - \cdot 67656 = \cdot 00117.$$

Now, 214 is the diff. for $10'$,

$\therefore$ 117 „ „ „ „ $\frac{1170}{214}$ or $5'28''$,

$\therefore \quad \cos 47°25'28'' = \cdot 67656$,

$\therefore \quad \cos(180° - 47°25'28'') = -\cos 47°25'28'' = -\cdot 67656.$

$\therefore \quad A = 180° - 47°25'28'' = 132°34'32''.$

Ex. 3. The sides of a triangle are proportional to 7, 12, 11; find the least angle, having given $L \tan 17°33' = 9 \cdot 500042$ and difference for $1' = 439$.

Let the sides a, b, c of the $\triangle ABC$ be proportional to 7, 12, 11.

Then $A : B : C = 7 : 12 : 11$ and A is the least angle.

Let $a = 7K$, $b = 12K$, $c = 11K$. $\therefore \ s = \frac{1}{2}(7K + 12K + 11K) = 15K$.

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{3K \times 4K}{15K \times 8K}} = \sqrt{\frac{1}{10}} = \left(\frac{1}{10}\right)^{\frac{1}{2}}$$

$$\therefore \quad L \tan\frac{A}{2} = 10 + \tfrac{1}{2}\log 1 - \tfrac{1}{2}\log 10 = 10 + 0 - \cdot 5 = 9 \cdot 5.$$

Hence, the difference here $= 9 \cdot 500042 - 9 \cdot 5 = \cdot 000042$ and the angle $\frac{A}{2}$ is less than $17°33'$.

Now, $439 =$ diff. for $1'$ or $60''$

$$\therefore \quad 42 = \text{diff. for } \frac{60'' \times 42}{439} \text{ or } 5 \cdot 7'';$$

$$\therefore \quad \frac{A}{2} = 17°33' - 5 \cdot 7'' = 17°32'54 \cdot 3'' \text{ (nearly)}$$

$$\therefore \quad A = 35°5'49'' \text{ (nearly)}.$$

Ex. 4. Given $a=\sqrt{3}-1$, $b=\sqrt{6}$, $c=2$, solve the triangle.

Here $\cos B=\dfrac{c^2+a^2-b^2}{2ca}=\dfrac{4+(\sqrt{3}-1)^2-6}{2\times 2(\sqrt{3}-1)}$

$$=\frac{4+4-2\sqrt{3}-6}{4(\sqrt{3}-1)}=\frac{2(1-\sqrt{3})}{4(\sqrt{3}-1)}=-\frac{1}{2}=\cos 120^\circ.$$

$\therefore\ B=120^\circ$.

Again, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{4-2\sqrt{3}+6-4}{2(\sqrt{3}-1)\times\sqrt{6}}=\dfrac{6-2\sqrt{3}}{2(\sqrt{3}-1)\sqrt{6}}$

$$=\frac{2\sqrt{3}(\sqrt{3}-1)}{2\sqrt{6}(\sqrt{3}-1)}=\frac{\sqrt{3}}{\sqrt{6}}=\frac{1}{\sqrt{2}}=\cos 45^\circ,$$

$\therefore\ C=45^\circ$. $\therefore\ A=180^\circ-120^\circ-45^\circ=15^\circ$.

Ex. 5. If two angles of a triangle be 45° and 75°, find the ratio of its sides.

Let ABC be the triangle in which $A=45^\circ$ and $B=75^\circ$.

Then $C=180^\circ-75^\circ-45^\circ=60^\circ$.

$\sin 45^\circ=\dfrac{1}{\sqrt{2}}$, $\sin 75^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin 60^\circ=\dfrac{\sqrt{3}}{2}$;

$\therefore\ a:b:c=\sin 45^\circ:\sin 75^\circ:\sin 60^\circ=\dfrac{1}{\sqrt{2}}:\dfrac{\sqrt{3}+1}{2\sqrt{2}}:\dfrac{\sqrt{3}}{2}$

$$=2:(\sqrt{3}+1):\sqrt{6}\ [\text{ multiplying by } 2\sqrt{2}\ ]$$

Ex. 6. The angles of a triangle are as 7 : 3 : 2, prove that the ratio of the least side to the greatest side is $\sqrt{2}:(\sqrt{3}+1)$.

Here the sum of the three angles is 180° and they are in the ratio 7 : 3 : 2.

$7+3+2=12$.

$\therefore$ The least angle $=\frac{2}{12}\times 180^\circ=30^\circ$,

and the greatest angle $=\frac{7}{12}\times 180^\circ=105^\circ$.

$\therefore$ the reqd. ratio $=\sin 30^\circ:\sin 105^\circ$

$$=\frac{1}{2}:\frac{\sqrt{3}+1}{2\sqrt{2}}=\sqrt{2}:(\sqrt{3}+1).$$

Ex. 7. The ratio of the smallest angle to the greatest angle of a trianale is as 2 : 5 and the third angle is $1\frac{1}{2}$ times the smallest angle. Compare the lengths of the sides.

Let ABC be a triangle in which A : B = 2 : 5.

$\because$ The third angle C is $1\frac{1}{2}$ times the least angle A,

$\therefore$ A : C = 2 : 3

$\therefore$ A : B : C = 2 : 5 : 3

$\therefore$ $A=\frac{2}{2+5+3}\times 180^\circ=36^\circ$. Similarly, $B=90^\circ$, $C=54^\circ$.

$\therefore$ $a : b : c = \sin A : \sin B : \sin C$

$$= \sin 36^\circ : \sin 90^\circ : \sin 54^\circ$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4} : 1 : \frac{\sqrt{5}+1}{4} \quad [\because \sin 54^\circ = \cos 36^\circ = \tfrac{1}{4}(\sqrt{5}+1)]$$

$$= \sqrt{(10-2\sqrt{5})} : 4 : (\sqrt{5}+1).$$

Exercise 13

1. If in a triangle $a=283$, $b=317$, $c=428$, find all its angles by the help of the Tables.

2. In a plane triangle $a=18$, $b=20$, $c=22$; calculate the value of $L \tan \frac{A}{2}$. Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$. [C. U. '15]

3. If $a=17$, $b=20$, $c=27$, find all the angles by using the tables.

4. The sides of a triangle are 9, 10 and 11, find the angle opposite to the side 10. Given $\log 2 = \cdot 30103$, $L \tan 29^\circ 30' = 9\cdot 7526420$, $L \tan 29^\circ 29' = 9\cdot 7523472$. [H. S. '64]

5. Find the greatest angle of the triangle whose sides are 5, 6, 7. Given $\log 6 = \cdot 7781513$, $L \cos 39^\circ 14' = 9\cdot 8890644$ and diff. for $1' = 1032$. [C. U. ; P. U.]

6. If $a=15$, $b=19$, $c=24$, find the greatest angle of the triangle ; given log 5·7=0·75587, L cos 88°59′=8·24903, and diff. for 1′=718. [C.U. '36]

7. The sides of a triangle are 7, 8 and 9. Determine all the angles, given log 2=·3010300, L tan 24°5′40″=9·6505069, L tan 24°5′50″=9·6505634, L tan 29°12′20″=9·7474183, L tan 29°12′30″=9·7474677. [C. U. '38]

8. The sides of a triangle are 4, 5, 6 ; find B having given log 2=·3010300, L cos 27°53′=9·9464040, diff. for 1′=·0000669 [C. U. '41 ; P. U. '44]

9. The sides of a triangle are proportional to 2, 3, 4. Find the greatest angle, having given log 2=·30103, log 3=·4771213, L tan 52°14′=10·1108395, L tan 52°15′=10·1111004.

10. Find the greatest angle of the triangle whose sides are 12, 15, 16. (Use log tables). [C. U. '57]

11. If the sides of a triangle are as 68 : 75 : 77, find the least angle. Given log 2=·3010300, L cos 26°34′=9·9515389 and diff. for 1′=632.

12. If $a=\sqrt{6}$, $b=2$ and $c=\sqrt{3}+1$, solve the triangle.

13. Solve the triangle in which $a=5\sqrt{3}$ and $b=c=5$.

14. The sides of a triangle are a, b and $\sqrt{a^2+ab+b^2}$ feet ; find the greatest angle.

15. If the sides of a triangle are 4, 5, 6 feet, show that the least angle is half of the greatest angle.

16. If one angle of a triangle is 45° and the ratio of the other two is 2 : 7, find the angles and the ratio of the sides.

17. In $\triangle ABC$, $A=45°$ and $B=60°$, find the ratio of the least side to the greatest.

18. The angles of a triangle are as 1 : 2 : 3 ; compare the magnitudes of the sides.

19. If $A=45°$, $B=60°$, show that $a+b\sqrt{2}=2c$.

20. The base angles of a triangle are 22°30′ and 112°30′, prove that the base is twice the height.

21. If $\cos A = \frac{5}{13}$ and $\cos B = \frac{12}{13}$, find $a : b : c$.

22. The angles of a triangle are as $3 : 4 : 5$; find the ratio of the least side to the greatest side.

23. The ratio of the smallest angle to the greatest angle of a triangle is $2 : 7$ and the other angle is half as much again as the smallest angle. Compare the magnitudes of the sides.

24. The angles of a triangle are 40°, 60°, 80°, and the greatest side is 22 ft.; find the least side, given that $L \sin 40^\circ = 9{\cdot}8080675$, $L \sin 80^\circ = 9{\cdot}9933515$, $\log 22 = 1{\cdot}3424227$, $\log 14359 = 4{\cdot}1571242$, diff. for $1 = {\cdot}0000302$. [B. U. 1899]

112. Case III. Two sides and the included angle given.

Let the two sides b and c and the included angle A of the $\triangle ABC$ be given. It is required to solve the triangle, *i.e.*, to determine the angles B, C and the side a. Here a definite solution is possible, for only one triangle can be drawn with the given parts.

I. If $b = c$, then $B = C$. So the values of B and C can be determined from the relation $A + B + C = 180^\circ$, or, $A + 2B = 180^\circ$. Thus three angles and two sides being known, the third side can also be found.

II. If b and c are unequal, then suppose $b > c$.

Now $B + C = 180^\circ - A$, $\dfrac{B+C}{2} = 90^\circ - \dfrac{A}{2}$ ......(1)

Again $\because \tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$,

$$\therefore L \tan \frac{B-C}{2} = 10 + \log\left(\frac{b-c}{b+c} \cot \frac{A}{2}\right)$$

$$= \log (b-c) - \log(b+c) + \log \cot \frac{A}{2} + 10$$

$$= \log (b-c) - \log(b+c) + L \cot \frac{A}{2} \text{......(2)}$$

Here, b, c and A being known, the value of the right hand side in (2) can be found. Thus the value of $L \tan \frac{B-C}{2}$ being obtained, the value of $\frac{B-C}{2}$ can be determined.

Now, $\frac{B+C}{2}$ and $\frac{B-C}{2}$ being known, the values of B and C can be determined.

Again, the three angles of the triangle being known, the side a can be found from the relation $\frac{a}{\sin A}=\frac{b}{\sin B}$,

or from $\frac{a}{\sin A}=\frac{c}{\sin C}$.

[N.B. $\because \quad \frac{a}{\sin A}=\frac{b}{\sin B}, \quad \therefore \quad a=b\frac{\sin A}{\sin B}$,

$$\therefore \quad \log a=\log b+\log \sin A-\log \sin B$$
$$=\log b+(10+\log \sin A)-(10+\log \sin B)$$
$$=\log b+L \sin A-L \sin B. \text{]}$$

[*Alternative Proof*] : The value of a can be found from the relation $a^2=b^2+c^2-2bc \cos A$, here b, c, A being known.

Now, a being known, B can be determined from the relation $\sin B=\frac{b \sin A}{a}$, or, from $\cos B=\frac{c^2+a^2-b^2}{2ca}$. Thus A and B being known, C can be easily found.

[N.B. It is not, however, convenient to use this method.]

113. Case IV. Two angles and a side given.

It is very easy to solve a triangle, when two of its angles and any side are given. Two angles being given the third angle can be found from the relation $A+B+C=180^\circ$.

Again, since any one of its sides (say a) is known, b and c can be obtained from the relation $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

Thus $b=\frac{a \sin B}{\sin A}$, $c=\frac{a \sin C}{\sin A}$.

Examples (14)

Ex. 1. If $a=1+\sqrt{3}$, $b=2$, $C=60°$, solve the triangle.

Here the values of c, A, B are to be found.

From the formula, $c^2=a^2+b^2-2ab\cos C$

$$=(1+\sqrt{3})^2+(2)^2-2(1+\sqrt{3}).2.\cos 60°$$

$$=4+2\sqrt{3}+4-2(1+\sqrt{3}).\ 2.\ \tfrac{1}{2}$$

$$=8+2\sqrt{3}-2-2\sqrt{3}=6.$$

$$\therefore\quad c=\sqrt{6}.$$

Again $\sin A=\dfrac{a\sin C}{c}=\dfrac{(1+\sqrt{3})\sin 60°}{\sqrt{6}}=\dfrac{(1+\sqrt{3})}{\sqrt{6}}.\dfrac{\sqrt{3}}{2}$

$$=\frac{1+\sqrt{3}}{2\sqrt{2}}=\sin 75°,\quad \therefore A=75°,$$

$\therefore\quad B=180°-A-C=180°-75°-60°=45°.$

Hence, $c=\sqrt{6}$, $A=75°$, $B=45°$.

Ex. 2. In a triangle ABC if $a=21$, $b=11$, $C=34°42'30''$; find A and B, given $\log 2=\cdot 30103$ and $L\tan 72°38'45''=10\cdot 50515$.

[B.H.U. '47]

Here $\dfrac{A+B}{2}=90°-\dfrac{C}{2}=90°-\tfrac{1}{2}(34°42'30'')$

$$=90°-17°21'15''=72°38'45''\quad\cdots(1)$$

Again, $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}=\dfrac{21-11}{21+11}\cot\dfrac{C}{2}$

$$=\frac{10}{32}\tan\frac{A+B}{2}=\frac{10}{32}\tan 72°38'45''\left[\because\quad \frac{A+B}{2}+\frac{C}{2}=90°\right]$$

$\therefore\quad L\tan\dfrac{A-B}{2}=10+\log 10-5\log 2+\log\tan 72°38'45''$

$$=1-5\times\cdot 30103+L\tan 72°38'45''$$

$$[\because\quad 10+\log\tan\theta=L\tan\theta]$$

$$=1-1\cdot 50515+10\cdot 50515=10.$$

$\therefore\quad 10+\log\tan\dfrac{A-B}{2}=10,\ \therefore\ \log\tan\dfrac{A-B}{2}=0=\log 1$

$\therefore\quad \tan\dfrac{A-B}{2}=1=\tan 45°,\quad \therefore\ \dfrac{A-B}{2}=45°,\cdots\cdots(2)$

Adding (1) and (2) we have

$$A=\frac{A+B}{2}+\frac{A-B}{2}=72°38'45''+45°=117°38'45''\ ;$$

and from (1) – (2) we have $B=72°38'45''-45°=27°38'45''$.

Ex. 3. The sides b and c of $\triangle ABC$ are as 5 : 3 and $A=60°30'$. Find the other angles; given $\log 2=\cdot30103$, $L\cot 30°15'=10\cdot23420$, $L\tan 23°13'=9\cdot63240$ and diff. for $1'=35$.

Here $\dfrac{A}{2}=31°15'$, $\therefore\ \dfrac{B+C}{2}=90°-\dfrac{A}{2}=59°45'\cdots\cdots(1)$

From the formula, $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}=\dfrac{5-3}{5+3}\cot\dfrac{A}{2}$

$$=\frac{1}{4}\cot\frac{A}{2}.$$

$\therefore\quad \log\tan\dfrac{B-C}{2}=\log 1-2\log 2+\log\cot\dfrac{A}{2}$

$$=-2\times\cdot30103+\log\cot\frac{A}{2},$$

$\therefore\quad L\tan\dfrac{B-C}{2}=-\cdot60206+L\cot 30°15'$

[Adding 10 to both sides]

$$=-\cdot60206+10\cdot23420=9\cdot63214.$$

Now, $L\tan 23°13'=9\cdot63240$ (given)

and $L\tan\dfrac{B-C}{2}=9\cdot63214$

$\therefore$ diff. $=$ 26.

$\because$ 35 is the diff. for $1'$ or $60''$,

$\therefore$ 26 ,, ,, $\dfrac{60''}{35}\times 26$ or $44''57$.

$$\therefore \quad \frac{B-C}{2}=23°13'-44{\cdot}57''=23°12'15'' \text{ (App.)}\cdots\cdots(2)$$

Now, adding (1) and (2) we have $B=59°45'+23°12'15''$
$=82°57'15''$

and subtracting (2) from (1) we have $C=36°32'45''$.

Ex. 4. $A=60°15'$, $B=54°30'$ and $c=100$ ft., find b; given $\log 8{\cdot}9646162={\cdot}9525317$, $L \sin 54°30'=9{\cdot}9106860$, $L \sin 65°15'=9{\cdot}9581543$.

Here $C=180°-(A+B)=180°-114°45'=65°15'$.

$$\because \quad \frac{b}{\sin B}=\frac{c}{\sin C}, \quad \therefore \ b=\frac{c \sin B}{\sin C}=\frac{100 \sin B}{\sin C},$$

$$\therefore \quad \log b=\log 10^2+\log \sin B-\log \sin C$$
$$=2+\log \sin 54°30'-\log \sin 65°15'$$
$$=2+L \sin 54°30'-10-(L \sin 65°15'-10)$$
$$=2+L \sin 54°30'-L \sin 65°15'$$
$$=2+9{\cdot}9106860-9{\cdot}9581543=1{\cdot}9525317.$$

Now since $\log b$ has the same mantissa as $\log 8{\cdot}9646162$ (given) and as the characteristic of $\log b$ is 1, $\therefore$ $b=89{\cdot}646162$ ft.

Ex. 5. If $A=70°$, $B=40°50'$ and $c=4{\cdot}85$, solve the triangle.

$\because$ $A+B=110°50'$, $\therefore$ $C=180°-(A+B)=69°10'$.

From tables, $L \sin 70°=9{\cdot}97299$

$L \sin 40°50'=9{\cdot}81549$

$L \sin 69°10=9{\cdot}97063$.

$$\because \quad \frac{a}{\sin A}=\frac{c}{\sin C}, \quad \therefore \ a=\frac{c \sin A}{\sin C}=\frac{4{\cdot}85 \times \sin 70°}{\sin 69°10'}$$

$$\therefore \quad \log a=\log 4{\cdot}85+L \sin 70°-L \sin 69°10'$$
$$={\cdot}68574+9{\cdot}97299-9{\cdot}97063={\cdot}68810$$
$$=\log 4{\cdot}88 \text{ (App.)}$$

$\therefore$ $a=4{\cdot}88$.

Again, $b=\dfrac{c \sin B}{\sin C}=\dfrac{4{\cdot}85\times\sin 40^\circ 50'}{\sin 69^\circ 10'}$

$\therefore \quad \log b=\log 4{\cdot}85+\text{L} \sin 40^\circ 50'-\log 69^\circ 10'$

$\qquad ={\cdot}68574+9{\cdot}81549-9{\cdot}97063$

$\qquad ={\cdot}53060=\log 3{\cdot}39$ (nearly)

$\therefore \quad b=3{\cdot}39.$

Hence, $a=4{\cdot}88$, $b=3{\cdot}39$ and $C=69^\circ 10'$.

Ex. 6. If $c=123$, $A=29^\circ 17'$, $B=135^\circ$, find the greatest side : given $\log 2={\cdot}3010300$, $\log 123=2{\cdot}0899051$, $\log 32110=4{\cdot}5066403$, diff. for $1=135$, L sin $15^\circ 42' 40''=9{\cdot}4327596$ and diff. for $1'=543$.

Here $A+B=29^\circ 17'+135^\circ=164^\circ 17'$

$\therefore \quad C=180^\circ-(A+B)=180^\circ-164^\circ 17'=15^\circ 43'.$

Hence B being the greatest angle, its opposite side b is the greatest side.

$\because \quad \dfrac{b}{\sin B}=\dfrac{c}{\sin C}, \quad \therefore \ b=\dfrac{c \sin B}{\sin C},$

or, $b=\dfrac{c \sin 135^\circ}{\sin 15^\circ 43'}=\dfrac{c \sin 45^\circ}{\sin 15^\circ 43'}$ [$\because \sin 135^\circ=\sin(180^\circ-45^\circ)$
$=\sin 45^\circ$]

$$=\frac{123\times\dfrac{1}{\sqrt{2}}}{\sin 15^\circ 43'}$$

$\therefore \quad \log b=\log 123+\log \dfrac{1}{\sqrt{2}}-\log \sin 15^\circ 43' \cdots\cdots(1)$

Again, $15^\circ 43'-15^\circ 42' 40''=20''$ (diff.)

$\because$ diff. for $60''$ is 543,

$\therefore$ diff. for $20''$ is $\dfrac{543}{60}\times 20$ or 181.

$\therefore \quad \text{L} \sin 15^\circ 43'=9{\cdot}4327596+{\cdot}0000181=9{\cdot}4327777.$

$\therefore \quad \log \sin 15^\circ 43'=9{\cdot}4327777-10=\bar{1}{\cdot}4327777,$

Now from (1) we have

$\log b = \log 123 + \log 1 - \frac{1}{2} \log 2 - \log \sin 15°43'$

$[\because \quad \sqrt{2} = (2)^{\frac{1}{2}}]$

$= 2{\cdot}0899051 - \frac{1}{2} \times {\cdot}3010300 - \bar{1}{\cdot}4327777$

$= 2{\cdot}0899051 - {\cdot}1505150 + 1 - {\cdot}4327777 = 2{\cdot}5066124.$

Here the mantissa of $\log b$ is ·5066124 and that of the given log 32110 is ·5066403,

the diff. of the mantissa $= {\cdot}5066403 - {\cdot}5066124 = 279$ (briefly).

$\therefore$ 135 is the diff. for 1,

$\therefore$ 279 is the diff. for $\frac{279}{135}$ or 2·066

$\therefore$ ·5066124 is the mantissa of the logarithm of (32110 − 2·066) or 32107·93.

Now, $\because$ the characteristic of $\log b$ is 2,

$\therefore$ $\log b = \log 321{\cdot}0793$, $\therefore$ $b = 321{\cdot}0793$.

Ex. 7. The base of a triangle is 7 ft. and the base angles are 129°23′ and 38°36′; find the length of its shorter side.

Let BC be the base of $\triangle$ ABC, i.e, $a = 7$,

B = 38°36′ and C = 129°23′. Hence the length of b, the side opposite to angle B, is to be found.

The remaining angle $= 180° - (38°36' + 129°23') = 12°1'$.

$\because \quad \frac{b}{\sin B} = \frac{a}{\sin A}, \quad \therefore \; b = \frac{a \sin B}{\sin A} = \frac{7 \times \sin 38°36'}{\sin 12°1'},$

$\therefore \quad \log b = \log 7 + \log \sin 38°36' - \log \sin 12°1'$

$= \log 7 + L \sin 38°36' - L \sin 12°1'$

$= {\cdot}84510 + 9{\cdot}79510 - 9{\cdot}31845$ [From tables]

$= 1{\cdot}32175.$

Now, the mantissa of 2097 is found to be ·32163 from the tables.

$$\begin{array}{r} {\cdot}32175 \\ {\cdot}32163 \\ \hline \end{array}$$

$\therefore$ diff. = 12, but from the table we have the diff. for 1 is 21, $\therefore$ 12 is the diff. for $\frac{12}{21}$ or ·57.

$\therefore$ ·32175 is the mantissa of log 2097·6 (app.) and as its characteristic is 1, $\therefore$ $\log b = \log 20{\cdot}976$. Hence, $b = 20{\cdot}976$.

Exercise 14

1. If $b=\sqrt{3}$, $c=1$ and $A=30°$, solve the triangle.

2. If $b=\sqrt{6}$, $a=1+\sqrt{3}$ and $C=45°$, solve the triangle.

3. Two sides of a triangle have lengths 1 and 3 ft. and the included angle is 40°. Find the other angles in degrees and minutes.

4. In a plane triangle $b=540$, $c=420$ and $A=52°6'$; find B and C, having given L tan $26°3'=9·6891430$,
L tan $14°20'=9·4074189$, L tan $14°21'=9·4079543$. [C. U. '34]

5. Two sides of a triangle are 3 and 5 feet and the included angle is 120°. Find the other angles, having given log $4·8=·6812412$, L tan $8°12'=9·1586706$, diff. for $60''=·0008940$. [B. U. E. '63 , C. U. '40, '49]

6. Two sides of a triangle are 18 ft. and 2 ft. and the included angle is 55°. Find the remaining angles; having given log $2=·3010300$, L cot $27°30'=10·2835233$,
L tan $55°46'=10·1863769$ and diff. for $1'=·0002763$. [C. U. '42]

7. If the sides a and b are in the ratio 7 : 3 and the angle C is 60°, find A and B, given log $2=·3010300$, log $3=·4771213$, L tan $34°42'=9·8403776$, diff. for $1'=2699$. [B. H. U. '40]

8. Two sides of a triangle are 14 and 11 and the included angle is 60°. Find the remaining angles, having given L tan $11°44'=9·3174299$, L tan $11°45'=9·3180640$, log $2=·3010300$, log $3=·4771213$. [C. U. '44]

9. In a triangle $b=2·25$, $c=1·75$, $A=54°$, find B and C, having given log $2=·301030$, L tan $63°=10·292834$, L tan $13°47'=9·389724$, L tan $13°48'=9·390270$. [C. U. '31]

10. Given $a=70$, $b=35$, $C=36°52'12''$, $\log 3=0.4771213$, $L \cot 18°26'6''=10.4771213$. Calculate the other two angles A and B. [C. U. '35, '37]

11. If $b=243$, $c=681$, $A=50°42'$, solve the triangle by the help of Mathematical tables.

12. In a triangle $b=80$, $c=100$ and $A=60°$, find the other angles, having given $\log 3=.47712$, $L \tan 10°53'36''=9.28432$. [C. U. '46]

13. If $a=204$, $b=91$ and $\tan\frac{A}{2}=\frac{17}{6}$, show that $c=125$.

14. If $a=19$, $B=52°28'$ and $C=93°40'$, find b, having given $\log 27038=4.4319746$, $\log 19=1.2787536$, $\log 27037=4.4319585$, $L\sin 52°28'=9.8992727$, $L\sin 33°52'=9.7460595$. [P. U. '36]

15. Given $b=10$, $A=45°$, $B=66°42'20''$; find a, given that $\log 2=.3010300$, $\log 7.698622=.8864131$,
$L\sin 66°42'=9.9630538$, diff. for $1'=544$. [C. U. 1906]

16. If $A=C=75°$, $b=\sqrt{8}$, solve the triangle.

17. The angles of a triangle are as $2:2:1$ and the least side is 2, solve the triangle.

18. If $a=39$, $A=81°35'$, $B=27°55'$; solve the triangle. [C. U. '33]

19. If $b=1000$, $A=45°$, $C=68°17'40''$, find the least side having given $\log 2=.3010300$, $\log 7.6986=.8864118$,
diff. for $1=57$, $L\sin 66°42'=9.9630538$, diff. for $1'=544$.

20. If $B=45°$, $C=10°$ and $a=200$ ft. ; find b, having given $\log 2=.30103$, $L\sin 55°=9.9133645$, $\log 1726.4=3.2371414$, $\log 1726.5=3.2371666$. [C. U. '47]

113. Case V. Two sides and an angle opposite to one of them given.

Let the two sides a and b of a $\triangle ABC$ and the angle A opposite to the side a be given. To solve the triangle, *i.e.*, to find the side c and the angles B and C.

From the relation $\dfrac{b}{\sin B}=\dfrac{a}{\sin A}$, we have $\sin B=\dfrac{b \sin A}{a}$,

$\therefore$ L sin B $=\log b+$ L sin A $-\log a$, whence B can be obtained.

Now since A and B are known, C can be found from the relation $A+B+C=180°$.

Again, $\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$,

$\therefore \quad c=\dfrac{a \sin C}{\sin A}$, or, $c=\dfrac{b \sin C}{\sin B}$,

$\therefore \quad \log c=\log a+$ L sin C $-$ L sin A,

or, $\log c=\log b+$ L sin C $-$ L sin B, from any of which c can be found. Thus the triangle can be solved.

Three different cases may arise when these data are given. The given parts may be such that (1) no triangle can be constructed with them and hence the triangle cannot be solved, or, (ii) only one triangle can be drawn; or, (iii) two triangles can be drawn and so two solutions will be obtained.

We shall now discuss these cases.

(1) *Case I.* $\sin B=\dfrac{b \sin A}{a}$, here if $b \sin A>a$, then

$\dfrac{b \sin A}{a}>1$, *i.e*, $\sin B>1$, which is not possible as the sine of an angle cannot be greater than 1. So B cannot be determined and no triangle can be drawn. Hence no solution of the triangle is possible.

(2) *Case II.* If b sin A $=$ a, then we have $\frac{b \sin A}{a}=1$,

i.e., $\sin B=1=\sin 90°$. $\therefore$ $B=90°$.

$\therefore$ In this case the triangle is right-angled, and b can be obtained from the relation $b^2=a^2+c^2$, *i.e.*, $b=\sqrt{a^2+c^2}$.

It is to be noted that construction of the triangle fails, if here $a=b$, or, $a>b$.

(3) *Case III.* If b sin A $<$ a, then sin B <1 and B can be determined.

We know that the sines of two supplementary angles are equal, so two values of B will be found here—one of which lies between 0° and 90° and the other between 90° and 180° (*i.e.*, one acute and the other obtuse). But both the two values obtained may not be always admissible.

Thus this case may give rise to some sub-cases.

As for example—

(i) *If* $a>b$, then A $>$ B and hence the obtuse value of B cannot be *admissible*, as the two angles A and B of a triangle cannot be both obtuse. So, only the acute value of B will be admissible and only one solution of the triangle is possible.

(ii) If a $=$ b, then A $=$ B and therefore here also only the acute value of B is admissible and only one solution is possible.

(iii) If a $<$ b, then A $<$ B and so B may be either acute or obtuse. Hence, both the values (acute and obtuse) being admissible here, *two triangles* may be constructed with the given parts and *two solutions* of the triangle will be obtained. This sub-case is said to be the **ambiguous case** in the solutions of triangles. Here A will be an acute angle and B will have two supplementary values.

The above *results* are briefly stated below :—

(1) When $b \sin A > a$, no triangle is possible ;

(2) When $b \sin A = a$, a right-angled triangle is obtained as solution ;

(3) When $a \geqslant b$ (so $a > b \sin A$), one solution having C acute is obtained.

[Here the symbol $\geqslant$ signifies that $a > b$, or, $a = b$.]

(4) If $b \sin A < a$ and $a < b$, we have two solutions, and this case is the ambiguous case.

114. Geometrical treatment of the ambiguous case.

Suppose the sides a, b and angle A of a $\triangle ABC$ are given. Take any st. line AX and at A draw $\angle XAY$ equal to the angle A.

From AY cut off $AC = b$ and draw $CD \perp AX$.

$\because \dfrac{CD}{AC} = \sin A$, $\therefore CD = AC \sin A = b \sin A$.

Now, draw a circle with centre C and with radius a. If C is joined with the point (say B) at which this circle cuts AX, then we get the required triangle ABC.

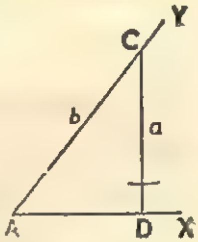


Fig. 12

Now, (1) if $a < CD$ (*i.e.*, $a < b \sin A$), then the circle will not meet AX at any point [Fig. 12] and hence no triangle can be drawn.

(2) If $a = CD$ (*i. e.*, $a = b \sin A$), then the circle will touch AX at D which here coincides with B, and hence the right-angled $\triangle ACD$ (*i.e.*, $\triangle ACB$) will be the triangle required. [Here a cannot be equal to b.]

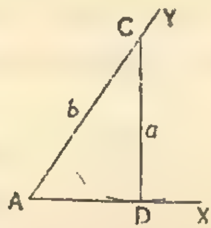


Fig. 13

(3) If $a > CD$ (*i.e.*, $a > b \sin A$), then we have two cases so that (i) either $a > b$, or (ii) $a = b$.

(i) If $a>b$, the circle will cut AX at two points B and B$'$ on opposite sides of A.

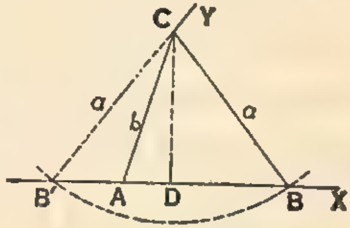


Fig. 14

Here in $\triangle AB'C$, though $AC=b$ and $CB'=a$, but $\angle CAB'$ is equal to the supplement of angle A and not equal to A. So this triangle is inadmissible and hence $\triangle ABC$ is the only solution.

(ii) Here, if $a=b$, then B$'$ will coincide with A and only one isosceles triangle ABC is obtained as the solution.

(4) If $a>CD$ (*i.e.* $a>b \sin A$) but less than b or AC, then the circle will cut AX at two points B_1 and B_2 on the same side of A. [Fig. 15]

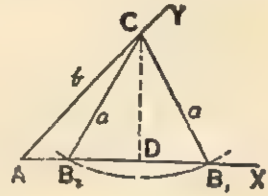


Fig. 15

Here the three parts of both the triangles ACB_1, and ACB_2 are equal to the three given parts ; for $\angle A=$the given angle A, $CA=b$, and $CB_1=CB_2=a$.

So we have two solutions in this case. Hence this is the ambiguous case.

[*N. B.* Of the two triangles obtained in the ambiguous case, one is acute-angled and the other obtuse-angled. In these triangles, $B_1+B_2=180°$, so they are supplementary angles and $\sin B_1=\sin B_2$.]

Examples (15)

Ex. 1. If $b=3$, $c=3\sqrt{3}$ and $B=30°$, find C and A.

Here $b<c$, $\therefore$ $B<C$. Hence angle C will be either acute or obtuse, and the two values of C will be supplementary.

Now, $\because$ $\dfrac{\sin C}{c}=\dfrac{\sin B}{b}$,

$$\therefore \sin C=\frac{c \sin B}{b}=\frac{c\times\sin 30°}{b}=\frac{3\sqrt{3}\times\frac{1}{2}}{3}=\frac{\sqrt{3}}{2}$$

$$=\sin 60° \text{ or } \sin(180°-60°), \quad \therefore \quad C=60° \text{ or } 120°.$$

If $C=60°$, then $A=180°-(B+C)=180°-(30°+60°)=90°$;
If $C=120°$, then $A=180°-(B+C)=180°-(30°+120°)=30°$;
Hence, $C=60°$, $A=90°$; or, $C=120°$, $A=30°$.

[*N.B.* Here, if $C=60°$, A will be $90°$ and so the $\triangle$ is right-angled. If $C=120°$, then $A=30°=B$ and so the $\triangle$ is isosceles.]

Ex. 2. If $a=6$, $c=2\sqrt{2}$ and $C=45°$, solve the triangle (if possible).

We have $\sin A=\dfrac{a \sin C}{c}$, ...(1)

Here $a \sin C=6\times\sin 45°=6\times\dfrac{1}{\sqrt{2}}=3\times\sqrt{2}$, and $c=2\sqrt{2}$;

$\therefore$ $a \sin C>c$, $\therefore$ from (i) $\sin A>1$, which is impossible. Hence, the solution is not possible here.

Ex. 3. If $b=100$, $a=b\sqrt{2}$ and $B=30°$, solve the triangle, giving two solutions in case of ambiguity.

Here $b<a$, $\therefore$ $B<A$. $\therefore$ A will have two supplementary values.

Hence, two solutions are possible here, it being a case of ambiguity.

$$\text{We have } \sin A=\frac{a \sin B}{b}=\frac{b\sqrt{2}\times\sin 30°}{b}=\sqrt{2}\times\tfrac{1}{2}$$

$$=\frac{1}{\sqrt{2}}=\sin 45° \text{ or } \sin(180°-45°)$$

$$=\sin 45° \text{ or } \sin 135°, \therefore A=45° \text{ or } 135°.$$

Now, (i) if $A=45°$, then $C=180°-(A+B)$

$$=180°-(30°+45°)=105° ;$$

$$\text{and } c=\frac{b \sin C}{\sin B}=\frac{100\times\sin 105°}{\sin 30°}=\frac{100\times\sin 75°}{\sin 30°}$$

$$=\frac{100\times\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{1}{2}}=\frac{200\times(\sqrt{3}+1)}{2\sqrt{2}}=\frac{100(\sqrt{3}+1)}{\sqrt{2}}$$

$$=\frac{100\times\sqrt{2}(\sqrt{3}+1)}{2}=50(\sqrt{6}+\sqrt{2})=193{\cdot}185.$$

(ii) If $A=135°$, then $C=180°-(30°+135°)=15°$

and then $c=\frac{b \sin C}{\sin B}=\frac{100\times \sin 15°}{\sin 30°}=\frac{100\times\frac{(\sqrt{3}-1)}{2\sqrt{2}}}{\frac{1}{2}}$

$$=50(\sqrt{6}-\sqrt{2})=51{\cdot}76.$$

Hence, we have two solutions here :—

(i) $A=45°$, $C=105°$, $c=193{\cdot}185$;

(ii) $A=135°$, $C=15°$, $c=51{\cdot}76$.

Ex. 4. If $C=26°26'$, $b=127$ and $c=85$, find B ;

given $\log 1{\cdot}27={\cdot}1038037$, $\log 8{\cdot}5={\cdot}9294189$,

$L \sin 26°26'=9{\cdot}6485124$ and $\sin 41°41'28''=\bar{1}{\cdot}8228972$.

Here $b \sin C<c$ and $c<b$, hence B may be acute or obtuse.

$\because$ $\log 1{\cdot}27={\cdot}1038037$, $\therefore$ $\log 127=2{\cdot}1038037$;

$\because$ $\log 8{\cdot}5={\cdot}9294189$, $\therefore$ $\log 85=1{\cdot}9294189$;

and $\because$ $\log \sin 41°41'28''=\bar{1}{\cdot}8228972$,

$\therefore$ $L \sin 41°41'28''=10+\bar{1}{\cdot}8228972=9{\cdot}8228972$.

Now, $\sin C=\frac{b \sin C}{c}=\frac{127 \sin 26°26'}{85}$,

$\therefore$ $L \sin B=\log 127+L \sin 26°26'-\log 85$

$$=2{\cdot}1038037+9{\cdot}6485124-1{\cdot}9294189$$

$$=9{\cdot}8228972=L \sin 41°41'28''$$

$\therefore$ $B=41°41'28''$ or $(180°-41°41'28'')$

$=41°41'28''$ or $138°18'32''$.

Ex. 5. If $a=4{\cdot}5$, $c=45$, find A so that C may be a right angle. Given that $L \sin 5°33'=8{\cdot}98157$ and the diff. for $1'=5260$.

Here $\sin A = \frac{a \sin C}{c} = \frac{4{\cdot}5 \times \sin 90^\circ}{45}$ [$\because$ C=90° (hyp.)]

$$= \frac{4{\cdot}5 \times 1}{45} = \frac{1}{10}.$$

$\therefore$ $\log \sin A = \log 1 - \log 10 = -1$ [$\because$ $\log 1 = 0$]

$\therefore$ $L \sin A = -1 + 10 = 9$

and $L \sin 5^\circ 33' = 8{\cdot}98157$ (given)

$\therefore$ diff. $= {\cdot}01843$

but 5260, *i.e.*, ·05260 is the diff. for 60″,

$\therefore$ ·01843 " " $\frac{60'' \times 1843}{5260}$ or 21″

$\therefore$ $L \sin A = L \sin 5^\circ 33' 21''$, $\therefore$ $A = 5^\circ 33' 21''$.

Ex. 6. If a, b and A of $\triangle$ABC be given, prove that in the ambiguous case the difference between the two values of c is $2\sqrt{a^2 - b^2 \sin^2 A}$. [U.P. B. '41]

From the formula we have $a^2 = b^2 + c^2 - 2bc \cos A$,

or, $c^2 - 2b \cos A . c + b^2 - a^2 = 0$, which is a quadratic equation in c. Solving it we have

$$c = \frac{2b \cos A \pm \sqrt{4b^2 \cos^2 A - 4(b^2 - a^2)}}{2}$$

$$= \frac{2b \cos A \pm 2\sqrt{b^2 \cos^2 A - b^2 + a^2}}{2}$$

$$= b \cos A \pm \sqrt{a^2 - b^2(1 - \cos^2 A)}$$

$$= b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

Hence, c has two values $b \cos A + \sqrt{a^2 - b^2 \sin^2 A}$ and $b \cos A - \sqrt{a^2 - b^2 \sin^2 A}$.

$\therefore$ the difference of the two values $= 2\sqrt{a^2 - b^2 \sin^2 A}$.

Ex. 7. If $b=573$, $a=394$, $B=112°4'$, find A and C, given $\log 57·3=1·7581546$, $\log 3·94=·5954962$,

$L \sin 39°35'=9·8042757$, diff. for $1'=1527$ and

$L \cos 22°4'=9·9669614$.

From the given log 57·3 and log 3·94 we have

$\log 573=2·7581546$ and $\log 394=2·5954962$.

Again, $B=112°4'=90°+22°4'$,

$$\therefore \quad \sin B=\sin(90°+22°4')=\cos 22°4'$$

$$\therefore \quad \sin A=\frac{a \sin B}{b}=\frac{394\times\sin 112°4'}{573}$$

$$=\frac{394\times\cos 22°4'}{573},$$

$$\therefore \quad L \sin A=\log 394+L\cos 22°4'-\log 573$$

$$=2·5954962+9·9669614-2·7581546$$

$$=9·8043030.$$

$$\begin{array}{lrl} \text{Now,} & L\sin A & =9·8043030 \\ \text{but} & L\sin 39°35' & =9·8042757 \\ \hline \therefore & \text{diff.}= & \quad 273 \quad \text{(briefly)} \end{array}$$

$\because$ 1527 is the diff. for 60″,

$$\therefore \quad 273 \quad ,, \quad ,, \quad \frac{60''}{1527}\times 273 \text{ or } 11'' \text{ (nearly)}$$

$$\therefore \quad L \sin A=L\sin 39°35'11''.$$

$$\therefore \quad A=39°35'11'' \text{ and } C=180°-(A+B)=28°20'49''.$$

Exercise 15

1. If $a=\sqrt{6}$, $c=2$, $A=60°$, find B and C.

2. If $B=60°$, $c=6$ and $b=3\sqrt{3}$, show that it may be a right-angled triangle.

Solve the following triangles :

3. $a=2$, $b=\sqrt{3}+1$ and $A=45°$.

4. $b=34$, $c=70$, $B=30°$. 5. $b=20°$, $c=20\sqrt{2}$, $B=30°$.

6. $C=30°$, $b=30$, $c=10\sqrt{3}$. 7. $a=5$, $c=5\sqrt{3}$, $C=60°$.

8. If in $\triangle ABC$, $c=36·5$ ft., $a=45$ ft. and $A=43°15'$, find AC using tables.

9. If $a=5$, $b=7$ and $A=30°$, find B in degrees and minutes, having given $\sin 44°=·6947$ and $\sin 45°=·7071$. [C. U. '29]

10. If $b=5$, $c=4$ and $B=45°$, find A and C ; given $\log 2=·30103$, $L \sin 34°26'=9·752575$.

11. If $b=112$, $c=175$, $B=36°20'$, find the other angles, having given $\log 2=·30103$, $L \sin 36°20'=9·77268$ and $L \sin 67°47'=9·96650$.

12. If $a=63$, $c=36$, $C=29°23'15''$, find A ; given $\log 2=·3010300$, $\log 7=·8450980$, $L \sin 29°23'=9·6907721$, diff. for $1'=2243$, $L \sin 59°10'=9·9338222$, diff. for $1'=755$.

13. If in a triangle $a=5$, $b=4$ and $A=45°$, find the remaining angles, having given $\log 2=·30103$, $L \sin 34°26 =9·7523919$, $L \sin 34°27'=9·7525761$. [B. H. U. '51]

14. If $a=5$ ft., $b=8$ ft., $A=35°$; find the smaller value of c, having given $\log 2=·30103$, $\log 456706=5·659637$, $L \sin 31°35'43'' =9·719261$, $L \sin 35°=9·758591$, $L \sin 66°35'=9·962672$, $L \sin 66°36'=9·962727$. [A. U. '13]

15. If $a=35$ and $b=350$, find A so that B may be a right angle, having given

$$L \sin 5°44'=8·9995595, \text{ diff. for } 1'=12565.$$

16. If $2b=3a$ and $\tan^2 A=\frac{3}{5}$, prove that the third side has two values, one being double the other.

17. In an obtuse-angled triangle $b=1325$, $c=1665$ and $B=52°19'$, solve the triangle.

18. If in an ambiguous case the angles A and C have two values A_1, A_2 and C_1, C_2 respectively, show that

$$\frac{\sin A_1}{\sin C_1}+\frac{\sin A_2}{\sin C_2}=2\cos B.$$

19. If a, b, A are given and if c_1, c_2 be the values of the third side in the ambiguous case, prove the following if $c_1>c_2$:

(i) $c_1-c_2=2a\cos B$ [B.H.U.I. '28]

(ii) $\cos\frac{C_1-C_2}{2}=\frac{b\sin A}{a}$ [A.I. '41]

(iii) $c_1^2+c_2^2-2c_1c_2\cos 2A=4a^2\cos^2 A.$

20. In the case that admits of two solutions prove that the two values of C satisfy the equation

$$\frac{(a+b)^2}{1+\cos C}+\frac{(b-a)^2}{1-\cos C}=\frac{2a^2}{\sin^2 A}.$$ [B. H. U. I. '42]

Heights and Distances

115. We have dealt with the problems on heights and distances in the first part of this book (for class IX). Here we shall discuss some harder problems on heights and distances.

It is to be noted here that when an observer views the angle of elevation or depression of a distant object his own height is not taken into consideration, his eye being taken as a point on the horizontal plane. But if in any problem the height is given, it cannot be ignored.

$$\therefore \quad AP = \frac{a \sin \phi}{\sin (\theta + \phi)} = a \sin \phi \operatorname{cosec} (\theta + \phi)$$

$$\therefore \quad h = PM = AP \sin \alpha = a \sin \alpha \sin \phi \operatorname{cosec} (\theta + \phi)$$

[Putting the value of AP]

and $d = AM = AP \cos \alpha = a \cos \alpha \sin \phi \operatorname{cosec} (\theta + \phi)$.

[*N.B.* (a) Here also the formulas are suitable for logarithmic calculations.

(*b*) The distance AB may be taken at right angles with AM.]

118. *To find the distance between two visible but inaccessible objects.*

Let P and Q be two inaccessible objects. The distance PQ between them is to be determined.

For observation take any two suitable points A and B on the horizontal plane and let a be the distance AB.

At A measure the angles PAQ, QAB and PAB and let them be α, β and θ respectively.*

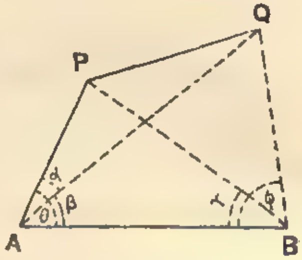


Fig. 19

* [*N.B.* If the pts. A, B, P, Q be coplanar, then $\angle PAB$ need not be measured, as in that case $\angle PAB = \angle PAQ + \angle QAB = \alpha + \beta$.]

Again at B measure the angles PBA and QBA and let them be γ and ϕ respectively.

Then $\angle APB = 180° - (\theta + \gamma)$ and $\angle AQB = 180° - (\phi + \beta)$.

Now, from $\triangle PAB$ we have $\dfrac{AP}{\sin \gamma} = \dfrac{AB}{\sin APB}$,

$$\text{or,} \quad \frac{AP}{\sin \gamma} = \frac{a}{\sin \{180° - (\theta + \gamma)\}} = \frac{a}{\sin (\theta + \gamma)} = a \operatorname{cosec} (\theta + \gamma).$$

$$\therefore \quad AP = a \sin \gamma \operatorname{cosec} (\theta + \gamma).$$

Similarly, from $\triangle ABQ$ we have

$$\frac{AQ}{\sin\phi}=\frac{AB}{\sin AQB}=\frac{a}{\sin\{180^\circ-(\phi+\beta)\}}=\frac{a}{\sin(\phi+\beta)}$$
$$=a\operatorname{cosec}(\phi+\beta).$$

$\therefore$ $AQ=a\sin\phi\operatorname{cosec}(\phi+\beta)$.

$\therefore$ From $\triangle PAQ$ we have $PQ^2=AQ^2+AP^2-2AP.AQ\cos\alpha$.

Thus PQ can be determined.

Examples (16)

Ex. 1. A ship sails m miles along a line having a bearing $N\alpha E$ and then another m miles in a direction $N\theta E$. Find the final distance of the ship from the starting point and the final bearing.

Suppose the ship starts from O, the point of intersection of NS and EW (*i.e.*, the North-South and the East-West lines), and sails $OP(=m)$ miles towards E (*i. e.*, to the East) making an angle α with ON. Then it again sails PQ $(=m)$ miles East of North. The distance OQ and $\angle NOQ$ are to be determined.

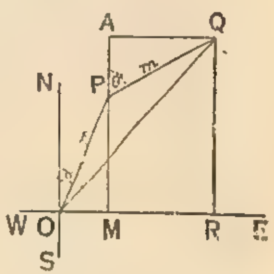


Fig. 20

Through P draw the vertical line AM on EW and draw $QA\perp AM$.

Then $\angle OPM=\angle NOP=\alpha$.

Again, $\angle OPQ=\angle OPM+\angle MPQ=\alpha+180^\circ-\theta$

$\therefore$ $\cos\angle OPQ=-\cos(\theta-\alpha)$.

$OP=PQ=m$ miles. $\therefore$ $\angle POQ=\angle OQP$

$=\frac{1}{2}(180^\circ-\angle OPQ)=\frac{1}{2}(180^\circ-\alpha-180^\circ+\theta)=\frac{1}{2}(\theta-\alpha)$

$\therefore$ The final bearing $=\angle NOQ=\angle NOP+\angle POQ$

$$=\alpha+\tfrac{1}{2}(\theta-\alpha)=\tfrac{1}{2}(\theta+\alpha).$$

Now, $OQ^2 = OP^2 + PQ^2 - 2PO.PQ \cos \angle OPQ$

$$= m^2 + m^2 - 2m^2 \times -\cos(\theta - \alpha)$$
$$= 2m^2 + 2m^2 \cos(\theta - \alpha) = 2m^2\{1 + \cos(\theta - \alpha)\}$$
$$= 2m^2 . 2\cos^2 \tfrac{1}{2}(\theta - \alpha) = 4m^2 . \cos^2 \tfrac{1}{2}(\theta - \alpha)$$

$\therefore$ the required distance $= OQ = 2m \cos \dfrac{\theta - \alpha}{2}$.

Ex. 2. A man observes the elevation of a hill top to be 15°, but after walking a mile directly towards it on level ground finds the elevation to be 75°. Find the height of the hill.

Let the hill PM stand on the ground AM and let 15° and 75° be the angles of elevation of the hill top P from the points A and B respectively on the base AM.

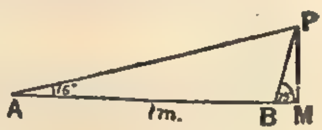


Fig. 21

Then $\angle A = 15°$ and $\angle PBM = 75°$.

$\therefore$ $\angle APB = 75° - 15° = 60°$.

Here $AB = 1$ mile. Let h be the height of PM.

Now, $PM = PB \sin 75°$, or, $h = PB \sin 75°$.

Again, from $\triangle APB$ we have $\dfrac{PB}{\sin 15°} = \dfrac{AB}{\sin 60°} = \dfrac{1}{\frac{\sqrt{3}}{2}}$ miles

$$= \frac{2}{\sqrt{3}} \text{ miles.}$$

$\therefore$ $PB = \dfrac{2 \sin 15°}{\sqrt{3}}$ mi. $= \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ mi. $= \dfrac{\sqrt{3}-1}{\sqrt{6}}$ mi.

$\therefore$ $h = PB \sin 75° = \dfrac{\sqrt{3}-1}{\sqrt{6}} \times \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ mi. $= \dfrac{1}{2\sqrt{3}}$ miles

$$= \frac{1}{2\sqrt{3}} \times 1760 \text{ yds.} = \frac{880}{\sqrt{3}} \text{ yds.} = 880\sqrt{3} \text{ feet.}$$

Ex. 3. A flagstaff standing on the top of a pillar 25 ft. high subtends an angle whose tangent is ·125 at a point 60 ft. from the foot of the pillar. Find the height of the flagstaff.

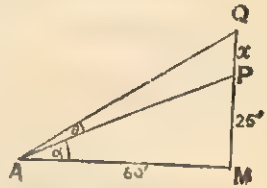


Fig 22

Let the flagstaff PQ standing on the pillar PM subtend an angle θ at A, 60 feet from M. To determine the length of PQ.

Here, we have $\tan\theta = \cdot 125 = \frac{1}{8}$, $PM = 25$ ft. and $AM = 60$ ft.

Let $PQ = x$ ft. and $\angle PAM = \alpha$.

Now, $\tan\alpha = \frac{25}{60} = \frac{5}{12}$ and

$AP^2 = (60)^2 + (25)^2$, $\therefore\ AP = \sqrt{3600+625} = \sqrt{4225} = 65$ ft.

Hence, $QM = AM \tan(\theta+\alpha) = 60 \tan(\theta+\alpha)$

$$= 60 \times \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{60\left(\frac{1}{8}+\frac{5}{12}\right)}{1-\frac{1}{8}\times\frac{5}{12}} = 34\tfrac{2}{7} \text{ ft.}$$

$\therefore$ The reqd. height $PQ = QM - PM = \left(34\frac{2}{7} - 25\right)$ ft. $= 9\frac{2}{7}$ ft.

Ex. 4. From a point on the horizontal plane, the elevation of the top of a hill is 45°. After walking 300 yards towards its summit up a slope inclined at an angle of 15° to the horizon the elevation is 75°. Find the height of the hill.

Let h be the height of the hill PM and 45° be the angle of elevation of the hill top P from the point A. Let AB be a slope inclined at an angle of 15° to the horizontal line AM, and let the elevation of P from the point B, which is 300 yards from A up the slope, be 75°. To find the height h.

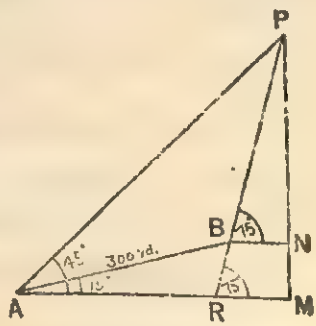


Fig. 23

Let BN be perpendicular to PM, so that BN ∥ RM.

Then, $\angle PBN = \angle PRM = 75°$, and

$\angle PAB = 45° - 15° = 30°$,

$\angle APB = 75° - 45° = 30°$,

$\therefore\ \angle ABP = 180° - (30° + 30°) = 120°$,

Now, $h = \text{PM} = \text{PA} \sin 45° = \frac{1}{\sqrt{2}}\,\text{PA}$.

Again, from $\triangle$PAB we have $\frac{\text{PA}}{\sin 120°} = \frac{\text{AB}}{\sin 30°}$,

$$\therefore \quad \text{PA} = \frac{\text{AB} \sin 120°}{\sin 30°} = \frac{300 \text{ yds.} \times \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 300\sqrt{3} \text{ yds.}$$

$$\therefore \quad h = \frac{1}{\sqrt{2}}\,\text{PA} = \frac{1}{\sqrt{2}} \times 300\sqrt{3} \text{ yds.} = 150\sqrt{6} \text{ yds.}$$

Ex. 5. A statue 10 ft. high stands on a pillar 30 ft. high. To an observer on a level with the top of the statue, the pillar and the statue subtend equal angles. Find the distance of the observer from the top of the statue.

Let the statue QP, 10 ft high, stand on the pillar PM 30 ft. high and let the observer be at the point A, Q and A being at the same height from the ground. PQ and PM subtend equal angles at A, *i.e.*, $\angle \text{PAQ} = \angle \text{PAM}$.

To find the distance AQ.

Let $\text{AQ} = d$ and $\angle \text{PAQ} = \theta$.

Now, $\because$ AP bisects $\angle$QAM,

$$\therefore \quad \frac{\text{AM}}{\text{AQ}} = \frac{\text{PM}}{\text{PQ}} = \frac{30}{10} = \frac{3}{1},$$

$$\therefore \quad \text{AM} = 3\text{AQ} = 3d.$$

Again, $\because$ Q is a right angle,

$$\therefore \quad \text{AM}^2 = \text{QM}^2 + \text{AQ}^2,$$

or, $(3d)^2 = 40^2 + d^2$, or, $8d^2 = 1600$,

or, $d^2 = 200$, $\therefore \; d = 10\sqrt{2}$.

$\therefore$ The required distance $= 10\sqrt{2}$ ft.

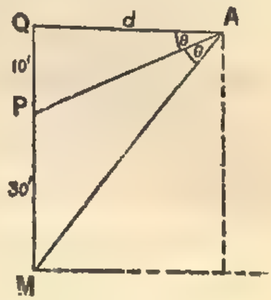


Fig. 24

Ex. 6. A person whose eye is 15 ft. above the ground finds that the angle of elevation of the top of a post is 17°19′ and the angle of depression of the foot of the post is 8°36′. Find the height of the post and its distance from the person.

Let h feet be the height of the post PM and the eye of the observer be at B, 15 ft. above the ground.

Let the angle of elevation of P and the angle of depression of M observed from B be 17°19′ and 8°36′ respectively.

To find h and the distance AM $(=x)$. Suppose $BD \perp PM$. Then $BD = AM = x$, $DM = AB = 15$ ft., $\angle PBD = 17°19'$ and $\angle MBD = 8°36'$.

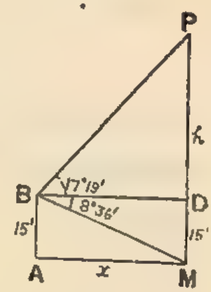


Fig. 25

Here $\tan 17°19' = \frac{PD}{BD} = \frac{h-15}{x}$, $\therefore\ x = \frac{h-15}{\tan 17°19'}$.

Again, $\tan 8°36' = \frac{DM}{BD} = \frac{15}{x}$, $\therefore\ x = \frac{15}{\tan 8°36'}$,

$$\therefore\ \frac{h-15}{\tan 17°19'} = \frac{15}{\tan 8°36'},\ \text{or, } h-15 = 15 \times \frac{\tan 17°19'}{\tan 8°36'}$$

$$\therefore\ h-15 = \frac{15 \times {}^{\cdot}31179}{{}^{\cdot}15123}\ [\text{ from log tables }] = 30{}^{\cdot}9,$$

$\therefore\ h = 30{}^{\cdot}9 + 15 = 45{}^{\cdot}9$, $\therefore$ the reqd. height $= 45{}^{\cdot}9$ ft.,

and the reqd. distance $= x = \frac{15}{\tan 8°36'} = \frac{15}{{}^{\cdot}15123}$ ft. $= 99{}^{\cdot}2$ ft.

Ex. 7. The shadow of a pillar 106 ft. high is 53 ft. on the horizontal plane on which it stands. Find the sun's altitude, having given $L \tan 63°26' = 10{}^{\cdot}3009994$, diff. for $1' = 3159$ and $\log 2 = {}^{\cdot}3010300$.

Let AM be the shadow of the pillar PM and let θ be the angle of elevation of the sun at A. To find θ. Here, $\tan \theta = \frac{106}{53} = 2$.

$\therefore$ L tan $\theta = \log 2 + 10 = 10\cdot3010300$,
but L tan $63^\circ 26' = 10\cdot3009994$

$\therefore$ diff. = 306 (briefly)

Again, 3159 is the diff. for 1′ or 60″

$\therefore$ 306 „ „ „ „ $\frac{60''}{3159} \times 306$ or $5\cdot8''$

$\therefore$ L tan θ = L tan $63^\circ 26' 5\cdot8''$, $\therefore$ $\theta = 63^\circ 26' 5\cdot8''$.

Ex. 8. The angle of elevation of an aeroplane from a point 200 ft. above a lake is 45° and the angle of depression of its reflection is 75°. Find the height of the aeroplane above the surface of the lake. Assume that the image is vertically below the aeroplane at a depth below the lake-surface equal to the height above the surface. [B. H. U. '43]

Let P be the aeroplane and P′ be the image or reflection of the aeroplane in the water below the surface BC of the lake.

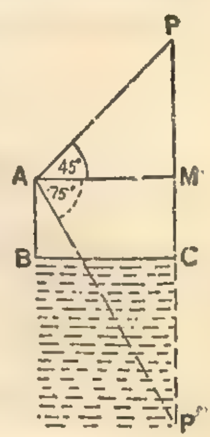


Fig. 26

Here PCP′ is a vertical line and PC = CP′. The point A is 200 ft. above the surface of the lake, *i.e.*, AB = 200 ft. From A the angle of elevation of P is 45° and the angle of depression of P′ is 75°.

To find CP, the height of the aeroplane.

Let $AM \perp PP'$, then $\angle PAM = 45^\circ$ and $\angle MAP' = 75^\circ$. $MC = AB = 200$ ft.

Now, $MP = AM \tan 45^\circ$ and $MP' = AM \tan 75^\circ$.

$$\therefore \quad \frac{MP'}{PM} = \frac{AM \tan 75^\circ}{AM \tan 45^\circ} = \frac{\tan 75^\circ}{\tan 45^\circ}$$

$$\therefore \quad \frac{MP' + PM}{MP' - PM} = \frac{\tan 75^\circ + \tan 45^\circ}{\tan 75^\circ - \tan 45^\circ} = \frac{\sin(75^\circ + 45^\circ)}{\sin(75^\circ - 45^\circ)}$$

$$= \frac{\sin 120^\circ}{\sin 30^\circ} = \frac{\sin 60^\circ}{\sin 30^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

Again, $MP'+PM=PP'=2CP$,
and $MP'-PM=MC+CP'-PM$
$=MC+(MC+PM)-PM=2MC$.

$\therefore \quad \dfrac{2CP}{2CM}=\sqrt{3}$, or, $\dfrac{CP}{CM}=\sqrt{3}$, or, $\dfrac{CP}{200}=\sqrt{3}$,

$\therefore \quad CP=200\sqrt{3}$. $\therefore$ the reqd. height $=200\sqrt{3}$ ft.

Ex. 9. A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer walking directly towards the foot observes the angle subtended by the flagstaff from two positions on his path to be the same namely θ. The distance between the two positions is d, and the angle subtended by the tower at his first position is α. Find the height of the tower and the length of the flagstaff.

Let PQ be the flagstaff fixed on the tower PM, and let PQ subtend an angle θ at the two points of observation A and B, *i.e.*, $\angle PAQ=\angle PBQ=\theta$. PM subtends an angle α at A and $AB=d$.

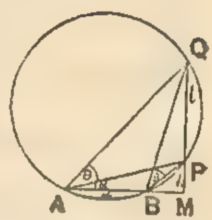


Fig. 27

To find PM and PQ.

Let $PM=h$ and $PQ=l$.

$\because$ PQ subtends equal angles at A and B on the same side of it (each $=\theta$),

$\therefore$ A, B, P and Q are concylic.

Hence, of the cyclic quadrilateral $ABPQ$, the exterior $\angle PBM=$ interior opposite $\angle AQP=90^\circ-\angle QAM=90^\circ-(\theta+\alpha)$,

Now, $d=AB=AM-BM=PM\cot PAM-PM\cot PBM$

$=h\cot\alpha-h\cot\{90^\circ-(\theta+\alpha)\}$

$=h\{\cot\alpha-\tan(\theta+\alpha)\}$

$=h\left\{\dfrac{\cos\alpha}{\sin\alpha}-\dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}\right\}$

$$=h\left\{\frac{\cos\alpha\cos(\theta+\alpha)-\sin\alpha\sin(\theta+\alpha)}{\sin\alpha\cos(\theta+\alpha)}\right\}$$

$$=h.\frac{\cos(\theta+2\alpha)}{\sin\alpha\cos(\theta+\alpha)},$$

$\therefore\quad h=d\sin\alpha\cos(\theta+\alpha)\sec(\theta+2\alpha).$

Again, from $\triangle APQ$ we have $\dfrac{l}{\sin\theta}=\dfrac{AP}{\sin AQP}=\dfrac{h\operatorname{cosec}\alpha}{\sin\{90^\circ-\theta(+\alpha)\}}$

$$=\frac{h}{\sin\alpha\cos(\theta+\alpha)}$$

$\therefore\quad l=\dfrac{h\sin\theta}{\sin\alpha\cos(\theta+\alpha)}=d\sin\theta\sec(\theta+2\alpha)$

[putting the value of h]

Ex. 10. A man, walking towards a building on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be the greatest when he is at a distance of d from the building. If θ be the observed greatest angle, find the length of the flagstaff and the height of the building. [P. U. '41]

Let PQ be the flagstaff fixed on the building PM. The point A is at a distance d from PM, so that, by the given condition the angle θ subtended by PQ at A is the greatest (*i.e.*, the angle subtended by PQ at any other point on AM is less than θ).

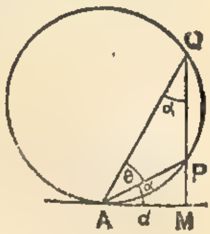


Fig 28

The lengths PQ and PM are to be determined.

It is found from geometry that the angle subtended by PQ at A on AM will be the greatest, if a circle drawn through A, P and Q touches the path AM at A.

Let $\angle PAM$ be α, then $\angle AQP=\angle PAM=\alpha$.

$\because\quad \angle QAM+\angle AQM=90^\circ,\ \therefore\ \theta+2\alpha=90^\circ.$

Now, $PQ=MQ-PM=AM\tan QAM-AM\tan PAM$

$$=d\tan(\theta+\alpha)-d\tan\alpha=d\left\{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}-\frac{\sin\alpha}{\cos\alpha}\right\}$$

$$=d\left\{\frac{\sin(\theta+\alpha)\cos\alpha-\sin\alpha\cos(\theta+\alpha)}{\cos(\theta+\alpha)\cos\alpha}\right\}$$

$$=d\,\frac{\sin\theta}{\cos(\theta+\alpha)\cos\alpha}=d\frac{2\sin\theta}{2\cos(\theta+\alpha)\cos\alpha}$$

$$=\frac{2d\sin\theta}{\cos(\theta+2\alpha)+\cos\theta}=\frac{2d\sin\theta}{\cos 90^\circ+\cos\theta}$$

$[\because\ \theta+2\alpha=90^\circ]$

$$=\frac{2d\sin\theta}{\cos\theta}\ [\because\ \cos 90^\circ=0]=2d\tan\theta.$$

Again, $\because\ \theta+2\alpha=90^\circ,\ \therefore\ \alpha=45^\circ-\frac{1}{2}\theta=\frac{\pi}{4}-\frac{\theta}{2}.$

$\therefore\quad PM=d\tan\alpha=d\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$

Ex. 11. A spherical balloon whose radius is r feet subtends at an observer's eye an angle α, when the angular elevation of its centre is β. Determine the height of the centre of the balloon. [C. U. 53]

Let the observer's eye be at A on the horizontal line AX, and let O be the centre of the spherical balloon, so that $\angle OAX=\beta$. Let AP and AQ be two tangents to the sphere so that PAQ is the angle subtended by the sphere at A.

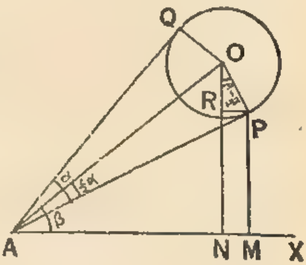


Fig. 29

Then, $\angle PAQ=\alpha$,

$\therefore\quad \angle PAO=\frac{1}{2}\angle PAQ=\frac{1}{2}\alpha.$

Draw $PM\perp AX$, $ON\perp AX$ and $PR\perp ON$.

Now, the height of the centre of the spherical balloon

$=ON=OR+RN=OR+PM$

$=OP\cos POR+AP\sin PAM$

$=r\cos(\beta-\frac{1}{2}\alpha)+OP\cot OAP\sin PAM$

$[\because\ \angle OPA=90^\circ]$

$$=r\cos(\beta-\tfrac{1}{2}\alpha)+r\cot\tfrac{1}{2}\alpha\sin(\beta-\tfrac{1}{2}\alpha)$$

$$=\frac{r}{\sin\frac{1}{2}\alpha}\{\sin\tfrac{1}{2}\alpha\cos(\beta-\tfrac{1}{2}\alpha)+\cos\tfrac{1}{2}\alpha\sin(\beta-\tfrac{1}{2}\alpha)\}$$

$$=\frac{r\sin(\frac{1}{2}\alpha+\beta-\frac{1}{2}\alpha)}{\sin\frac{1}{2}\alpha}=\frac{r\sin\beta}{\sin\frac{1}{2}\alpha}.$$

Exercise 16

1. A man walks one mile bearing an angle ϕ_1 with a fixed direction, and then another mile bearing ϕ_2 with the same direction. Find (*a*) his final distance from the starting point and (*b*) the final bearing. [C. U. '50]

2. PQ is a line 1000 yds. long; Q is due north of P and from P a distant point R bears 70° east of north; at P it bears 41°22′ east of north. Find the distance from P to R.

3. The elevation of a tower due north of a point A is θ and at a point B due west of A is ϕ. Show that its altitude is $\dfrac{AB\sin\theta\sin\phi}{\sqrt{(\sin^2\theta-\sin^2\phi)}}$. [P. U. '38 Sup.]

4. A person walking along a straight road observes that at two consecutive milestones the angles of elevation of a hill in front of him are 30° and 75°; find the height of the hill.

5. The upper two-thirds of a ship's mast subtends at a point on the deck an angle whose tangent is ·5. Find the tangent of the angle subtended by the other part of the mast at the point.

6. A post standing on a wall subtends an angle whose tangent is $\frac{3}{5}$ at a pt. on the ground and the ratio of their heights is 3 : 1. Find the tangent of the angle of elevation of the top of the post at the point.

7. From an aeroplane vertically over a straight horizontal road, the angles of depression of two consecutive milestones on the opposite sides of the aeroplane are observed to be α and β. Show that the height in miles of the aeroplane above the road is given by $\dfrac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$. [C. U. '52]

8. The altitude of a hill is observed to be 43° and after walking 500 ft. towards it up a slope inclined at 28° to the horizon the altitude is 73°. Find the vertical height of the hill above the first point of observation, given $\sin 43^\circ = \cdot 682$.

9. The upper half of a pole, seen from a point on a level with the foot of the pole, subtends an angle whose tangent is $\frac{1}{3}$. Find the tangent of the angle subtended at the same point by the whole post. [B. H. U. '42]

10. A statue 6 ft. high stands on a column 10 ft. high. To an observer on a level with the top of the statue, the column and the statue subtend equal angles. Find the distance of the observer from the top of the statue.

11. A light post on a pillar 20 ft. high subtends an angle of $\frac{\pi}{4}$ at a point on the ground and it also subtends the same angle at a point which is 20 feet nearer to the pillar. Find the height of the lightpost.

12. Determine the height of a mountain if the elevation of its top at an unknown distance from the base is 28° and at a distance 3 miles 77 yards further off from the mountain along the same line the elevation is 16°; given $\log 1\cdot 6071 = \cdot 2060$, $L \sin 12^\circ = 9\cdot 3179$, $L \sin 16^\circ = 9\cdot 4403$ and $L \sin 28^\circ = 9\cdot 6716$.

13. A post is fixed on the top of a wall standing on a horizontal plane. A person finds that the angles subtended at a point on the plane by the wall and the post are 15° and 30°. He then walks 200 feet directly towards the wall and finds that the post again subtends an angle 30°. Find the heights of the wall and the post.

14. A tower subtends an angle α at a point P on the same level as the foot of the tower, and at a second point Q, which is h feet vertically above P, the depression of the foot of the tower is β. Find the height of the tower. [C. U. '50]

15. The elevation of a hill at a place A due East of it is $45°$; at a place B due South of A the elevation is $30°$. If the distance AB is 500 yards, find the height of the hill.

16. An object is observed from 3 points A, B, C lying in a horizontal straight line which passes directly underneath the object. The angular elevation at B is twice that at A and at C three times that at A; if $AB = a$, $BC = b$, show that the height of the object is $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

17. A temple 50 yds. high stands on a hill 80 ft. high. At what point on the plane passing through the foot of the hill should an observer stand so that the temple and the hill may subtend equal angles, the height of his eye being 5 feet?

18. The angular altitude of a lighthouse from a point on the shore is $12°32'$, and from a point 500 feet nearer the altitude is $26°34'$. Find its height above the sea-level.

19. The angles of elevation of an aeroplane from two stations a mile apart and from a point half-way between the two are $60°$, $30°$ and $45°$ respectively. Find the height of the aeroplane.

20. Two towers stand on a horizontal plane and their distance apart is 120 ft. A person standing successively at the bases observes that the angular elevation of one is double that of the other, but when halfway between them, their elevations appear to be complementary. Show that the heights are 90 ft. and 40 ft. respectively. [P. U. '39 sup.]

21. A tower stood at the foot of a plane inclined to the horizon at $12°$. At a point 1000 ft. straight up the incline from the foot of the tower, the tower subtended an angle of $57°$. Find the height of the tower having given $\log 2 = \cdot 30103$, $\log 11\cdot857 = 1\cdot074105$ and $L \sin 57° = 9\cdot92359$.

22. The angles of depression of two objects 360 ft. apart from the top of a hill are $27°12'$ and $18°24'$ respectively. Find

the height of the hill, assuming the objects and the top of the hill are in the same vertical plane.

[Given, log 360 = 2·5563, log 339·4 = 2·5308,
log sin 27°12′ = $\bar{1}$·6600, log sin 18°24′ = $\bar{1}$·4992,
log sin 8°48′ = $\bar{1}$·1847] [C. U. '58]

23. The angle of elevation of the top of a tower AB from a station P due south of it (and on the same level with the base A of the tower) is θ; from another station Q due west of the former, the elevation is ϕ. If a is the distance between the stations, show that $h^2 = \dfrac{a^2}{\cot^2\phi - \cot^2\theta}$.

24. The angle of elevation of a balloon from a point h feet above a lake is ϕ and the angle of depression of its reflection in the lake is θ. Prove that the height of the balloon above the lake is $h\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}$ assuming that the image is vertically as much below the surface as the balloon is above it.

[U. P. B.]

25. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ and that at the middle point it is ϕ. Prove that the vertical height of the peak is $\dfrac{a\sin\theta\sin\phi}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}}$. [U. P. B. '55 ; P.U. '40]

26. The angles of elevation of a bird flying in a horizontal straight line from a fixed point at four successive observations are α, β, γ, δ. The observations being taken at equal intervals of time. Assuming the speed of the bird to be uniform, show that $\cot^2\alpha - \cot^2\delta = 3(\cot^2\beta - \cot^2\gamma)$. [P. U. '41]

27. The elevation of a steeple at a place due south of it is 45°, and at another place due west of the former place the elevation is 30°. If the distance between the two places be a, find the height of the steeple.

28. A person walking along a straight road observes that the greatest angle which two objects subtend is α. From the spot he walks a distance c and the objects now appear as one, their direction making an angle β with the road. Shew that the distance between the objects is $\dfrac{2c \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$.

29. On the bank of a river is a column 200 ft. high supporting a statue 30 ft. high. To an observer on the opposite bank with his eye on the level of the ground the statue subtends an angle equal to that subtended by a man 6 ft. high standing at the base of the column ; determine the breadth of the river.

Changes in the Trigonometrical ratios.

119. We are going to diccuss the changes in the trigonometrical ratios of an angle as it increases from $0°$.

Let the st. lines XOX' and YOY' intersect at right angles at O. Let a circle be drawn with centre O and with any radius OP. Here OP traces out different angles revolving from its original position OX in the anti-clockwise direction. So OP is the radius vector or the angle-generating line.

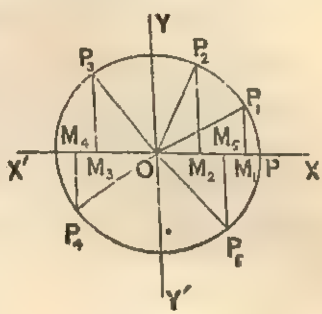


Fig. 30

When OP is coincident with OX, $\angle XOP$ is $0°$. It gradually traces out the positive angles POP_1, POP_2, etc. in different positions. Here OP_1, OP_2, etc are always taken to be positive and equal to the radius OP. P_1M_1, P_2M_2, etc are drawn perpendicular to XOX'. These perpendiculars are positive in the first two quadrants and negative in the other two. OM_1, OM_2, OM_5 are positive and OM_3, OM_4 are negative. These have been discussed before.

120. Changes in sine.

Sine $= \dfrac{\text{perpendicular}}{\text{hypotenuse}}$. When $\angle XOP = 0°$, the hypotenuse OP coincides with OX and the perpendicular (PM) from P to OX becomes zero and therefore sin $XOP = 0$. Now, in the first quadrant, as the angle intreases from 0° to 90°, the perpendicular P_1M_1 increases gradually, for the hypotenuse always remains equal to OP and both are positive.

When $\angle XOP = 90°$, both OP_1 and P_1M_1 coincide with OY and both OP_1 and P_1M_1 being equal, the sine of the angle is 1, which is the maximum value of sin XOP.

Again, in the second quadrant, as the angle increases from 90° to 180°, the hypotenuse (OP_3) remains the same but the perpendicular (P_3M_3) diminishes gradually, remaining positive. So the sin XOP will gradually diminish. Thus when $\angle XOP = 180°$, OP_3 coincides with OX′, but P_3M_3 becomes zero and consequently sin XOP is then zero. Hence as $\angle XOP$ increases from 90° to 180°, its sine diminishes from 1 to 0.

Now, if $\angle XOP$ increases further from 180° to 270° (*i. e.*, in the third quadrant), the perpendicular (P_4M_4) gradually increases numerically from zero but is negative. So sin XOP really decreases gradually from 0. Thus, when $\angle XOP = 270°$, the perpendicular (P_4M_4) coincides with OY′ and becomes equal to the hypotenuse (OP_4) which also coincides with OY′, remaining always constant ($=$OP) and positive. Then sin XOP $=$ sin $270° = -1$ (P_5M_5 being negative). Hence, as the angle increases from 180° to 270°, its sine gradually diminishes from 0 to -1.

In the fourth quadrant, as $\angle XOP$ increases from 270° to 360°, the perpendicular (PM) is negative and gradually diminishes to zero. So sin XOP numerically diminishes from 1 to 0, but really it increases from -1 to 0.

Now the discussion above may be summed up as follows, taking the angle to be θ :

As θ *increases from* $0°$ *to* $90°$, sin θ *increases* from 0 to 1.

As θ *increases from* $90°$ *to* $180°$, sin θ *diminishes* from 1 to 0.

As θ *increases from* $180°$ *to* $270°$, sin θ *diminishes* from 0 to -1.

As θ *increases from* $270°$ *to* $360°$, sin θ *increases* from -1 to 0.

121. Changes in cosine.

$\text{Cosine} = \dfrac{\text{base}}{\text{hypotenuse}}$. It appears from art. 119 that in the first quadrant as $\angle XOP$ (*i. e.*, θ) increases from $0°$ to $90°$, its base (OM) gradually decreases (see OM_1, OM_2 in fig. 30), but its hypotenuse (OP) remains unaltered ($OP = OP_1 = OP_2 = \cdots$) and both are positive. So, cos θ diminishes gradually.

When $\theta = 0°$, the base (OM) and the hypotenuse (OP) are equal and so cos $\theta = 1$. When $\theta = 90°$, both OP and OY coincide and the base (OM) becomes zero, so then cos $\theta = 0$. Hence, as θ increases from $0°$ to $90°$, cos θ gradually diminishes from 1 to 0.

In the second quadrant, as θ increases from $90°$ to $180°$, the base (OM) gradually increases numerically but is negative and when $\theta = 180°$, the base and the hypotenuse are equal. Hence, as θ increases from $90°$ to $180°$, cos θ gradually diminishes from 0 to -1.

In the third quadrant, as θ increases, the base gradually diminishes numerically, remaining negative. When $\theta = 270°$, the base becomes zero. Hence as θ increases from $180°$ to $270°$, cos θ increases from -1 to 0.

Again, in the fourth quadrant, the base (OM) is positive and gradually increases. When $\theta = 360°$, the base OM = hypotenuse OP, and then cos $\theta = 1$. Hence, as θ increases from $270°$ to $360°$, cos θ increases from 0 to 1.

The above results may be summed up as follows :

As θ increases from 0° to 90°, $\cos\theta$ diminishes from 1 to 0.

As θ increases from 90° to 180°, $\cos\theta$ diminishes from 0 to -1

As θ increases from 180° to 270°, $\cos\theta$ increases from -1 to 0.

As θ increases from 270° to 360°, $\cos\theta$ increases from 0 to 1.

122. Changes in tangent.

$$\text{Tangent} = \frac{\text{perpendicular}}{\text{base}}.$$

In the first quadrant, as θ increases, the perpendicular (PM) increases but the base (OM) decreases, both remaining positive. So the tangent increases gradually.

When $\theta = 0^\circ$, the hypotenuse OP coincides with OX and then the perpendicular$=0$ and the base$=$OP, and therefore $\tan\theta = \frac{0}{OP} = 0.$

When $\theta = 90^\circ$, OP coincides with OY and so the perpendicular$=$OP and the base$=0$, and therefore $\tan 90^\circ = \frac{OP}{0} = \infty.$

Hence in this quadrant $\tan\theta$ increases from 0 to ∞.

In the second quadrant, the perpendicular (PM) gradually diminishes, but the base is negative and increases numerically. Here we observe that as soon as the radius vector OP passes OY and enters the second quadrant, the value of $\tan\theta$ suddenly passes from ∞ to $-\infty$, *i.e.*, there is a sudden break in the value of $\tan\theta$ which changes from a very large positive to a very large negative value.

As OP coincides with OX$'$ when θ is 180°, the perpendicular becomes zero and the base$=$OP. Then $\tan\theta = \frac{0}{OP} = 0.$ Hence, in the second quadrant the value of the tangent numerically diminishes from ∞ to 0 (*i.e.*, really increases from $-\infty$ to 0).

In the third quadrant, the perpendicular increases numerically from 0 to OY′, while the base decreases numerically from OX′ to 0, both being negative. So $\tan\theta$ is positive. Since OP coincides with OY′ when θ is $270°$, the base $=0$ and the perpendicular $=$ OP, and therefore $\tan\ 270° = \frac{OP}{0} = \infty$. Hence, in the third quadrant $\tan\theta$ increases from 0 to ∞.

In the fourth quadrant the base is positive but the perpendicular is negative and so the tangent is negative. Again, in this quadrant, the perpendicular gradually diminishes while the base increases and therefore the tangent diminishes numerically. It is seen here that as soon as the radius vector (OP) passes OY′, $\tan\theta$ suddenly passes from ∞ to $-\infty$. So here also there is a sudden break in the value of $\tan\theta$. When $\theta = 360°$, OP and OX coincide, the perpendicular (PM) $=0$ and the base is equal to OP, $\therefore$ $\tan\ 360° = \frac{0}{OP} = 0$. Hence in this quadrant the tangent gradually increases from $-\infty$ to 0.

To state briefly :

As θ increases from $0°$ to $90°$, $\tan\theta$ increases from 0 to ∞, but as θ passes through $90°$, $\tan\theta$ suddenly changes from ∞ to $-\infty$.

As θ increases from $90°$ to $180°$, $\tan\theta$ increases from $-\infty$ to 0.

As θ increases from $180°$ to $270°$, $\tan\theta$ increases from 0 to ∞, but as θ passes through $270°$, $\tan\theta$ suddenly changes from ∞ to $-\infty$.

As θ increases from $270°$ to $360°$, $\tan\theta$ increases from $-\infty$ to 0.

123. Changes in cotangent.

The value of cot θ is the reciprocal of the value of tan θ, *i.e.*, $\cot\theta=\frac{1}{\tan\theta}$. Hence the changes in cot θ can be obtained from the changes in tan θ as follows :

As θ increrases from 0° to 90°, cot θ decreases from ∞ to 0 ;

As θ increases from 90° to 180°, cot θ diminishes from 0 to $-\infty$, and as θ passes through 180°, cot θ suddenly changes from $-\infty$ to $+\infty$;

As θ increases from 180° to 270°, cot θ diminishes from ∞ to 0 ;

As θ increases from 270° to 360°, cot θ diminishes from 0 to $-\infty$, but as θ passes through 360°, cot θ suddenly changes from $-\infty$ to $+\infty$.

124. Changes in secant.

As $\sec\theta=\frac{1}{\cos\theta}$, the changes in secant can be determined from those in cosine as follows :

As θ increases from 0° to 90°, sec θ increases from 1 to ∞ and immediately after sec θ suddenly changes from $+\infty$ to $-\infty$.

As θ increases from 90° to 180°, sec θ increases from $-\infty$ to -1.

As θ increases from 180° to 270°, sec θ diminishes from -1 to $-\infty$ and immediately after sec θ suddenly changes from $-\infty$ to $+\infty$.

As θ increases from 270° to 360°, sec θ diminishes from ∞ to 1.

125. Chages in cosecant.

As cosec $\theta = \frac{1}{\sin\theta}$, the changes in cosecant can be obtained from the changes in sine as follows :

As θ increases from 0° to 90°, cosec θ diminishes from ∞ to 1 ;

As θ increases from 90° to 180°, cosec θ increases from 1 to ∞ and then cosec θ suddenly changes from $+\infty$ to $-\infty$;

As θ increases from 180° to 270°, cosec θ increases from $-\infty$ to -1 ;

And as θ increases from 270° to 360°, cosec θ diminishes from -1 to $-\infty$ and as θ passes through 360°, cosec θ suddenly changes from $-\infty$ to $+\infty$.

126. We know that if an angle increases by any complete multiple of 2π (or 360°), all its trigonometrical ratios remain the same (*i.e.*, unaltered). If the radius vector OP of angle θ revolves further and makes complete revolutions the trigonometrical ratios of θ will be repeated for each complete revolution. Since there is a recurrence of the values of trigonometrical functions (ratios), they are called *periodic functions* (they being repeated after each period of 2π).

Graphs of Trigonometrical Functions

127. The graphs of trigonometrical functions ($\sin x$, $\cos\theta$, etc) can be drawn just like those of algebraic functions.

Here also the straight lines XOX' and YOY' intersecting at right angles at O are taken as the axes of co-ordinates, O being

the origin. The positive and negative directions of the axes are the same as in algebraic graphs.

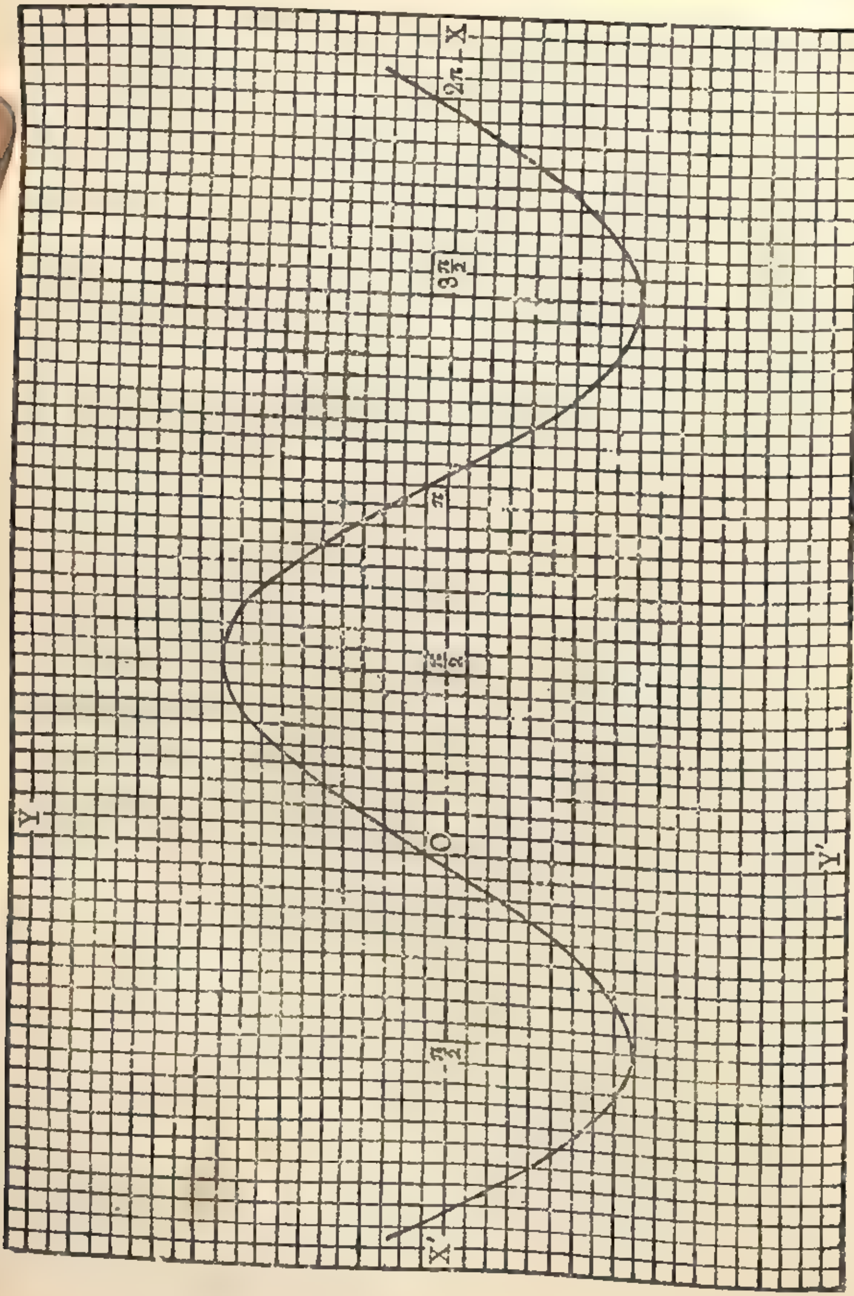
Here the values of the angles are represented by chosen lengths along the x-axis and the corresponding values of the trigonometrical ratios along the y-axis. Thus each pair of the corresponding values will be the co-ordinates of a point. Then a suitable scale being chosen, several points are plotted on the graph paper.

Now joining the plotted points free-hand we obtain the required graphs of the given trigonometrical function.

128. Sine graph or graph of sin *x*.

Let $y = \sin x$. The values of $\sin x$ (*i.e.*, y) corresponding to the different values of x can be found from the table of natural sines. Here the values of $\sin x$ (correct to 2 places of decimals) corresponding to the values of x, differing by 10°, are tabulated below.

x	-90°	-80°	-70°	-60°	-50°	-40°	30°	-20°
y or $\sin x$	-1	$-\cdot 98$	$-\cdot 94$	$-\cdot 87$	$-\cdot 77$	$-\cdot 64$	$-\cdot 50$	$-\cdot 34$
x	-10°	0°	10°	20°	30°	40°	50°	60°
y or $\sin x$	$-\cdot 17$	0	$\cdot 17$	$\cdot 34$	$\cdot 50$	$\cdot 64$	$\cdot 77$	$\cdot 87$
x	70°	80°	90°	100°	110°	120°	...	...
y or $\sin x$	$\cdot 94$	$\cdot 98$	1	$\cdot 98$	$\cdot 94$	$\cdot 87$	...	...



Graph 1 [sine graph ($-\pi$ to 2π)]

Let one small division along the x-axis represent 10° and 10 small divisions along the y-axis represent unity (*i.e.*, *1*). Now, plotting the points $(-90^\circ, -1)$, $(-80^\circ, -\cdot 98)$, etc tabulated above and joining them free-hand, we get the required graph [See graph 1].

[*N.B.* The sines of angles from 0° to 90° are given in the table of natural sines. To find the sines of angles less than 0° and greater than 90°, we take the help of the formulas $\sin(-\theta) = -\sin\theta$, $\sin(180^\circ-\theta)=\sin\theta$, $\sin(180^\circ+\theta)=-\sin\theta$, $\sin(360^\circ-\theta)=-\sin\theta$. (2) The values of the angles may be taken at intervals of 10°, 5°, 15°, or at any other suitable intervals. (3) It is convenient to represent unity (*i. e.*, *1*) by 10 or 5 small divisions along y-axis. (4) This process will be followed in drawing other graphs.]

Some special features of sine graph

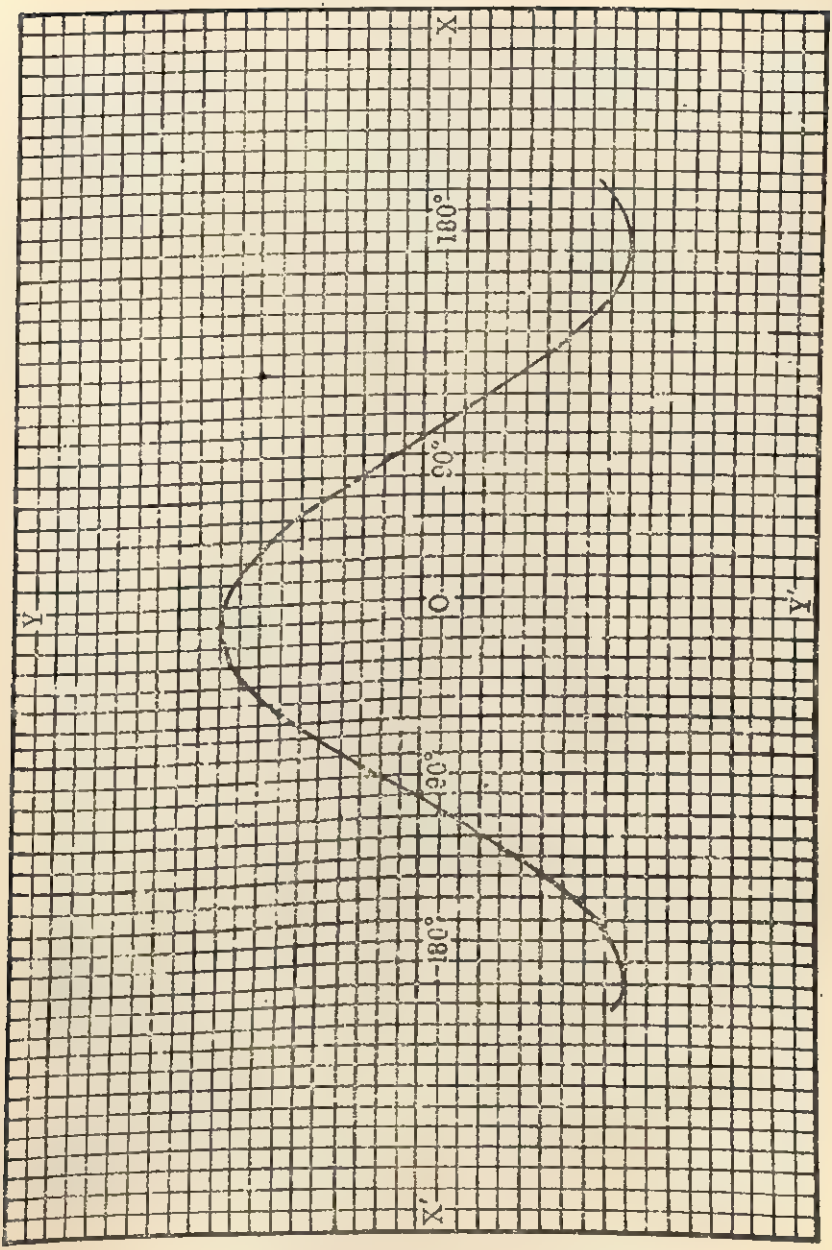
It appears from the graph that (i) it is a continuous graph extending on either side in symmetrical wave form. (ii) The ordinate of no point on the graph exceeds $+1$ or is less than -1, and so the maximum value of $\sin x$ is $+1$ and its minimum value is -1, and these values occur when the values of x are odd multiplies of 90°. (iii) At the origin and at points where x is an even multiple of 90°, the value of $\sin x$ is 0, for the graph cuts the x-axis at these points. (iv) $\because \sin(2n\pi+x)=\sin x$, $\therefore$ the portion of the graph from 0° to 2π goes on being repeated on either side.

129. Cosine graph or graph of cos *x*.

Let $y=\cos x$. Here the values of x at intervals of 15° and the corresponding values of $\cos x$ (to 2 places of decimals), obtained from the natural cosine table, are tabulated below :

x	−90°	−75°	−60°	−45°	−30°	−15°	0°	15°
y or $\cos x$	0	·26	·5	·71	·87	·97	1	·97
x	30°	45°	60°	75°	90°	105°	120°	135°
y or $\cos x$	·87	·71	·5	·26	0	−·26	−·5	−·71
x	150°	165°	180°	195°	210°	225°	240°	...
y or $\cos x$	−·87	−·97	−1	−·97	−·87	−·71	−·5	...

Let one small division along the x-axis represent 10° and 10 small divisions along the y-axis represent 1. Now, plotting the above points and joining them free-hand, we obtain the required graph. [see graph 2]



Graph 2 [cosine graph ($-180°$ to $180°$)]

Some special features of cosine graph

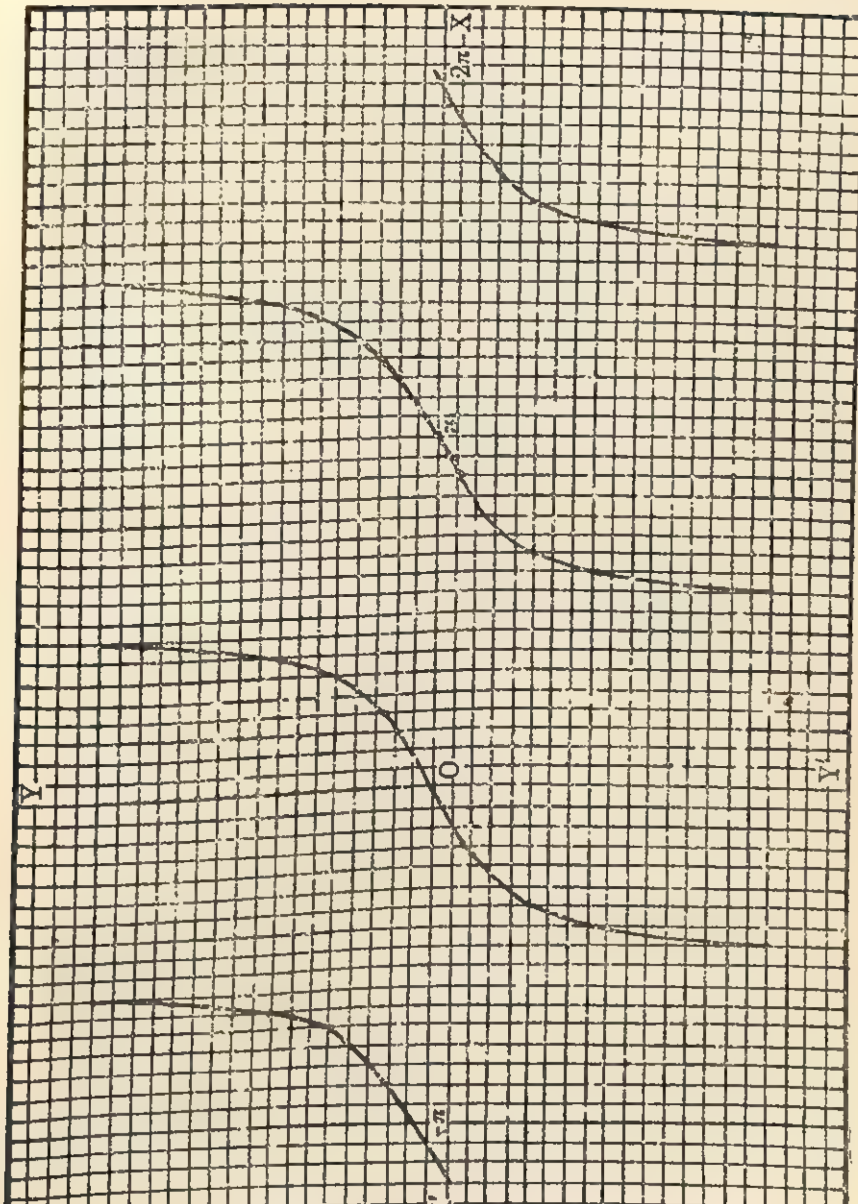
If appears from the graph that (i) the portion of the graph from $-90°$ to $90°$ is symmetrical about the y-axis. This happens because $\cos(-x)=\cos x$. (ii) As $\cos(2n\pi+x)=\cos x$, the graph repeats itself at intervals of 360° (iii) The cosine graph becomes exactly the same as the sine graph, only if the sine graph is moved to the left through 90°-space, *i.e.*, if the origin O (in graph 1) be shifted 9 small divisions to the left and this new position of O be taken as the origin. This is so because $\cos x=\sin(90°+x)$.

130. Tangent graph or graph of tan *x*.

Let $y=\tan x$. Here the values of x differing by 10° and the corresponding values of $\tan x$ (to 2 places of decimals) are tabulated below from the natural tangent table.

x	−120°	−110°	−100°	−90°	−80°	70°	−60°
y or $\tan x$	1·73	2·75	5·76	∞, $-\infty$	−5·67	−2·75	−1·73
x	−50°	−40°	−30°	−20°	−10°	0°	10°
y or $\tan x$	−1·19	−·84	−·58	−·36	−·18	0	·18
x	20°	30°	40°	50°	60°	70°	80°
y or $\tan x$	·36	·58	·84	1·19	1·73	2·75	5·67
x	90°	100°	110°	120°	...		...
y or $\tan x$	∞, $-\infty$	−5·67	−2·75	−1·73	...	...	...

Let 1 small division along the x-axis represent 10° and 3 small divisions along y-axis represent unity. Now plotting the above tabulated points and joining them free-hand, we obtain the graph of $\tan x$ [see graph 3].



Graph 3 [tanget graph ($-\pi$ to 2π)]

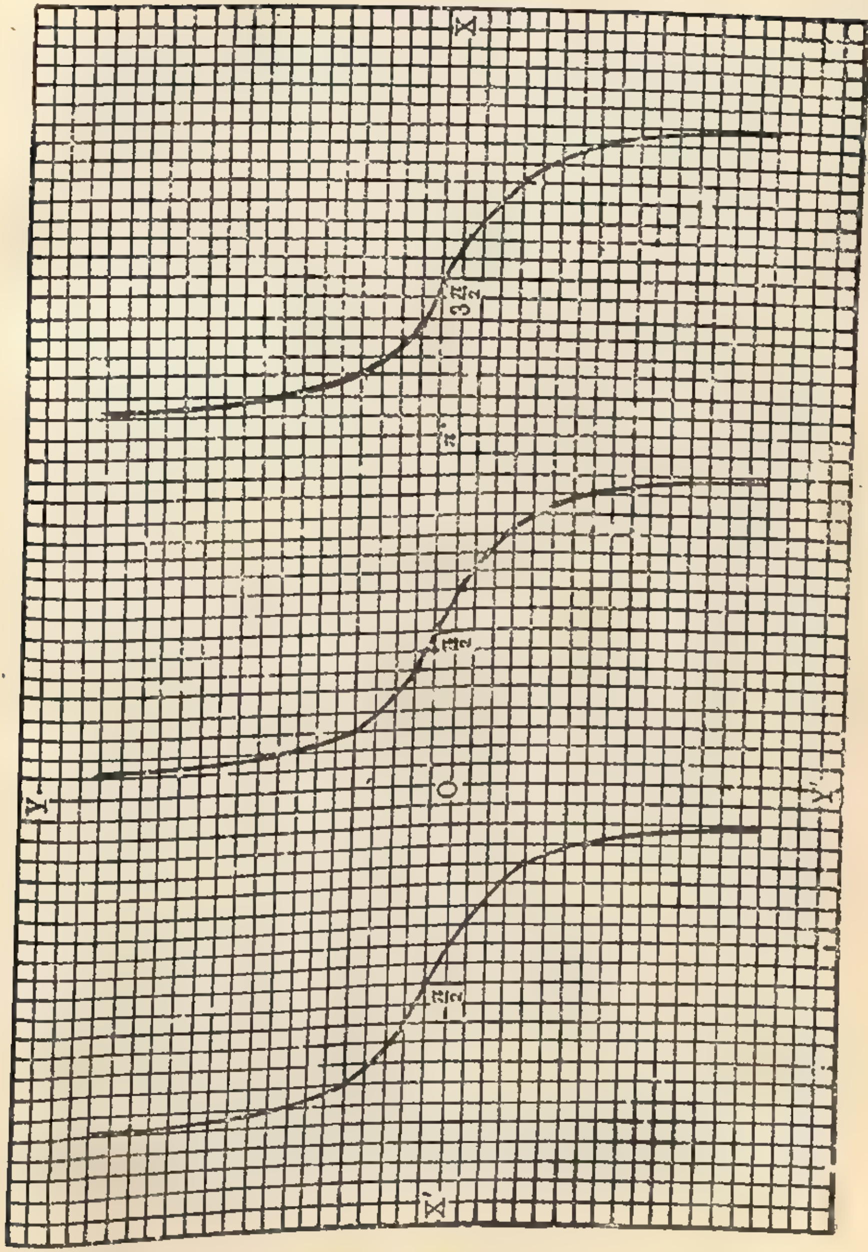
Some special features of tangent graph

If appears from the graph that (i) it is not a continuous curve, but consists of an infinite number of similar separate portions or branches, parallel to one another. The graph is disconnected at points where the value of x is an odd multiple of 90°. As x passes through these points from the left to the right, the value of tan x suddenly changes from ∞ to $-\infty$. (ii) The graph continually approaches the lines parallel to the y-axis at points corresponding to the odd multiples of 90° on either side of the x-axis. These lines are called the *asymptotes* to the curve of the graph. (iii) $\because$ $\tan(n.180^\circ + x) = \tan x$ (where n is any integer), $\therefore$ each branch is repeated at an interval of 180°, the repetition being of the branch from -90° to 90° on both sides (left and right).

131. Cotangent graph or graph of cot x.

Let $y = \cot x$. Here the values of x at intervals of 10° and the corresponding values of y (*i.e.*, of cot x) taken from the natural cotangent table are tabulated below.

x	-120°	-110°	-100°	-90°	-80°	-70°	-60°
y or $\cot x$	·58	·36	·18	0	−·18	−·36	−·58
x	-50°	-40°	-30°	-20°	-10°	0°	10°
y or $\cot x$	−·84	−1·19	−1·73	−2·75	−5·67	$-\infty$, ∞	5·67
x	20°	30°	40°	50°	60°	70°	80°
y or $\cot x$	2·75	1·73	1·19	·84	·58	·36	·18
x	90°	100°	110°	120°	...	...	...
y or $\cot x$	0	−·18	−·36	−·58	...	...	...



Graph 4 [cotangent graph $(-\pi$ to $2\pi)$]

Let 1 small division along x-axis represent 10° and 3 small divisions along y-axis represent unity or 1. Plotting the above points and joining them free-hand, we obtain the required graph [see graph 4].

Some special features of cotangent graph

(i) It is also a discontinuous graph. The continuity breaks at points where $x = 0^\circ$ or any multiple of 180°. (ii) The tangent graph becomes the cotangent graph, being shifted through 90° either to the left or to the right. (iii) $\because$ $\cot(n.\ 180^\circ + x) = \cot x$, $\therefore$ the portion of the graph between 0° and 180° are repeated again and again on either side. (iv) The graph continually approaches the lines parallel to the y-axis on both sides of the x-axis at points where $x = 0^\circ$ or any multiple of 180°, but never actually meets them. These lines are called the asymptotes to the curve.

132. Cosecant graph or graph of cosec *x*.

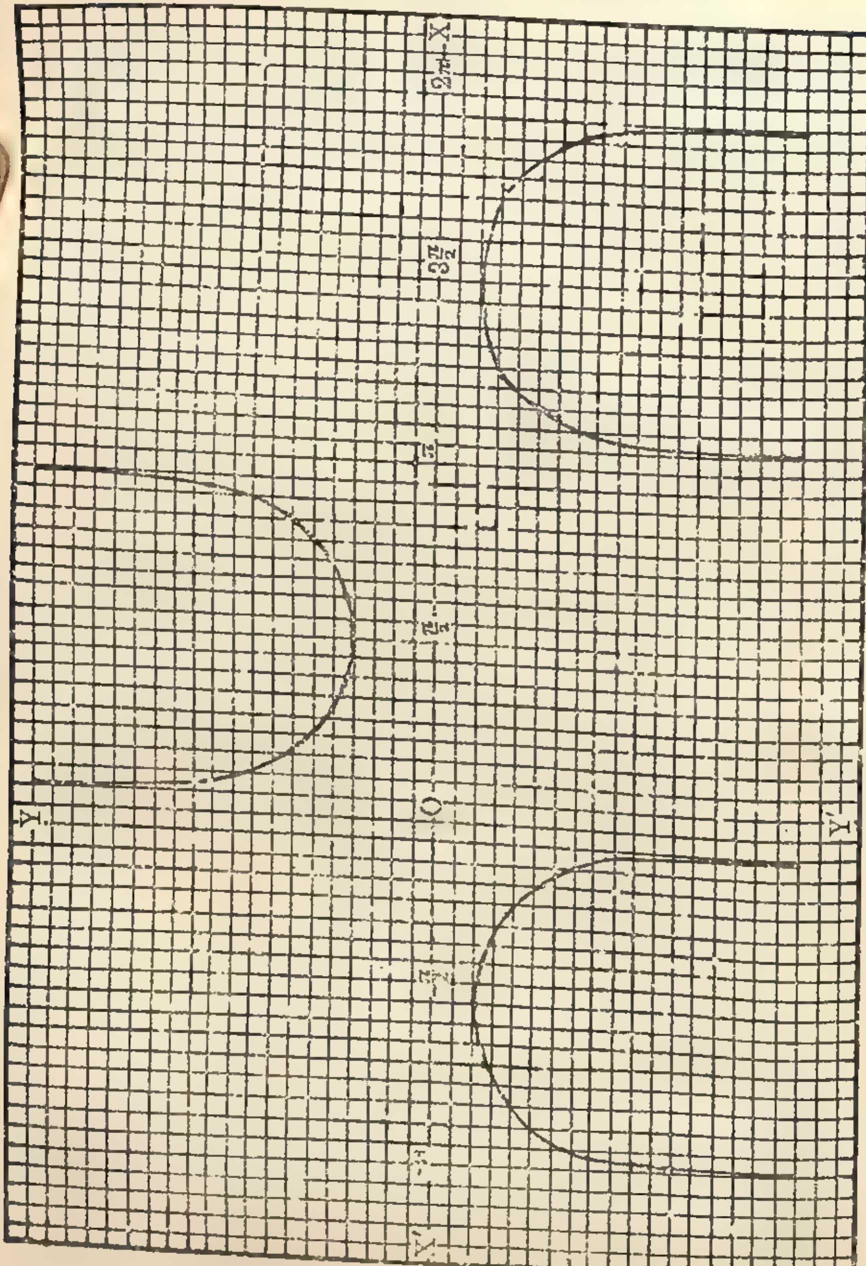
Let $y = \operatorname{cosec} x$. Here the values of x at intervals of 15° and the corresponding values of cosec x are tabulated below from the natural cosecant table [if this table be not available, the values of cosec x can be found from the natural sine table, as $\operatorname{cosec} x = \frac{1}{\sin x}$.]

x	$-105°$	$-90°$	$-75°$	$-60°$	$-45°$	$-30°$	$-15°$
y or cosec x	$-1·04$	-1	$-1·04$	$-1·15$	$-1·41$	-2	$-3·86$
x	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$
y or cosec x	3·86	2	1·41	1·15	1 04	1	1·04
x	$120°$	$135°$	$150°$	$165°$	$195°$	$210°$	$225°$
y or cosec x	1·15	1·41	2	3·86	$-3·86$	-2	$-1·41$

Let one small division along the x-axis denote $10°$ and 3 such divisions along the y-axis denote 1. Now, plotting the above points and joining them free-hand, we obtain the required graph. [see graph 5].

Some special features of cosecant graph

(i) This graph also is not continuous. It consists of an infinite number of detached branches. At angle $0°$ and at each multiple angle of $180°$, the graph is disconnected. At these points the lines parallel to the y-axis are asymptotes to the curve.



Graph 5 [cosecant graph ($-\pi$ to 2π)]

(ii) No part of the graph lies between $y=1$ and $y=-1$, for y is always greater than 1 or less than -1. So the graph lies above the x-axis within the limits of $0°$ and $180°$ and below the x-axis within the limits of $180°$ and $360°$.

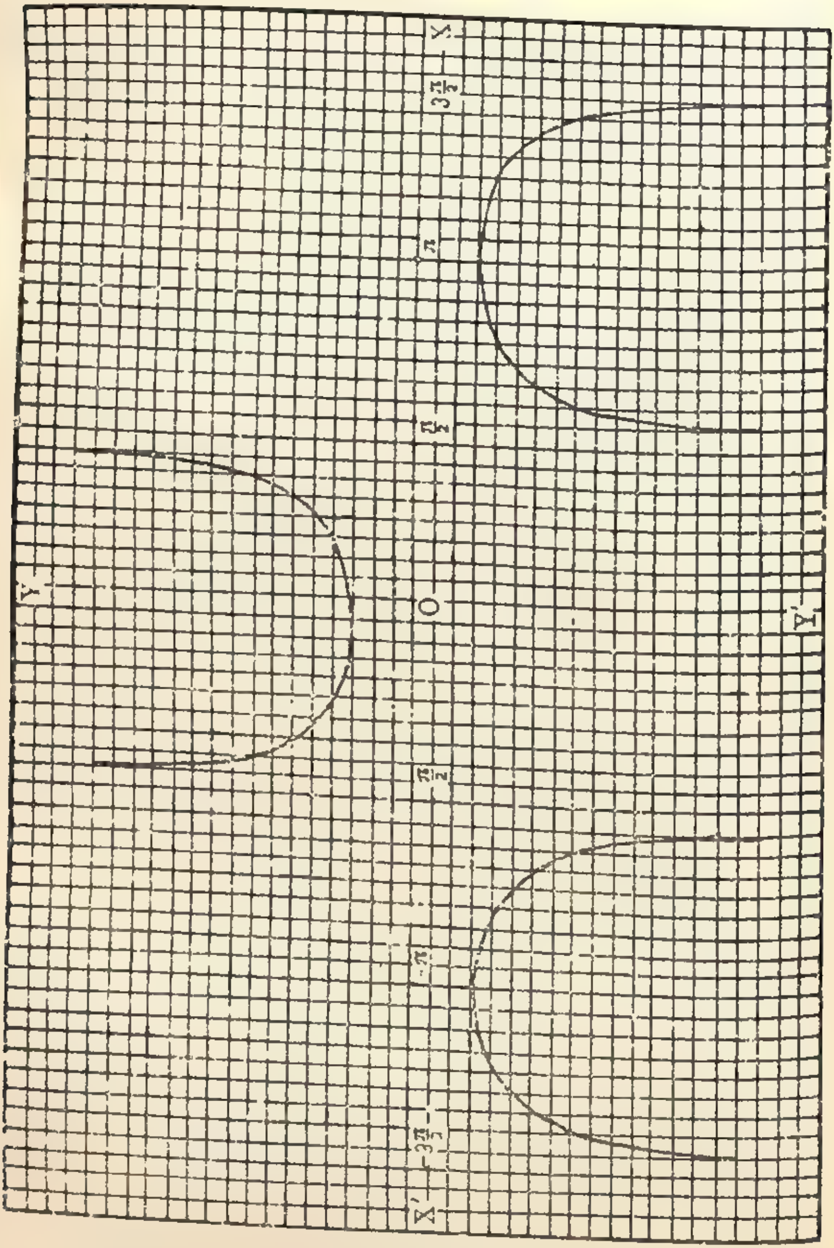
(iii) $\because$ cosec $(n.360°+x)=$ cosec x, $\therefore$ the position of the graph between $0°$ and $360°$ is repeated again and again on either side.

133. Secant graph or graph of sec *x*.

Let $y=\sec x$. Here the values of y or $\sec x$ corresponding to the values of x at intervals of $15°$ are tabulated from the natural secant table $\left[\text{or, from the natural cosine table, as } \sec x=\dfrac{1}{\cos x}.\right]$

x	$-105°$	...	$-75°$	-60	$-45°$	$-30°$	-15	$0°$
y or $\sec x$	$-3·86$	...	$3·86$	2	$1·41$	$1·15$	$1·04$	1
x	$15°$	$30°$	$45°$	$60°$	$75°$	$105°$	$120°$	$135°$
y or $\sec x$	$1·04$	$1·15$	$1·41$	2	$3·86$	$-3·86$	-2	$-1·41$
x	$150°$	$165°$	$180°$	$195°$	$210°$	$225°$	...	...
y or $\sec x$	$-1·15$	$-1·04$	-1	$-1·04$	$-1·15$	$-1·14$	...	...

Let 1 small division along the x-axis represent $10°$ and 3 such divisions along the y-axis represent unity. Now, plotting the above points and joining them free-hand, the required graph is obtained [see graph 6].



Graph 6 $\left[\text{secant graph}\left(-\frac{3\pi}{2} \text{ to } \frac{3\pi}{2}\right)\right]$

Some special features of secant graph

(i) This graph also is not continuous. There is discontinuity at points where x is an odd multiple of $90°$. The lines parallel to the y-axis at these points are asymptotes.

(ii) $\because$ Sec $(n.\ 360°+x)=\sec x$, $\therefore$ there is repetition of the graph at intervals of $360°$.

(iii) Since cosec $(90°+x)=\sec x$, the cosecant graph being shifted to the left through a $90°$-space will become the secant graph.

134. Graphs of other trigonometrical functions such as $\sin 2x$, $2\cos 3x$, etc can be drawn in a similar way. In the tabulation in these cases, the values of x are noted in the first row, the values of $2x$ or $3x$ in the second and those of the whole function ($\sin 2x$, $2\cos 3x$, etc) in the third row. The graphs are drawn, plotting the points on a chosen scale.

Again, to draw the graph of a trigonometrical expression, such as $\tan x+\cot x$, the values of x are noted in the first row of the tabulation, those of $\tan x$ and $\cot x$ in the second row and the third row respectively and the sum of the second and third rows is noted in the fourth row.

135. Graphical solution of equations.

Like algebraic equations, trigonometrical equations also can be solved graphically. Suitable methods may be adopted in different cases. (i) Draw two graphs of the expressions on the two sides of the equation. The abscissæ of the points of

intersection of the two graphs will give the required solution. *Or,* (ii) Transpose all the terms to the left hand side of the equation. Thus the right hand side will be 0. Then draw the graph of the expression on the left side. Now the abscissæ of the points, where the graph intersects the x-axis, will give the required solution of the equation.

Examples (17)

Ex. 1. Draw the graph of $y=\sin x+\cos x$ between the range $x=0$ to $x=2\pi$, and find from the graph the values of x for which (i) $y=0$, (ii) y is maximum, (iii) y is minimum.

[C. U. '34]

Here the values of x at intervals of 15° or $\frac{\pi}{12}$ and the corresponding values of sin x and cos x taken from the table are tabulated. The sum of sin x and cos x is noted in the fourth row.

x	0°	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$\sin x$	0	·26	·5	·7	·87	·97	1
$\cos x$	1	·97	·87	·7	·5	·26	0
y or $\sin x + \cos x$	1	1·23	1·37	1·4	1·37	1·23	1
x	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π	$\frac{13\pi}{12}$
$\sin x$	·97	·87	·7	·5	·26	0	−·26
$\cos x$	−·26	−·5	−·7	−·87	−·97	−1	−·97
y or $\sin x + \cos x$	·71	·37	0	−·37	−·71	−1	−1·23

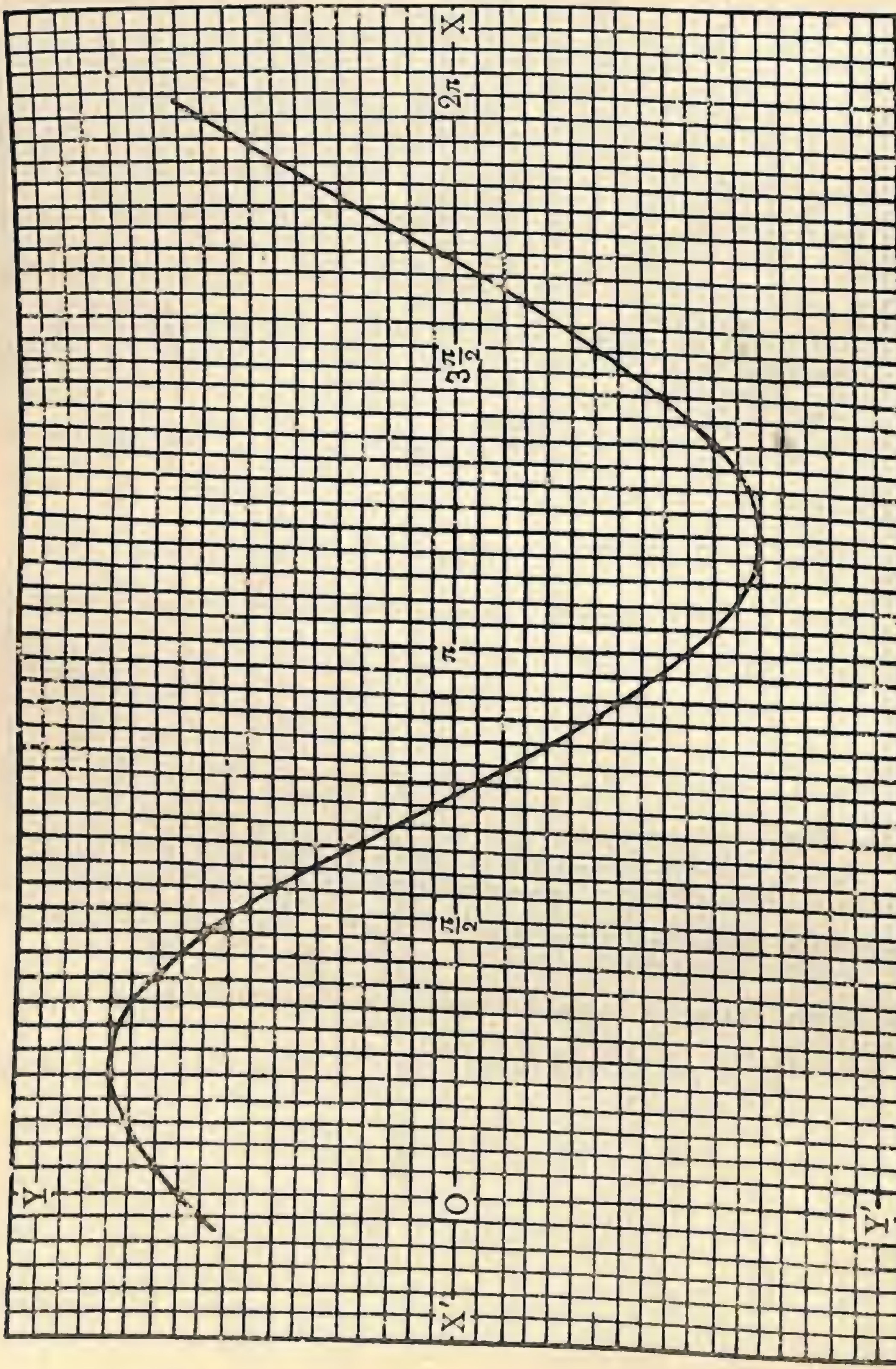
x	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$
$\sin x$	$-\cdot5$	$-\cdot7$	$-\cdot87$	$-\cdot97$	-1	$-\cdot97$	$-\cdot87$
$\cos x$	$-\cdot87$	$-\cdot7$	$-\cdot5$	$-\cdot26$	0	$\cdot26$	$\cdot5$
y or $\sin x + \cos x$	$-1\cdot37$	$-1\cdot4$	$-1\cdot37$	$-1\cdot23$	-1	$-\cdot71$	$-\cdot37$
x	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π	...	...	...
$\sin x$	$-\cdot7$	$-\cdot5$	$-\cdot26$	0	...	...	...
$\cos x$	$\cdot7$	$\cdot87$	$\cdot97$	1	...	...	...
y or $\sin x + \cos x$	0	$\cdot37$	$\cdot71$	1	...	...	...

Let 2 small divisions along x-axis represent 15° and 10 small divisions along y-axis represent 1 (unity). Now, plotting the above points and joining them free-hand, the required graph is obtained [see graph 7].

Again, (i) $y=0$ at the points where the graph intersects the x-axis. Here $x=\frac{3\pi}{4}$ and $x=\frac{7\pi}{4}$ at the two points where the graph cuts the x-axis. Hence $x=\frac{3\pi}{4}$ and $x=\frac{7\pi}{4}$, for which $y=0$.

(ii) It appears from the graph that the abscissa of the highest point of the graph is $\frac{\pi}{4}$. Hence y is maximum when $x=\frac{\pi}{4}$.

(iii) Again, the abscissa of the lowest point of the graph is seen to be $\frac{5\pi}{4}$. Hence y is minimum, when $x=\frac{5\pi}{4}$ or 225°.



Graph 7 [graph of $y = \sin x + \cos x$]

[*N.B.* Here the equation may be put as $y=\sin x+\cos x$

$$=\sqrt{2}\left(\sin x.\ \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}.\cos x\right)$$

$$=\sqrt{2}\,(\sin x\cos 45^\circ+\sin 45^\circ\cos x)$$

$$=\sqrt{2}\sin(45^\circ+x).$$

Hence, if the values of $\sin(45^\circ+x)$ as obtained from the tables be multiplied by $\sqrt{2}$ or 1·414, the corresponding values of y can be had.]

Ex. 2. Solve graphically the equation $\cot\theta-\tan\theta=2$ between $\theta=0$ and $\theta=\pi$. [C. U. '49]

Here $\cot\theta-\tan\theta=2$,

or, $\dfrac{1}{\tan\theta}-\tan\theta=2$, or, $1-\tan^2\theta=2\tan\theta$,

or, $\dfrac{2\tan\theta}{1-\tan^2\theta}=1$, or, $\tan 2\theta=1$.

Now, by drawing the graphs of $y=\tan 2\theta$ and $y=1$, the solution is obtaind from the abscissæ of the points of their intersection.

(i) $y=\tan 2\theta$.

θ	0°	15°	22·5°	30°	37·5°	52·5°	60°	75°
2θ	0°	30°	45°	60°	75°	105°	120°	150°
$\tan 2\theta$ or y	0°	·58	1	1·73	3·73	− 3 73	− 1·73	—·58
θ	90°	105°	120°	127·5°	142·5°	150°	180°	...
2θ	180°	210°	240°	255°	285°	300°	360°	...
$\tan 2\theta$ or y	0	·58	1·73	3·73	− 3·73	− 1·73	0	...

10. Solve graphically the equation $\tan x = \cos x$, between $x=0$ and $x=\frac{1}{2}\pi$. [C. U. '56]

11. Solve graphically the equation $\tan x = 2x$ between the values $x=0$ and $x=\frac{\pi}{2}$. [C. U. '39]

12. Solve graphically $\sin 2x = \sin x$, giving only those values of x which lie between $x=0$ and $x=2\pi$. [C. U. '40]

13. Solve graphically the equation $x - \tan x = 0$ between $x=0$ and $x=\frac{\pi}{2}$. [Ans. $x=0$] [C.U. '45]

14. Solve graphically the equation $5 \sin \theta + 2 \cos \theta = 5$ between $\theta = 0^\circ$ to $\theta = 270^\circ$. [Ans. $46^\circ 25'$ (App.) and 90°] [C. U. '47]

15. Sketch a period of the tangent graph $y = \tan x$, including $x = \frac{1}{2}\pi$ and discuss the behaviour of the graph near $x = \frac{\pi}{2}$. [C. U. '51]

16. Sketch the graphs of $y=x$, $y=\sin x$ and $y=\tan x$ in the range $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ with reference to the same axes of x and y. From the nature of the graph near the origin, can you suggest any relation among them at the origin? [C. U. '52]

CHAPTER III

CO-ORDINATE GEOMETRY

Some important Results

[Circle] :

1. (*a*) Equation of a circle :

(i) standard form : $x^2+y^2=a^2$, where centre is $(0, 0)$, radius $=a$.

(ii) general form : $x^2+y^2+2gx+2fy+c=0$, where centre is $(-g, -f)$, radius$=\sqrt{g^2+f^2-c}$.

(*b*) Equation of a circle with the st. line joining the points (x_1, y_1) and (x_2, y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

2. (*a*) Equation of the tangent to the circle $x^2+y^2=a^2$ at (x_1, y_1) is $xx_1+yy_1=a^2$.

(*b*) Equn. of tangent to $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

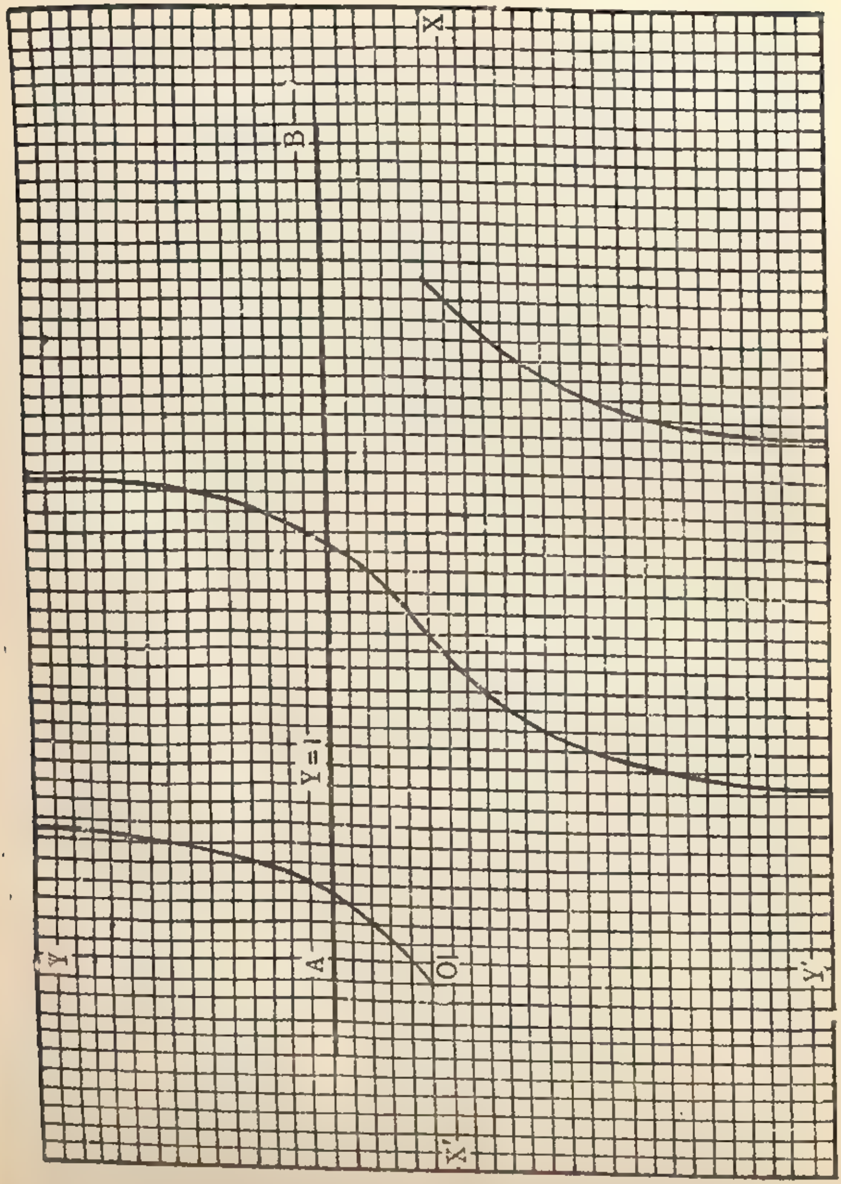
(*c*) Condition that the line $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$ is $c=\pm a\sqrt{1+m^2}$:

(i) $y=mx+a\sqrt{1+m^2}$ and (ii) $y=mx-a\sqrt{1+m^2}$ are tangents to $x^2+y^2=a^2$ for all values of m, and in (i) the point of contact is $\left(-\dfrac{am}{\sqrt{1+m^2}}, \dfrac{a}{\sqrt{1+m^2}}\right)$ and in (ii) the point of contact is $\left(\dfrac{am}{\sqrt{1+m^2}}, \dfrac{-a}{\sqrt{1+m^2}}\right)$.

3. (i) Equation of the normal to the circle $x^2+y^2=a^2$ at (x_1, y_1) is $\dfrac{x}{x_1}=\dfrac{y}{y_1}$ or $xy_1-yx_1=0$.

(ii) Equn. of the normal to $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1) is $x(y_1+f)-y(x_1+g)-fx_1+gy_1=0$.

Equation of the chord which is bisected at a given point
h(x-h) + k (y-k) = 0.



Graph 8

Let 1 small division along the x-axis denote 5° and 5 such divisions along the y-axis denote 1. Now plotting the points and joining them free-hand, the graph of $y=\tan 2\theta$ is obtained [see graph 8].

(ii) Again, the st. line, parallel to the x-axis on positive side at a distance of 1 unit from it, is the graph of $y=1$.

This st. line (the second graph) cuts the first graph at two points (within the given limits) whose abscissæ are $22{\cdot}5^\circ$ and $112{\cdot}5^\circ$.

Hence, the sotution is $\theta=22{\cdot}5^\circ$ and $112{\cdot}5^\circ$.

Ex. 3. Solve graphically the equation $\operatorname{cosec} x=\cot x+\sqrt{3}$ between $x=0$ and $x=\pi$. [C. U. '42]

Here $\operatorname{cosec} x=\cot x+\sqrt{3}$, or, $\dfrac{1}{\sin x}=\dfrac{\cos x}{\sin x}+\sqrt{3}$,

or, $1=\cos x+\sqrt{3}\sin x$, or, $\frac{1}{2}=\cos x.\frac{1}{2}+\dfrac{\sqrt{3}}{2}.\sin x$,

or, $\frac{1}{2}=\cos x\cos 60^\circ+\sin x\sin 60^\circ$, or, $\frac{1}{2}=\cos(x-60^\circ)$.

Hence, by drawing the graphs of $y=\frac{1}{2}={\cdot}5$ and $y=\cos(x-60^\circ)$, the solution can be obtained from the points of intersection of the graphs.

(i) The graph of $y={\cdot}5$ is the st. line AB parallel to the x-axis at a distance of ${\cdot}5$ units from it on its positive side [see graph 9].

4. (a) Length of the chord of the circle $x^2+y^2=a^2$ intercepted by the line $y=mx+c$ is $\frac{2}{\sqrt{1+m^2}}\cdot\sqrt{a^2(1+m^2)-c^2}$.

(b) Length of the tangent from an external point (x_1, y_1) to the circle (i) $x^2+y^2=a^2$ and (ii) $x^2+y^2+2gx+2fy+c=0$ is

(i) $\sqrt{x^2+y^2-a^2}$ and (ii) $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.

[Parabola] :

1. Equation of the Parabola :

Standard form $y^2=4ax$, where latus rectum$=4a$, focus is $(a, 0)$, directrix is $x+a=0$, and extremities of the latus rectum are $(a, \pm 2a)$.

2. (i) Equation of the tangent at (x_1, y_1) is $yy_1=2a(x+x_1)$.

(ii) Condition that $y=mx+c$ may touch the parabola is $c=\frac{a}{m}$ (where m is not zero).

(iii) The line $y=mx+\frac{a}{m}$ is a tangent to the parabola for all values of m (except 0), and the pt. of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

3. Equation of the normal at (x_1, y_1) is $y-y_1=-\frac{y_1}{2a}(x-x_1)$.

4. Length of the chord of the parabola intercepted by the st. line $y=mx+c$ is $\frac{4}{m^2}\sqrt{a(a-mc)(1+m^2)}$.

5. Equation of the diameter is $y=\frac{2a}{m}$.

6. Parametric representation (i) $x=at^2$, $y=2at$.

(ii) Equation of the tangent is $y=\frac{x}{t}+at$.

(iii) Equation of the normal is $y+xt=2at+at^3$.

[**Ellipse**] :

1. Equation of the ellipse in standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) Here latus rectum $=2a(1-e^2)=2\frac{b^2}{a}$;

(ii) eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}$, or, $\sqrt{1-\frac{b^2}{a^2}}$

(iii) the directrices are $x=\pm\frac{a}{e}$

(iv) focus S is $(ae, 0)$, S′ is $(-ae, 0)$

(v) the focal distances of P (x_1, y_1) are SP $=a-ex_1$, S′P $=a+ex_1$. $\therefore$ their sum $=2a$.

2. (i) Equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(ii) Condition that the st. line $y=mx+c$ is a tangent to the ellipse is $c=\pm\sqrt{a^2m^2+b^2}$.

(iii) The st. lines $y=mx\pm\sqrt{a^2m^2+b^2}$ are tangents to the ellipse for all values of m their points of contact being

$$\left(-\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right),$$

and $$\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, -\frac{b^2}{\sqrt{a^2m^2+b^2}}\right).$$

3. Equation of the normal at (x_1, y_1) is $\dfrac{x-x_1}{\frac{x_1}{a^2}}=\dfrac{y-y_1}{\frac{y_1}{b^2}}$.

4. Length of the chord intercepted by the st. line $y=mx+c$ on the ellipse is $\dfrac{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}$

5. Equation of the diameter is $y=-\frac{b^2}{a^2m}\cdot x$.

6. (i) Auxiliary circle is $x^2+y^2=a^2$.

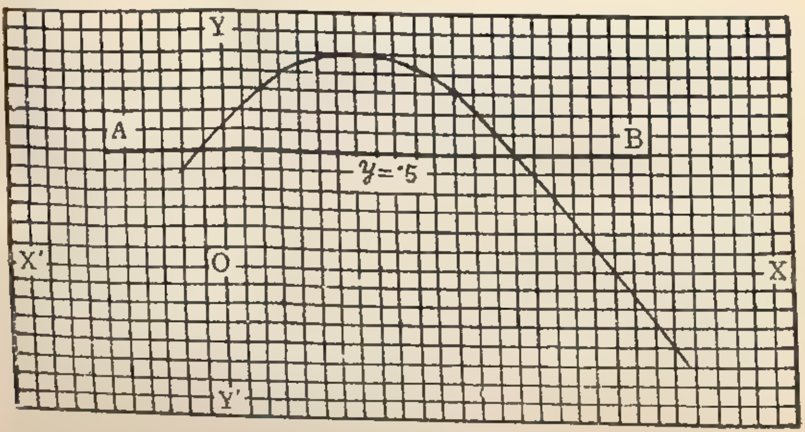
(ii) Director circle is $x^2+y^2=a^2+b^2$.

7. Parametric representation : $x=a\cos\theta$, $y=b\sin\theta$.

(ii) $y = \cos(x - 60°)$, from which we have

x	0°	15°	30°	45°	60°	75°	90°
$x - 60°$	$-60°$	$-45°$	$-30°$	$-15°$	0°	15°	30°
y or $\cos(x-60°)$	·5	·71	·87	·97	1	·97	·87
x	105°	120°	135°	150°	165°	180°	...
$x - 60°$	45°	60°	75°	90°	105°	120°	...
y or $\cos(x-60°)$	·71	·5	·26	0	$-$·26	$-$·5	...

Now, let one small division along the x-axis denote 7·5° and that along y-axis denote ·1 unit. On this scale the points (0°, ·5), (15°, ·71), etc are plotted and joined free-hand, and the graph of $y = \cos(x - 60°)$ is obtained [graph 9].



Graph 9

Here, the two graphs intersect at two points within the given limits and the abscissæ of the points are 0° and 120°.

Hence, the required solution is $x=0^\circ$ and 120°.

Ex. 4. Draw the graph of $3 \sin x+4 \cos x$. What is its maximum value? [C. U. '50]

Let, $y=3 \sin x+4 \cos x$.

Here, the values of x at intervals of 15° and the corresponding values of $\sin x$ and $\cos x$ are first determined from the table and then the values of x and y are tabulated.

x	-15°	0°	15°	30°	45°	60°	75°	90°	105°	120°
$3 \sin x$	$-\cdot78$	0	$\cdot78$	1·5	2·12	2·61	2·91	3	2·91	2·61
$4 \cos x$	3·86	4	3·86	3·46	2·83	2·00	1·03	0	$-1\cdot04$	$-2\cdot00$
y	3·08	4	4·64	4·96	4·95	4·61	3·94	3	1·87	·61

Let two small divisions along the x-axis and 4 small divisions along the y-axis represent 15° and unity (*i. e.*, 1) respectively. Now, plotting the above points and joining them free-hand, the required graph is obtained. [see graph 10].

[Hyperbola] :

1. Equation of hyperbola in standard form is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Here, (i) latus rectum $=2a(e^2-1)=2\frac{b^2}{a}$;

(ii) eccentricity $e=\frac{\sqrt{a^2+b^2}}{a}$ [for rectangular hyperbola $a=b$, $\therefore$ $e=\sqrt{2}$.]

(iii) focus S is $(ae, 0)$, S' is $(-ae, 0)$.

(iv) Directrices are $x=\pm\frac{a}{e}$.

(v) focal distances of $P(x_1, y_1)$ are

$$SP=ex_1-a, \quad S'P=ex_1+a.$$

2. (i) Equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(ii) Condition that the st. line $y=mx+c$ is a tangent to the hyperbola is $c=\pm\sqrt{a^2m^2-b^2}$.

(iii) The st. lines $y=mx\pm\sqrt{a^2m^2-b^2}$ are tangents to the hyperbola for all values of m, their points of contact being $\left(-\frac{a^2m}{\sqrt{a^2m^2-b^2}}, -\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$ and $\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

3. The equation of the normal at the pt. (x_1, y_1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{-\frac{y_1}{b^2}}.$$

4. The length of the chord of the hyperbola intercepted by the line $y=mx+c$ is $\frac{2ab\sqrt{1+m^2}\sqrt{c^2-a^2m^2+b^2}}{a^2m^2-b^2}$.

5. Equation of the diameter is $y=\frac{b^2}{a^2m}x$.

6. Equation of the asymptotes are $y=\pm\frac{b}{a}x$.

CIRCLE

Definition : A circle is the locus of a point which moves so as to be always at a constant distance from a fixed point.

The fixed point is called the centre and the given distance is called the radius of the circle.

The line traced out by the point is called the circumference of the circle. It is to be noted that the term circle strictly denotes the space bounded by the circumference, but the term is often used to denote the circumference itself, if there be no confusion.

137. *To find the equation of a circle whose centre is the origin.*

Let XOX′ and YOY′ be the two axes of co-ordinates, the origin O be the centre of the circle and a be its radius.

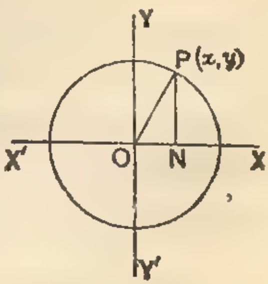


Fig. 1

Let P be any point on the circumference of the circle and let its co-ordinates be (x, y).

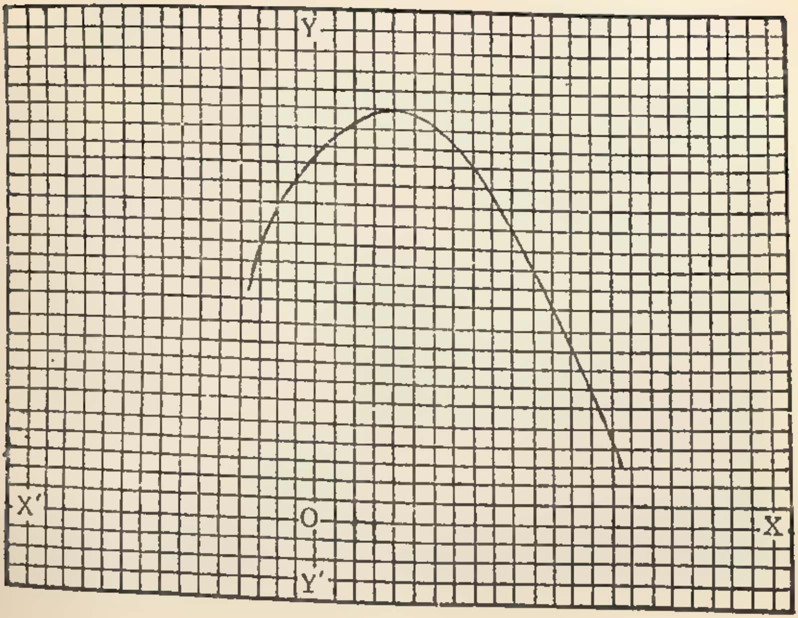
Join OP and draw PN perpendicular to OX. Then ON $= x$ and PN $= y$.

Now, in the right-angled $\triangle$OPN,

$ON^2 + PN^2 = OP^2$,

$\therefore\ \ \boldsymbol{x^2 + y^2 = a^2}$, this is the required equation of the circle.

[N. B. The equation of the second degree represents a circle, if the coefficients of x^2 and y^2 are equal and there is no 'xy' term.]



Graph 10

It is apparent from the graph that the maximum value of y is 5. Hence, the required maximum value$=5$.

Ex. 5. Solve graphically the equation $2\sin^2 x = \cos 2x$, giving only those solutions of x which lie between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

[C. U. '46, '48]

Here, $2\sin^2 x = \cos 2x$, or, $2\sin^2 x - \cos 2x = 0$,

or, $1 - \cos 2x - \cos 2x = 0$, or, $1 - 2\cos 2x = 0$.

Let, $y=1-2\cos 2x$, then $y=0$. The required solution will be obtained from the points of intersection of the graphs of the two equations.

The x-axis itself is the graph of the equation $y=0$.

For drawing the graph of $y=1-2\cos 2x$, the values of x at intervals of 15° and the corresponding values of y are tabulated from the natural cosine table.

x	-90°	-75°	-60°	-45°	-30°	-15°	0°
$\cos 2x$	-1	$-{\cdot}87$	$-{\cdot}5$	0	${\cdot}5$	${\cdot}87$	1
$1-2\cos 2x$ or y	3	2 74	2	1	0	$-{\cdot}74$	-1
x	15°	30°	45°	60°	75°	90°	105°
$\cos 2x$	${\cdot}87$	${\cdot}5$	0	$-{\cdot}5$	$-{\cdot}87$	-1	$-{\cdot}87$
$1-2\cos 2x$ or y	$-{\cdot}74$	0	1	2	$2{\cdot}74$	3	$2{\cdot}74$
x	120°	135°	150°	165°	180°	195°	210°
$\cos 2x$	$-{\cdot}5$	0	${\cdot}5$	${\cdot}87$	1	${\cdot}87$	${\cdot}5$
$1-2\cos 2x$ or y	2	1	0	$-{\cdot}74$	-1	$-{\cdot}74$	0
x	225°	240°	255°	270°	...	...	...
$\cos 2x$	0	$-{\cdot}5$	$-{\cdot}87$	-1	...	...	...
$1-2\cos 2x$ or y	1	2	$2{\cdot}74$	3	...	...	..

Let 2 small divisions along the x-axis and 5 small divisions along the y-axis represent 15° and 1 (unity) respectively. Plotting the above points and joining them free-hand, the graph of $y=1-2\cos 2x$ is obtained [graph 11].

138. *To find the equation of a circle, the co-ordinates of its centre being* (α, β).

Let C be the centre and a the radius of the circle. Take any point $P(x, y)$ on the circumference.

Draw CM and PN perpendicular to OX and draw $CR \perp PN$.

Then, $OM = \alpha$, $CM = \beta$, $ON = x$, $PN = y$, and $CR = MN$, $PR = PN - RN = PN - CM$.

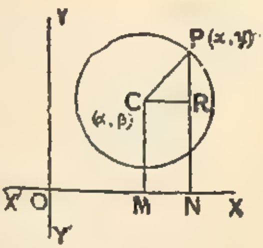


Fig. 2

Now, in the right-angled $\triangle CPR$,

$$CP^2 = CR^2 + PR^2$$
$$= (ON - OM)^2 + (PN - CM)^2$$
$$\therefore \quad a^2 = (x - \alpha)^2 + (y - \beta)^2.$$

$\therefore$ The required equation of the circle is $(\boldsymbol{x} - \alpha)^2 + (\boldsymbol{y} - \beta)^2 = \boldsymbol{a}^2$.

Corollary : If $(-\alpha, \beta)$, $(\alpha, -\beta)$ or $(-\alpha, -\beta)$ be the centre of the circle, then its equation will be $(x + \alpha)^2 + (y - \beta)^2 = a^2$, $(x - \alpha)^2 + (y + \beta)^2 = a^2$, or, $(x + \alpha)^2 + (y + \beta)^2 = a^2$ respectively.

Conversely, (α, β) is the centre of the circle

$$(x - \alpha)^2 + (y - \beta)^2 = a^2.$$

[**N. B.** In the Art. 138, (i) if the origin O be on the circumference, then $OM^2 + MC^2 = a^2$, i.e., $\alpha^2 + \beta^2 = a^2$. So the equation of the circle will be $x^2 + y^2 - 2\alpha x - 2\beta y = 0$.

(ii) If the origin be the centre, then $\alpha = 0$ and $\beta = 0$; so the equation of the circle is $x^2 + y^2 = a^2$.

(iii) If the origin is not on the circumference and the centre lies on the x-axis, then $\beta = 0$, so the equation becomes $(x - \alpha)^2 + y^2 = a^2$.

(iv) If the origin lies on the circumference and the diameter of the circle coincides with the x-axis, then $\beta = 0$ and $a = \alpha$; so the equation will be $x^2 + y^2 - 2\alpha x = 0$.]

139. *To show that the equation* $x^2+y^2+2gx+2fy+c=0$ *always represents a circle.*

Transposing c, and adding g^2 and f^2 to both sides of the given equation we have $(x^2+2gx+g^2)+(y^2+2fy+f^2)$ $=g^2+f^2-c$, or, $(x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$, and evidently this represents a circle whose centre is the point $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$.

[N. B. If $g^2+f^2>c$, then the radius of the circle will be real, i.e., the circle will be real ;

If $g^2+f^2<c$, then the radius and therefore the circle will be imaginary ;

If $g^2+f^2=c$, then the radius vanishes, and so the circle becomes a point coinciding with $(-g, -f)$. Such a circle is called a point-circle.]

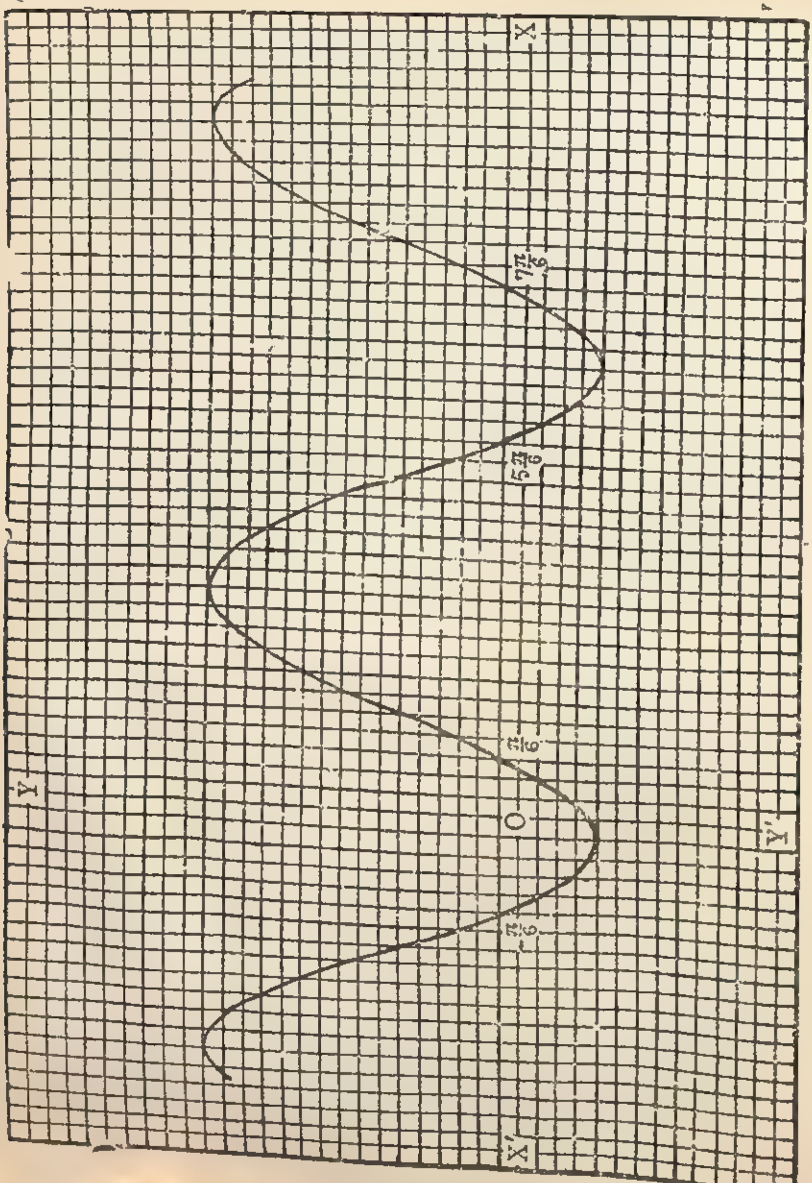
140. *To find the condition that the general equation of the second degree* $ax^2+by^2+2hxy+2gx+2fy+c=0$ *may represent a circle.*

Let us take the equation $ax^2+ay^2+2gx+2fy+c=0$.

Dividing it by a we have $x^2+y^2+2.\frac{g}{a}x+2.\frac{f}{a}y+\frac{c}{a}=0$, this is the equation of a circle whose centre is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius is $\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$.

Thus we find that $ax^2+by^2+2hxy+2gx+2fy+c=0$ will represent a circle if $a=b$ and $h=0$.

Hence the general equation of the second degree represents a circle if the coefficients of x^2 and y^2 be the same and the coefficient of xy be zero.



Graph 11

This graph intersects the x-axis (*i.e.*, the graph of $y=0$) at four points between the given limits. The values of x at these points are $-30°$, $30°$, $150°$, $210°$, the ordinates being 0.

Hence, the required solution is $x=-\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$.

Exercise 17

Draw the graphs of :—

1. Sin 2θ and cos 2θ between $\theta=0°$ and $\theta=210°$.

2. $\frac{1}{\cos 2x}$ and $\frac{1}{\cot 2x}$ between $x=-\frac{\pi}{2}$ and $x=\pi$.

3. Sin $x+\cos x$ and cos $2x$ between $x=0°$ and $x=\pi$.

4. Tan $3x$ between $x=0°$ and $x=\frac{\pi}{2}$.

5. Sec $\frac{x}{2}$ between $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$.

6. Trace the graph of $y=\sec x$ from $x=0°$ to $x=90°$, tabulating at the intervals of $10°$.

7. Draw the graphs of $y=\sin x$ and $y=\cos x$ between $x=0$ and $x=\pi$. Find the points where the graphs intersect.

[Ans. $x=\frac{1}{4}\pi$] [C. U. '36]

8. Obtain graphically a solution of the equation tan $x=1$ between $x=0$ and $x=\frac{1}{2}\pi$. [Ans. $\frac{1}{4}\pi$] [C. U. '37]

9. Draw the graphs of $y=\sin x$ and $y=\cos x$ between $x=0$ and $x=\pi$. Find the values of x between these limits which satisfy the equation $\sin x=\cos x$.

141. *To find the equation of a circle whose diameter is the line joining the two given points* (x_1, y_1) and (x_2, y_2).

Let A (x_1, y_1) and B (x_2, y_2) be the given points,

and P (x, y) be a point on the circumference of the circle.

Join AB, AP and BP.

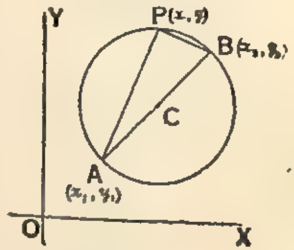


Fig. 3

Now, the gradient of AB is $\dfrac{y-y_1}{x-x_1}$;

and the gradient of BP is $\dfrac{y-y_2}{x-x_2}$;

but since AB is a diameter, $\therefore$ $\angle$APB in the semicircle is a right angle, i.e., AP, BP are perpendicular to each other. $\therefore$ the product of their gradients $= -1$.

$$\therefore \quad \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1,$$

or, $\quad (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$,

$\therefore$ the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Examples (18)

Ex. 1. Find the equation of a circle whose centre is the origin and radius is 4.

Here from the formula $x^2+y^2=a^2$, we have $x^2+y^2=4^2$.

$\therefore$ the required equation is $x^2+y^2=16$.

Ex. 2. Find the equation of a circle passing through the point (2, 3) and having the centre at the origin.

Let $x^2+y^2=a^2$ be the equation.

$\because$ the circle passes through the point (2, 3),

$\therefore \quad 2^2+3^2=a^2, \quad \therefore \quad a^2=13.$

$\therefore$ the required equation is $x^2+y^2=13$.

Ex. 3. Find the equation of a circle whose centre is at the point (3, 4) and radius is 6.

Evidently the required equation is $(x-3)^2+(y-4)^2=6^2$,

or, $x^2-6x+9+y^2-8y+16=36$,

or, $x^2+y^2-6x-8y-11=0$.

Ex. 4. Find the equation of a circle passing through the point $(-1, 2)$ and having the centre at the point $(2, -3)$.

Let $(x-2)^2+(y+3)^2=a^2$ be the equation.

Since the circle passes through the point $(-1, 2)$,

$\therefore$ $(-1-2)^2+(2+3)^2=a^2$, $\therefore$ $a^2=9+25=34$.

$\therefore$ the required equation is $(x-2)^2+(y+3)^2=34$,

or, $x^2-4x+4+y^2+6y+9=34$,

or, $x^2+y^2-4x+6y-21=0$.

Ex. 5. Find the co-ordinates of the centre and the radius of the circle $x^2+y^2-4x+6y-12=0$.

$\because$ $x^2+y^2-4x+6y-12=0$, $\therefore$ transposing -12 and adding $(4+9)$ or 13 to both sides we have

$$(x^2-4x+4)+(y^2+6y+9)=25,$$

or, $(x-2)^2+(y+3)^2=25=(5)^2$.

$\therefore$ the required co-ordinates of the centre are $(2, -3)$ and the radius is 5.

Ex. 6. Find the co-ordinates of the centre and the radius of the circle $3x^2+3y^2-5x-6y+4=0$.

Here $3x^2+3y^2-5x-6y+4=0$,

or, $x^2+y^2-\frac{5}{3}x-2y+\frac{4}{3}=0$ (dividing by 3)

or, $(x^2-2.\frac{5}{6}x+\frac{25}{36})+(y^2-2y+1)=1+\frac{25}{36}-\frac{4}{3}$

or, $\left(x-\frac{5}{6}\right)^2+(y-1)^2=\frac{13}{36}=\left(\frac{\sqrt{13}}{6}\right)^2$.

$\therefore$ the co-ordinates of the centre are $\left(\frac{5}{6}, 1\right)$

and the radius$=\frac{\sqrt{13}}{6}$.

Ex. 7. Find the equation of the circle passing through the points (0, 1), (1, 0), (2, 1).

Let $x^2+y^2+2gx+2fy+c=0$ be the equation.

Since the circle passes through the points (0, 1), (1, 0), (2, 1),

$\therefore$ the equation is satisfied by these co-ordinates.

$\therefore$ $1+2f+c=0 \cdots\cdots(1)$, $\quad 1+2g+c=0\cdots\cdots(2)$

and $5+4g+2f+c=0\cdots\cdots(3)$.

Now, subtracting (2) from (1) we have $2f-2g=0$, $\therefore$ $f=g$.

Subtracting (3) from (1) we have $-4-4g=0$, $\therefore$ $g=-1$;

$\therefore$ $f=-1$.

$\therefore$ from (1) we have $1-2+c=0$, $\therefore$ $c=1$.

$\therefore$ the required equation is $x^2+y^2-2x-2y+1=0$.

Ex. 8. Find the equation of the circle passing through the vertices of the triangle formed by joining the points (0, 0), $(a, 0)$, $(0, b)$.

Let $x^2+y^2+2gx+2fy+c=0$ be the equation.

$\because$ the circle passes through the points (0,0), $(a, 0)$, $(0, b)$,

$\therefore$ the equation must be satisfied by these co-ordinates.

Hence

$$\left.\begin{aligned} c&=0\cdots(1)\\ a^2+2ag+c&=0\cdots(2)\\ b^2+2bf+c&=0\cdots(3)\end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} a^2+2ag&=0\ldots(4)\\ b^2+2bf&=0\ldots(5)\end{aligned}\right\}$$

Now from (4) we have $2ag=-a^2$, $\therefore$ $g=-\frac{a}{2}$,

and from (5) we have $f=-\frac{b}{2}$.

$\therefore$ the required equation is $x^2+y^2-ax-by=0$.

Ex. 9. Find the equation of the circle passing through the points (1, 2), (3, -4) and (5, -6).

Let the equation be $x^2+y^2+2gx+2fy+c=0$.

Since the circle passes through the given points,

$$\therefore \quad 5+2g+4f+c=0\cdots(\text{i})$$
$$25+6g-8f+c=0\cdots(\text{ii})$$
$$61+10g-12f+c=0\ldots(\text{iii})$$

Solving them we have $g=-11$, $f=-2$, $c=25$.

$\therefore$ the required equation is $x^2+y^2-22x-4y+25=0$.

Ex. 10. Show that the four points $(0, 0)$, $(1, 1)$, $(5, -5)$ and $(6, -4)$ are concyclic and find the equation of the circle.

Suppose the circle $x^2+y^2+2gx+2fy+c=0$ passes through the points $(0, 0)$, $(1, 1)$ and $(5, -5)$.

Hence
$$\left.\begin{aligned} c&=0\cdots(\text{i})\\ 2+2g+2f&=0\cdots(\text{ii})\\ 50+10g-10f&=0\cdots(\text{iii})\end{aligned}\right\}\quad\text{or,}\quad \left.\begin{aligned} c&=0\cdots(\text{iv})\\ 1+g+f&=0\cdots(\text{v})\\ 5+g-f&=0\cdots(\text{vi})\end{aligned}\right\}$$

Solving the equations we have $g=-3$, $f=2$, $c=0$.

$\therefore$ the circle $x^2+y^2-6x+4y=0$ passes through these three points.

Now, we find that the equation $x^2+y^2-6x+4y=0$ is satisfied by $(6, -4)$ also. $\therefore$ the point $(6, -4)$ lies on the circle.

$\therefore$ the four points are concyclic and the equation of the circle is $x^2+y^2-6x+4y=0$.

Ex. 11. Find the equation of a circle passing through the points $(1, -2)$ and $(4, -3)$ and having its centre on the straight line $3x+4y=7$.

Let the circle be $x^2+y^2+2gx+2fy+c=0$, so its centre is $(-g, -f)$.

$\because$ the circle passes through the points $(1, -2)$ and $(4, -3)$,

$\therefore$ $5+2g-4f+c=0\cdots(\text{i})$ and $25+8g-6f+c=0\cdots(\text{ii})$.

Again, since the centre $(-g, -f)$ is on the straight line $3x+4y=7$, $\therefore$ $-3g-4f=7\cdots\cdots(\text{iii})$.

Solving the equations (i), (ii), (iii) we have

$$g=-\tfrac{47}{15},\ f=\tfrac{3}{5},\ c=\tfrac{11}{3}.$$

$\therefore$ the required equation is $x^2+y^2-\frac{94}{15}x+\frac{6}{5}y+\frac{11}{3}=0$,

$$\text{or,}\quad 15x^2+15y^2-94x+18y+55=0.$$

Ex. 12. Find the equation of a circle which passes through the origin and cuts off intercepts 3 and 4 from the axes.

Let C be the centre of the circle passing through the origin and let it cut the x-axis at A and the y-axis at B.

Draw $CM \perp OA$ and $CN \perp OB$.

Now, $OA = 3$, $OB = 4$.

$\because$ $OM = \frac{1}{2}OA = \frac{3}{2}$ and $ON = \frac{1}{2}OB = 2$,

$\therefore$ the co-ordinates of the centre are $(\frac{3}{2}, 2)$.

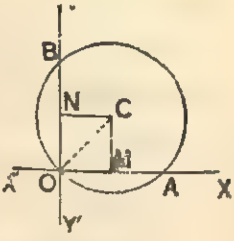


Fig. 4

Again, $OC^2 = OM^2 + CM^2 = (\frac{3}{2})^2 + (2)^2$
$= \frac{9}{4} + 4 = \frac{9+16}{4} = \frac{25}{4}$,

$\therefore$ the radius $= OC = \frac{5}{2}$.

$\therefore$ the required equation of the circle is

$(x - \frac{3}{2})^2 + (y-2)^2 = \frac{25}{4}$,

or, $x^2 - 3x + \frac{9}{4} + y^2 - 4y + 4 = \frac{25}{4}$,

or, $x^2 + y^2 - 3x - 4y + \frac{25}{4} = \frac{25}{4}$, or, $x^2 + y^2 - 3x - 4y = 0$.

Ex. 13. Show that the centres of the three circles $x^2 + y^2 - 10x + 9 = 0$, $x^2 + y^2 - 6x + 2y + 1 = 0$ and $x^2 + y^2 - 18x - 4y + 21 = 0$ lie on a straight line and also find the equation of the straight line.

Here the equations of the three circles are respectively $(x-5)^2 + y^2 = 16$, $(x-3)^2 + (y+1)^2 = 9$, $(x-9)^2 + (y-2)^2 = 64$.

$\therefore$ their centres are $(5, 0)$, $(3, -1)$ and $(9, 2)$ respectively.

Now, the equation of the straight line passing through $(5, 0)$ and $(3, -1)$ is $y - 0 = \dfrac{0-(-1)}{5-3}\left(x - 5\right)$

or, $y = \frac{1}{2}(x-5)$, or, $2y = x - 5$, or, $x - 2y = 5$.

We find that this equation is satisfied by the co-ordinates $(9, 2)$.

$\therefore$ the three given centres lie on a straight line whose equation is $x - 2y = 5$.

Ex. 14. Find the equation of the circle concentric with the circle $x^2+y^2+3x-4y+5=0$ and passing through the point $(-1, 2)$.

The equation of the circle concentric with the circle $x^2+y^2+3x-4y+5=0$ is $x^2+y^2+3x-4y+c=0$.

Since this circle passes through the point $(-1, 2)$,

$\therefore\quad 1+4-3-8+c=0\quad(\because\ x=-1, y=2),\quad\therefore\ c=6.$

Hence the required equation of the circle is

$$x^2+y^2+3x-4y+6=0.$$

Ex. 15. Find the radius of the circle whose centre is at the point $(1, -2)$ and which passes through the point of intersection of the straight lines $3x+y=14$ and $2x+5y=18$.

[C. U. '45]

Solving the given equations, we have $x=4$, $y=2$.

$\therefore$ The given straight lines intersect at the point $(4, 2)$.

Hence the distance between the given point $(1, -2)$ and the point $(4, 2)$ is the required radius.

$\therefore$ the required radius $=\sqrt{(4-1)^2+(2+2)^2}=5.$

Exercise 18

1. Find the equation of the circle

(i) whose centre is $(0, 0)$ and radius is 3 ;

(ii) whose centre is $(0, -3)$ and radius is 5 ;

(iii) whose centre is $(2, 3)$ and radius is 4 ;

(iv) whose centre is $(-3, 4)$ and radius is 3 ;

(v) whose centre is $(-3, 2)$ and radius is $\sqrt{6}$;

(vi) whose centre is $(-1, -2)$ and which passes through the point $(1, -3)$;

(vii) whose centre is $(2, 3)$ and which passes through the point $(5, 7)$.

[C. U. '57]

2. Find the *centre* and *radius* of the following circles :

(i) $x^2+y^2=4$; (ii) $x^2+y^2=5$;

(iii) $x^2+y^2+2x-4y+3=0$;

(iv) $x^2+y^2-4x-6y-12=0$;

(v) $2x^2+2y^2+3x-5y-2=0$;

(vi) $x^2+y^2-6x+14y+33=0$. [C. U. '51]

3. Find the area of the triangle formed by joining the centres of the three circles

$$x^2+y^2-10x-6y+30=0,\ x^2+y^2+4x-2y=0 \text{ and}$$
$$x^2+y^2-6x+6y+9=0.$$

4. Find the area of the triangle formed by joining the centres of the circles $x^2+y^2-2x-8y+12=0$,

$$x^2+y^2+2x+4y-4=0,\ x^2+y^2-4x-14y+17=0\ ;$$

what is your conclusion from the result you find ?

5. Show that the centres of the circles

$$x^2+y^2-18x-6y+26=0,\ x^2+y^2+10x+8y+25=0$$

and $x^2+y^2-6x+5=0$ are collinear.

6. Prove that the centres of the three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=1$, $x^2+y^2-12x+4y=1$ lie on one straight line, and find the equation of the straight line.

[C. U. (B. Sc.) '20 ; H. S. '68]

7. Show that the centres of the circles $x^2+y^2-6x+3=0$, $x^2+y^2+14x-8y+55=0$ and $x^2+y^2-16x+4y+19=0$ lie on a straight line and find the equation of that straight line.

8. Determine the centres and radii of the circles $x^2+y^2-2x+2y-7=0$, $x^2+y^2-6x-2y-6=0$ and $x^2+y^2-8x-4y-5=0$, and verify that the centres lie on a straight line, whose equation you are required to obtain.

[C. U. '50]

9. Find the equation of the circle which passes through the points :—

(i) $(0, 0), (4, 0), (0, 5)$ (ii) $(1, 2), (3, -4), (5, -6)$

(iii) $(1, 1), (2, -1), (3, 2)$ (iv) $(2, -1), (2, 3), (4, -1)$ [C. U. '12]

(v) $(0, 0), (0, 5), (-2, -1)$. [Mysore '46]

10. Show that the four points $(2, 0)$, $(5, -3)$, $(2, -6)$ and $(-1, -3)$ lie on a circle and find the equation of that circle.

11. Find the equation of the circle passing through the points $(-3, 7)$, $(2, 2)$ and having the centre on the straight line $2x-3y=13$.

12. Find the equation of the circle which passes through the points $(4, 3)$ and $(-2, 5)$ and has its centre on the line $2x-3y=4$. [C. U. '50]

√13. Find the equation of the circle passing through the origin and making intercepts 6 and 8 on the axes.

14. Find the equation of the circle passing through the point $(0, 0)$ and the points at which the straight line $3x+4y=12$ meets the axes.

15. Find the equation of the circle concentric with the circle $x^2+y^2-3x+4y+6=0$ and passing through the point $(3, 2)$.

16. Find the equation of the circle concentric with the circle $2x^2+2y^2+5x-6y+2=0$ and having radius equal to $\frac{\sqrt{5}}{4}$.

17. Find the equation to the circle whose centre is at the origin and which meets the straight line $\frac{x}{5}-\frac{y}{6}=1$ on the axis of y. [C. U. '40]

√18. Obtain the equation of the circle which passes through the two points on the axis of x which are at a distance 2 from the origin and whose radius is 5. [C. U. '52]

19. Obtain the equation of the circle which passes through the two points on the axis of x which are at a distance 4 from the origin and whose radius is 5. [C. U. '52]

20. ABCD is a square whose side is a; taking AB and AD as axes prove that the equation to the circle circumscribing the square is $x^2+y^2=a(x+y)$. [C. U. 1951]

21. Find the area of the equilateral triangle inscribed in the circle $x^2+y^2+2gx+2fy+c=0$.

[**Hints**: If ABC be the equilateral triangle and O be the centre of the circle; then $\triangle ABC=3\triangle OBC=\frac{3}{2}OB.OC \sin BOC$.]

22. Obtain the equation of the circle whose centre is the point (2, 3) and which passes through the intersection of the lines $3x-2y-1=0$ and $4x+y-27=0$. [C. U. '47]

23. A triangle has its vertices at the points (0, 1), (−2, 0) and (1, 0). Find the equation of the straight lines forming the sides. Find also the equation of the circle which has as a diameter that side of the triangle which lies in the first quadrant. [C. U. '50 (Sup.)]

24. Find the equation of the circle circumscribing the triangle formed by the lines $x+y=6$, $2x+y=4$ and $x+2y=5$. [Agra '45]

[**Hints**: Solving the equations two at a time, find the points of intersection of the three pairs. Let $x^2+y^2+2gx+2fy+c=0$ be the required equation. It is satisfied by the co-ordinates of the points of intersection. Now find the values of g, f, c......]

25. Prove that the radii of the circles $x^2+y^2=1$, $x^2+y^2-2x-6y=6$, $x^2+y^2-4x-12y=9$ are in A.P. [C.U. '17]

142. *To find the points of intersection of a straight line with a circle.*

Let $x^2+y^2=a^2\cdots(1)$ be the circle and $y=mx+c\cdots(2)$ be the straight line.

Putting $mx+c$ for y from (2) in (1) we have
$x^2+(mx+c)^2=a^2$, or, $(m^2+1)x^2+2mcx+(c^2-a^2)=0$, this is a quadratic equation in x. Two values of x can be had from this equation. Then substituting these values in

$y=mx+c$, we have two values of y also. Thus we have the co-ordinates of two points, and these are the co-ordinates of the two points of intersection of a straight line and a circle.

If $4m^2c^2-4(m^2+1)(c^2-a^2)>0$, i.e., if $a^2(1+m^2)-c^2>0$, then the two points are distinct and real.

If $a^2(1+m^2)-c^2=0$, then the two points will be coincident.

If $a^2(1+m^2)-c^2<0$, then the two points will be imaginary, i.e., the straight line will not meet the circle.

142. (*b*) *To find the length of the chord intercepted on a given straight line by a given circle.*

Let the equations of the circle and the straight line be $x^2+y^2=a^2$...(1) and $y=mx+c$......(2) respectively. Let them intersect at the points P and Q whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively.

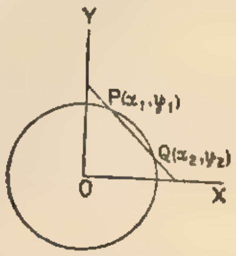


Fig. 4 (a)

Now substituting $mx+c$ for y from (2) in (1) we have $x^2+(mx+c)^2=a^2$, or, $(m^2+1)x^2+2mcx+(c^2-a^2)=0$...(3), it is a quadratic equation in x. The two roots of this equation will be the abscissæ of P and Q. Hence x_1 and x_2 will be the roots of the equation-(3).

$\therefore$ the sum of the roots

$$x_1+x_2=-\frac{2mc}{1+m^2}, \text{ and their product } x_1x_2=\frac{c^2-a^2}{1+m^2}.$$

$$\therefore \quad x_2-x_1=\sqrt{(x_1+x_2)^2-4x_1x_2}=\sqrt{\frac{4m^2c^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{1+m^2}}$$

$$=\frac{2}{1+m^2}\sqrt{m^2c^2-(1+m^2)(c^2-a^2)}$$

$$=\frac{2}{1+m^2}\sqrt{a^2(1+m^2)-c^2}\text{......(4)}$$

Since the points P and Q are on the straight line-(2), the equation $y=mx+c$ is satisfied by the co-ordinates of P and Q.

$\therefore$ $y_1=mx_1+c$ and $y_2=mx_2+c$.

$\therefore$ $y_2-y_1=m(x_2-x_1)$.

$\therefore$ the length of the chord $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}=(x_2-x_1)\sqrt{1+m^2}$

$=\dfrac{2}{1+m^2}\cdot\sqrt{a^2(1+m^2)-c^2}\cdot\sqrt{1+m^2}$ [putting the value of x_2-x_1 from (4)]

$=\dfrac{2}{\sqrt{1+m^2}}\cdot\sqrt{a^2(1+m^2)-c^2}$......(5).

Alternative Proof : Let the st. line $y=mx+c$ cut the circle $x^2+y^2=a^2$ at P and Q.

To find the length of the chord PQ.

From the centre O draw OM perpendicular to the chord PQ.

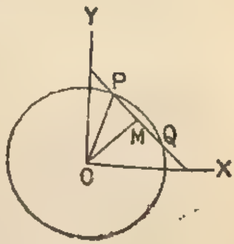


Fig. 4 (b)

Then $OM=\dfrac{m.0-0+c}{\sqrt{1+m^2}}=\dfrac{c}{\sqrt{1+m^2}}$.

and $OP=a$ [the radius of the circle]

Since the perpendicular from the centre of a circle to a chord bisects the chord, $\therefore$ $PQ=2PM$.

$\because$ $PM^2=OP^2-OM^2$, $\angle M$ being a rt. angle,

$\therefore$ $PM=\sqrt{OP^2-OM^2}=\sqrt{a^2-\dfrac{c^2}{1+m^2}}$

$=\dfrac{1}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2)-c^2}$.

$\therefore$ The length of the chord $=2PM$

$=\dfrac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2)-c^2}$.

[N. B. The length of the chord PQ gradually diminishes as the two points of intersection P and Q of the chord gradually approach each other and thus when P and Q will coincide at a point, the chord will be a tangent to the circle and in this position the length of the chord $PQ=0$.

Hence the condition for the st. line-(2) being a tangent to the circle-(1) is $PQ=0$, i.e., $\frac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2)-c^2}=0$,

or, $c^2=a^2(1+m^2)$, i.e., $c=\pm a\sqrt{1+m^2}$.]

143. Tangent

A straight line intersects a circle at two points. In the case the straight line meets the circle in two coincident points (i. e., when the two points of intersection coincide), it is a tangent to the circle. The point at which the tangent meets the circle is called the *point of contact.*

The tangent is always perpendicular to the radius drawn through the point of contact.

144. *To find the condition that the straight line* $y=mx+c$ *may touch a circle* $x^2+y^2=a^2$.

The given st. line is $y=mx+c$.

Putting this value of y in the equation $x^2+y^2=a^2$, we have $x^2+(mx+c)^2=a^2$,

or, $(m^2+1)x^2+2mcx+(c^2-a^2)=0\ldots(1)$

The given st. line will touch the circle if the roots of the equation-(1) are equal, the condition for which is

$$4m^2c^2-4(m^2+1)(c^2-a^2)=0,$$

or, $m^2c^2-(m^2c^2-m^2a^2+c^2-a^2)=0$,

or, $m^2a^2-c^2+a^2=0$, or, $c^2=a^2(m^2+1)$,

or, $c=\pm a\sqrt{m^2+1}$.

Hence the condition for the st. line $y=mx+c$ being a tangent to (i.e., touching) the circle $x^2+y^2=a^2$ is $c=\pm a\sqrt{m^2+1}$.

[N. B. We, therefore, have that each of the two st. lines $y=mx\pm a\sqrt{m^2+1}$ is always a tangent to the circle $x^2+y^2=a^2$.]

145. (*a*) *To find the equation of a tangent to the circle $x^2+y^2=a^2$ at the point (x_1, y_1) on it.*

Suppose (x_2, y_2) to be another point on the given circle.

Now, the equation of the straight line passing through the pts. (x_1, y_1) and (x_2, y_2) is $y-y_1=\dfrac{y_1-y_2}{x_1-x_2}(x-x_1)$; but since (x_1, y_1) and (x_2, y_2) lie on the circle $x^2+y^2=a^2$,

$\therefore$ we have $\left.\begin{array}{r} x_1^2+y_1^2=a^2 \\ \text{and } x_2^2+y_2^2=a^2 \end{array}\right\}$ from which we have

$$(x_1^2-x_2^2)+(y_1^2-y_2^2)=0,$$

or, $(x_1+x_2)(x_1-x_2)+(y_1+y_2)(y_1-y_2)=0,$

$$\therefore \quad \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}.$$

$\therefore$ The equation of the chord passing through those two points is $y-y_1=-\dfrac{x_1+x_2}{y_1+y_2}(x-x_1)$, and this st. line will be a tangent to the circle when the two points coincide, and in this position $x_2=x_1$ and $y_2=y_1$.

Hence the equation of the tangent is

$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1)=-\frac{x_1}{y_1}(x-x_1),$$

or, $yy_1-y_1^2=-xx_1+x_1^2$, or, $xx_1+yy_1=x_1^2+y_1^2$.

Now, since $x_1^2+y_1^2=a^2$, (x_1, y_1) being on the circle,

$\therefore$ the required equation of the tangent is

$\boldsymbol{xx_1+yy_1=a^2}$.

[**Alternative proof**]

We know that the radius OT through the point of contact is perpendicular to the tangent PTQ.

The co-ordinates of T and O are (x_1, y_1) and $(0, 0)$ respectively.

$\therefore$ The equation of the st. line OT is

$$y=\frac{y_1}{x_1}.x \left(\frac{y_1}{x_1} \text{ being its gradient}\right).$$

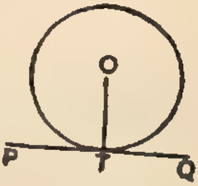


Fig. 5

Let the equation of PTQ be

$$y-y_1=m(x-x_1).$$

Now, since OT is perpendicular to PTQ, $\therefore$ the product of their gradients $=-1$.

$\therefore \quad m.\frac{y_1}{x_1}=-1, \quad \therefore \quad m=-\frac{x_1}{y_1}.$

$\therefore$ The equation of the tangent is $y-y_1=-\frac{x_1}{y_1}(x-x_1)$,

or, $yy_1-y_1^2=-xx_1+x_1^2$, or, $xx_1+yy_1=x_1^2+y_1^2$.

$\because$ the pt. (x_1, y_1) is on the circle, $\therefore \quad x_1^2+y_1^2=a^2$.

$\therefore$ the equation of the tangent to the circle $x^2+y^2=a^2$ at the pt. (x_1, y_1) is $xx_1+yy_1=a^2$.

145. (*b*) *To find the co-ordinates of the point of contact when $y=mx+c$ touches the circle $x^2+y^2=a^2$.*

Let the co-ordinates of the point of contact be (x', y'). Then the equation of the tangent to the circle $x^2+y^2=a^2$ at the point (x', y') is $xx'+yy'=a^2$, or, $y=-\frac{x'}{y'}.x+\frac{a^2}{y'}\cdots(1)$.

Now, since the line $y=mx+c$ is a tangent to the given circle at (x', y'), $\therefore$ the equation (1) and $y=mx+c$ must be the same equation, i.e., we have

$$m=-\frac{x'}{y'} \text{ and } c=\frac{a^2}{y'}. \quad \therefore \quad y'=\frac{a^2}{c} \text{ and } x'=-my'=-\frac{ma^2}{c}.$$

$\therefore$ the required co-ordinates are $\left(-\frac{ma^2}{c}, \frac{a^2}{c}\right)$.

Again, since $c=a\sqrt{1+m^2}$, the co-ordinates may be written as $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$.

146. *To find the equation of the tangent to the circle* $x^2+y^2+2gx+2fy+c=0$ *at the point* (x_1, y_1) *on it.*

Let (x_2, y_2) be another point on the circle. Now, the equation of the st. line passing through the pts. (x_1, y_1) and (x_2, y_2) is $y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)$; but since (x_1, y_1) and (x_2, y_2) are points on the circle,

$$\therefore \quad x_1^2+y_1^2+2gx_1+2fy_1+c=0, \text{ and}$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0.$$

$$\therefore \quad (x_1^2-x_2^2)+(y_1^2-y_2^2)+2g(x_1-x_2)+2f(y_1-y_2)=0,$$

or, $$(x_1-x_2)(x_1+x_2+2g)+(y_1-y_2)(y_1+y_2+2f)=0,$$

$$\therefore \quad \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}.$$

$\therefore$ The equation of the chord passing through the points (x_1, y_1) and (x_2, y_2) is

$$y-y_1=-\frac{x_1+x_2+2g}{y_1+y_2+2f}(x-x_1).$$

When the pt. (x_2, y_2) approaching towards the pt. (x_1, y_1) ultimately coincides with it, then the above chord will become a tangent to the circle and in this position $x_2=x_1$ and $y_2=y_1$.

$\therefore$ The equation of the tangent is

$$y-y_1=-\frac{2x_1+2g}{2y_1+2f}(x-x_1),$$

or, $$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1),$$

or, $$yy_1+fy-y_1^2-fy_1=-xx_1-gx+x_1^2+gx_1,$$

or, $$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1.$$

Now adding gx_1+fy_1+c to both sides we have

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c=0.$$

$\therefore$ The required equation of the tangent is

$$\boldsymbol{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.}$$

[N. B. We find here that the equation of the tangent is obtained by substituting xx_1 for x^2, yy_1 for y^2, $(x+x_1)$ for $2x$ and $(y+y_1)$ for $2y$ in the given equation of the circle. This rule will be applicable for obtaining at sight the equation of the tangent at (x_1, y_1) to any of the curves which will be dealt with later on.]

147. Normal : The normal at any point of a curve is the straight line which passes through the point and is perpendicular to the tangent at the point.

148. *To find the equation to the normal at the point* (x_1, y_1) *of the circle* $x^2+y^2=a^2$.

The normal will be a st. line perpendicular to the tangent at the point of contact (x_1, y_1).

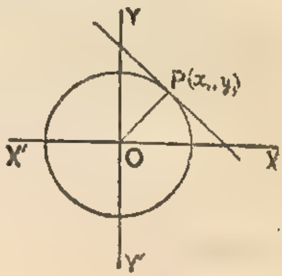


Fig 6

Here the origin is the centre of the given circle and the radius OP drawn through the point of contact P is perpendicular to the tangent at P.

The equation of the line passing through the points (0, 0) and (x_1, y_1) is $y-0=\dfrac{y_1-0}{x_1-0}(x-0)$, or, $y=\dfrac{y_1}{x_1}.x$, or, $xy_1=yx_1$.

$\therefore$ The required equation to the normal is $\boldsymbol{xy_1-yx_1=0}$.

Alternative method : The equation of the tangent at the point (x_1, y_1) is $xx_1+yy_1=a^2$,

or, $y=-\dfrac{x_1}{y_1}.x+\dfrac{a^2}{y_1}\cdots$(i) and its gradient is $-\dfrac{x_1}{y_1}$.

Again, the equation of the line passing through (x_1, y_1) is $y-y_1=m(x-x_1)\cdots$(ii).

Since the lines (i) and (ii) are perpendicular to each other,

$\therefore \quad m\left(-\dfrac{x_1}{y_1}\right)=-1, \quad \therefore \; m=\dfrac{y_1}{x_1}.$

$\therefore$ The equation of the normal is $y-y_1=\dfrac{y_1}{x_1}(x-x_1)$

or, $xy_1-yx_1=0.$

149. *To find the equation to the normal at the point (x_1, y_1) of the circle $x^2+y^2+2gx+2fy+c=0.$*

The co-ordinates of the centre of the given circle are $(-g, -f)$, and the normal passes through it.

The equation of the line passing through the points

$(-g, -f)$ and (x_1, y_1) is $y-y_1=\dfrac{y_1+f}{x_1+g}(x-x_1)$,

or, $y(x_1+g)-y_1(x_1+g)=x(y_1+f)-x_1(y_1+f)$,

or, $x(y_1+f)-y(x_1+g)+gy_1-fx_1=0$, which is the equation of the normal.

Alternative method : The equation of the tangent at the pt. (x_1, y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$,

or, $y=-\dfrac{x_1+g}{y_1+f}.x-\dfrac{gx_1+fy_1+c}{y_1+f}.$

Again, the equation of the st. line through (x_1, y_1) is

$y-y_1=m(x-x_1), \quad \therefore \quad m\left(-\dfrac{x_1+g}{y_1+f}\right)=-1, \therefore \; m=\dfrac{y_1+f}{x_1+g}.$

$\therefore$ The equation of the normal is $y-y_1=\dfrac{y_1+f}{x_1+g}(x-x_1)$,

or, $x(y_1+f)-y(x_1+g)+gy_1-fx_1=0.$ or $\dfrac{x-x_1}{x_1+g}=\dfrac{y-y_1}{y_1+f}$

[N. B. (i) It is evident from the above equation that the equation of the normal is satisfied by the co-ordinates $(-g, -f)$ of the centre of the circle. Hence the normal

at any point of the circle passes through the centre. (ii) Conversely, any st. line passing through the centre is a normal to the circle.]

150. *Two tangents can be drawn to a circle from an external point.*

The st. line $y=mx+a\sqrt{m^2+1}$ is a tangent to the circle $x^2+y^2=a^2$. If this tangent passes through an external point (x', y'), then $y'=mx'+a\sqrt{m^2+1}$,

or, $(y'-mx')^2=a^2(m^2+1)$,

or, $(x'^2-a^2)m^2-2x'y'm+(y'^2-a^2)=0$, which is a quadratic equation in m.

$\therefore$ Two values of m can be obtained from it, and therefore two tangents can be had by substituting the two values of m in $y=mx+a\sqrt{m^2+1}$.

151. Chord of contact.

If from an external point two tangents be drawn to a circle, the straight line joining the two points of contact is called the *chord of contact* of the tangents from the point.

152. *To find the equation of the chord of contact of the point (x', y') with respect to the circle $x^2+y^2=a^2$.*

The equations of the tangents to the given circle at the points (x_1, y_1) and (x_2, y_2) are

$$xx_1+yy_1=a^2 \text{ and } xx_2+yy_2=a^2.$$

Since these tangents pass through the point (x', y'),

$\therefore$ we have $x'x_1+y'y_1=a^2$ and $x'x_2+y'y_2=a^2$.

Hence it is evident that the st. line $xx'+yy'=a^2$ passes through the points of contact (x_1, y_1) and (x_2, y_2).

$\therefore$ The equation of the chord of contact of the point (x', y') is $\boldsymbol{xx'+yy'=a^2}$.

[N. B. The tangent at the point (x', y') and the chord of contact with respect to the point (x', y') are represented by the same equation $xx'+yy'=a^2$. The only distinction is that the equation represents the tangent, if the pt. (x', y') is on the circle and it represents the chord of contact, if the point (x', y') is outside the circle.]

153. The length of the tangent.

To find the length of the tangent from an external point (x', y') to the circle (i) $x^2+y^2=a^2$ and

(ii) $x^2+y^2+2gx+2fy+c=0$.

Let PT be a tangent from $P(x', y')$ to the circle whose centre is O. Join OT and OP.

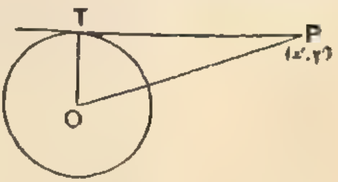


Fig. 7

Now, $\because$ $\angle OTP$ is a right angle,

$\therefore$ $PT^2=OP^2-OT^2$.

(i) The co-ordinates of the points P and O are (x', y') and $(0, 0)$ respectively.

$\therefore$ $OP^2=(x-0)^2+(y-0)^2=x^2+y^2$.

Again, $OT=\text{radius}=a$ (here),

$\therefore$ $PT^2=OP^2-OT^2=x^2+y^2-a^2$.

$\therefore$ The $(\text{tangent})^2=x^2+y^2-a^2$.

(ii) Here, the co-ordinates of P and O are (x', y') and $(-g, -f)$.

$\therefore$ $OP^2=(x'+g)^2+(y'+f)^2$.

Again, $OT=\text{radius}=\sqrt{g^2+f^2-c}$ (here)

$\therefore$ $PT^2=(x'+g)^2+(y'+f)^2-(g^2+f^2-c)$

i.e., $(\text{Tangent})^2=x'^2+y'^2+2gx'+2fy'+c$.

[N. B. We find from (i) and (ii) that if the equation of the circle be arranged with all the terms on the left side and zero on the right, then the square of the length of the tangent is obtained by substituting x', y' for x, y.]

154. *To find the equation of the chord of a circle which is bisected at a given point.*

Let $x^2+y^2=a^2$ be the circle whose centre is O and P (h, k) be the given point.

The equation of the st. line through P (h, k) is $y-k=m(x-h)$.

The equation of the st. line OP is $y=\frac{k}{h}x$.

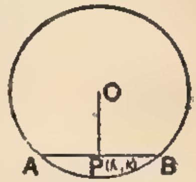


Fig. 8

Again, since P is the mid point of the chord AB,

$\therefore$ OP$\perp$AB, $\therefore$ $m\times\frac{k}{h}=-1$, $\therefore$ $m=-\frac{h}{k}$.

$\therefore$ The equation of the chord, whose middle point is (h, k), is $y-k=-\frac{h}{k}(x-h)$, or, $h(x-h)+k(y-k)=0$.

155. *To find the locus of the middle points of a system of parallel chords of a circle.*

Let $x^2+y^2=a^2 \cdots$(i) be the circle and $y=mx+c\cdots$(ii) be the equation of any one of the parallel chords.

Since the chords are parallel here, the gradient m is constant and the values of c are different for the different chords.

Eliminating y from (i) and (ii), [i.e., putting the value of x from (ii) in (i)] we have $x^2+(mx+c)^2=a^2$,

or, $x^2(1+m^2)+2mcx+(c^2-a^2)=0\cdots$(iii), it is a quadratic equation in x and its two roots represent the abscissæ of the two points of intersection of (i) and (ii).

Let x_1 and x_2 be the roots of the equation-(iii).

Then $x_1+x_2=$ the sum of the roots $=-\frac{2mc}{1+m^2}$.

Let (h, k) be the middle point of the chord-(ii).

$\therefore \quad h=\frac{1}{2}(x_1+x_2)=-\frac{mc}{1+m^2}\cdots$(iv).

Again, $\because$ the pt. (h, k) lies on the chord-(ii),

$\therefore \quad k=mh+c \quad \cdots$(v).

Now, eliminating c from (iv) and (v) we have $h=-\frac{m}{1+m^2}(k-mh)$, or, $h+mk=0$; as it is independent of c, it will be true for every one of the parallel chords.

Hence the required locus is $x+my=0$. Since this is a linear equation in x and y, the locus is a straight line. Again, since this equation is independent of the constant term, it will pass through the origin. But as here the origin is the centre of the circle, so it passes through the centre.

Hence the locus is a diameter of the circle.

156. *Find the locus of the middle points of the chords of the circle $x^2+y^2=a^2$ which pass through the fixed point (h, k).*

[C. U. (B. Sc) '37]

Let (x_1, y_1) be the co-ordinates of the middle point of any chord passing through the point (h, k). Then the equation of the chord is $xx_1+yy_1=x_1^2+y_1^2$.

Again, $\because$ the chord passes through the pt. (h, k),

$\therefore$ from the equation we have $hx_1+ky_1=x_1^2+y_1^2$.

Hence the locus of the middle point (x_1, y_1) is $hx+ky=x^2+y^2$, or, $x^2+y^2-hx-ky=0$, and it is evidently the equation of a circle. Thus the required locus is a circle which passes through the origin, the equation being independent of the constant term.

Examples (19)

Ex. 1. To find the equation of a circle which touches each axis at a distance of 3 from the origin.

It is evident from the figure that ON $=3$, OM $=3$.

$\therefore$ The co-ordinates of the centre C are $(3, 3)$ and the radius $=3$.

$\therefore$ the equation of the circle is $(x-3)^2+(y-3)^2=3^2$,

or, $x^2+y^2-6x-6y+9=0$.

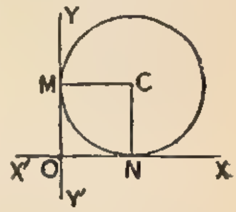


Fig. 9

Ex. 2. To find the equation of a circle of radius 5 that touches each axis.

[It is evident from the above figure] CN $=$ CM $=$ radius $=5$.

$\therefore$ the co-ordinates of C are $(5, 5)$.

$\therefore$ the equation of the circle is $(x-5)^2+(y-5)^2=5^2$,

or, $x^2+y^2-10x-10y+25=0$.

Ex. 3. Write down the equations of the tangents to the circles (i) $x^2+y^2=5$ (ii) $x^2+y^2+7x+12y+12=0$ at the point $(1, -2)$.

(i) Here the equation of the tangent at the piont $(1, -2)$ is $x.1+y(-2)=5$, or, $x-2y=5$.

(ii) Here $x^2+y^2+7x+12y+12=0$,

or, $x.x+y.y+\frac{7}{2}(x+x)+6(y+y)+12=0$.

$\therefore$ the equation of the tangent at the pt. $(1, -2)$ is

$$x.1+y(-2)+\tfrac{7}{2}(x+1)+6(y-2)+12=0,$$

or, $x-2y+\frac{7}{2}x+6y+\frac{7}{2}-12+12=0$,

or, $\frac{9}{2}x+4y+\frac{7}{2}=0$, or, $9x+8y+7=0$.

Ex. 4. Find the equations to the tangents to the circle $x^2+y^2=20$, which are parallel to the line $x+2y+5=0$.

The equation of the st. line parallel to the line $x+2y+5=0$ is $x+2y=c$, or, $x=c-2y$.

Putting $x=c-2y$ in the equation of the circle we have
$(c-2y)^2+y^2-20=0$, or, $5y^2-4cy+c^2-20=0$.

Now, if $x+2y=c$ be a tangent, the roots of the quadratic equation will be equal.

$\therefore$ $16c^2-4.5(c^2-20)=0$, or, $4c^2-5(c^2-20)=0$,

or, $c^2=100$, $\therefore$ $c=\pm 10$.

$\therefore$ the equations of the tangents are $x+2y=\pm 10$,

i.e., $x+2y=10$ and $x+2y+10=0$.

Ex. 5. Find the equations to the tangents to the circle $x^2+y^2-3x+10y-15=0$, which are perpendicular to the line $12x+5y=1$ and also find their points of contact.

The equation of any st. line perpendicular to the line $12x+5y=1$ is $5x-12y=c$, or, $x=\dfrac{c+12y}{5}$.

Now putting $\frac{1}{5}(c+12y)$ for x in the given equation of the circle we have

$\frac{1}{25}(c+12y)^2+y^2-\frac{3}{5}(c+12y)+10y-15=0$,
or, $c^2+144y^2+24cy+25y^2-15c-180y+250y-375=0$,
or, $169y^2+2(12c+35)y+(c^2-15c-375)=0 \cdots\cdots(1)$.

Now, (1) is a quadratic equation in y. Its two roots represent the ordinates of the two points of intersection of the st. line $5x-12y=c$ and the given circle. This st. line will be a tangent to the circle when the two points coincide, i.e., when the roots of the equation-(1) are equal.

$\therefore$ $4(12c+35)^2=169\times 4(c^2-15c-375)$,

or, $144c^2+840c+1225=169c^2-2535c-63375$,

or, $25c^2-3375c-64600=0$, or, $c^2-135c-2584=0$,

or, $(c+17)(c-152)=0$, $\therefore$ $c=-17$ or 152.

$\therefore$ the equations of the tangents are $5x-12y+17=0$ and $5x-12y=152$.

Again, substituting -17 for c in (1) we have

$169y^2+2(-17\times12+35)y+(289-15\times-17-375)=0,$

or, $169y^2-2.169y+169=0$, or, $y^2-2y+1=0$,

or, $(y-1)^2=0$, $\therefore$ $y=1, 1$.

Putting these values of y in the equation of the first tangent we have $x=-1, -1$.

$\therefore$ The co-ordinates of the point of contact of the first tangent are $(-1, 1)$.

Again, putting $c=152$ in (1) and simplifying we have $169y^2+2.1859y+20449=0$, or, $y^2+2.11y+121=0$,

or, $(y+11)^2=0$, $\therefore$ $y=-11, -11$.

Putting these values of y in the equation of the second tangent we have $x=4, 4$.

$\therefore$ The co-ordinates of the pt. of contact of the second tangent are $(4, -11)$.

Ex. 6. Find the length of the chord cut off from the straight line $y=2x-5$ by the circle $x^2+y^2-6x+8y-5=0$.

The given equation of the circle is $x^2+y^2-6x+8y-5=0$,

or, $(x-3)^2+(y+4)^2=30$.

$\therefore$ the centre is the pt. $(3, -4)$ and the radius $=\sqrt{30}$.

Now, the length (OM) of the perpendicular from the centre $(3, -4)$ to the given st. line $(y=2x-5)$

$$=\frac{2.3-(-4)-5}{\sqrt{1^2+2^2}}=\frac{5}{\sqrt{5}}=\sqrt{5}$$

[vide Fig. 4(*b*) in Art. 142 (b)]

$\therefore$ $PM^2=OP^2-OM^2=30-5=25$, $\therefore$ $PM=5$.

$\therefore$ The required length of the chord $=2.5=10$.

Ex. 7. Show that the line $x-y+2=0$ touches the circle $x^2+y^2=2$ and find the co-ordinates of the point of contact.

[H. S. '62 (Compl.)]

From the given line we have $y=x+2$.

Putting this value of y in the equation of the circle we have

$x^2+(x+2)^2=2$, or, $2x^2+4x+2=0$,

or, $x^2+2x+1=0$, or, $(x+1)^2=0$, $\therefore$ $x=-1, -1$.

Substituting these values of x in the equation of the st. line we have $y=1, 1$. So we find that the two points at which the st. line cuts the circle are coincident, their co-ordinates being $(-1, 1)$.

Hence the st. line $x-y+2=0$ is a tangent to the circle $x^2+y^2=2$ and the co-ordinates of the point of contact are $(-1, 1)$.

Ex. 8. For what values of k, the line $x+3y=k$ touches the circle $x^2+y^2-3x-3y+2=0$ and in that case, find the point of contact.

Putting $x=k-3y$ in the equation of the circle we have

$$(k-3y)^2+y^2-3(k-3y)-3y+2=0,$$

or, $10y^2+6(1-k)y+(k^2-3k+2)=0 \cdots\cdots$(i)

This is a quadratic equation in y. Now the two roots of this equation are the ordinates of the two points of intersection of the given st. line and the given circle. The given st. line will be a tangent to the circle when the two points of intersection are coincident. The condition for this is that the two roots of (i) must be equal.

$\therefore$ $36(1-k)^2-40(k^2-3k+2)=0$,

or, $9(1-2k+k^2)-10(k^2-3k+2)=0$,

or, $k^2-12k+11=0$, or, $(k-1)(k-11)=0$,

$\therefore$ $k=1$ or 11.

Putting $k=1$ in (i) we have $10y^2=0$, or, $y=0$.

$\therefore$ The co-ordinates of the pt. of contact of the tangent $x+3y=1$ are $(1, 0)$.

Again putting $k=11$ in (i) we have $10y^2-60y+90=0$,

or, $y^2-6y+9=0$, or, $(y-3)^2=0$, $\therefore$ $y=3$.

$\therefore$ the co-ordinates of the point of contact of the tangent $x+3y=11$ are $(2, 3)$.

Ex. 9. Find the equation to a circle touching the straight line $5x+12y=1$ and having its centre at the point $(3, 4)$.

Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the circle.

Here $g=-3, f=-4$.

$\therefore$ the equation of the circle becomes

$$x^2+y^2-6x-8y+c=0,$$

and its radius $=\sqrt{g^2+f^2-c}=\sqrt{9+16-c}=\sqrt{25-c}$.

The length of the perpendicular from the pt. $(3, 4)$ to the given st. line $(5x+12y=1)$

$$=\frac{15+48-1}{\sqrt{25+144}}=\frac{62}{13}, \quad \therefore \tfrac{62}{13}=\sqrt{25-c},$$

or, $\frac{3844}{169}=25-c, \quad \therefore \quad c=25-\frac{3844}{169}=\frac{381}{169}$.

$\therefore$ The equation of the circle is

$$x^2+y^2-6x-8y+\tfrac{381}{169}=0.$$

Ex. 10. A circle, the co-ordinates of whose centre are both positive, touches both the axes of co-ordinates. If it also touches the line $3x-4y+6=0$, find its equation and the co-ordinates of its point of contact with the given line.

Since the co-ordinates, *i. e.*, the abscisa and the ordinate of the centre are both positive, the circle lies in the first quadrant.

Again, since the circle touches the axes, the axes are equidistant from the centre, the distance being equal to the radius.

Let h be the radius of the circle.

Then the equation of the circle is $(x-h)^2+(y-h)^2=h^2$.

By the given condition, the distance of the line

$3x-4y+6=0$ from the centre (h, h) is also equal to h.

$$\therefore \quad h=\frac{3h-4h+6}{\sqrt{3^2+4^2}}=\frac{6-h}{5}, \text{ or, } 5h=6-h, \quad \therefore \quad h=1.$$

So the required equation of the circle is

$$(x-1)^2+(y-1)^2=1, \quad \text{or,} \quad x^2+y^2-2x-2y+1=0.$$

Again, the point common to the equation of the circle and the given line is the point of contact.

From the equation of the st. line we have $x=\dfrac{4y-6}{3}$.

Substituting this value of x in the equation of the circle we have $\left(\dfrac{4y-6}{3}-1\right)^2+(y-1)^2=1$,

or, $\left(\dfrac{4y-9}{3}\right)^2+y^2-2y+1=1$, or, $25y^2-90y+81=0$,

or, $(5y-9)^2=0$, $\therefore$ $y=\frac{9}{5}$.

$$\therefore \quad x=\frac{\dfrac{4\times 9}{5}-6}{3}=\frac{6}{5\times 3}=\frac{2}{5}.$$

$\therefore$ The required co-ordinates of the pt. of contact are $\left(\frac{2}{5}, \frac{9}{5}\right)$.

Ex. 11. Find the equation of the circle which cuts off intercepts equal to 5 and 2 on the axes of x and y respectively and whose centre lies on the line $2x-y=6$. Show that there are two such circles satisfying the given conditions.

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \cdots (1).$$

Since its centre $(-g, -f)$ lies on the line $2x-y=6$, we have $-2g+f=6 \cdots (2)$. The equation of the x-axis is $y=0$;

$\therefore$ The roots of the quadratic equation

$x^2+0+2gx+2f.0+c=0$, or, $x^2+2gx+c=3$ are the abscissæ of the points at which the x-axis intersects the circle-(1). Let x_1 and x_2 be the roots.

$\therefore$ $x_2+x_1=$ the sum of the roots $=-2g$ and $x_1x_2=c$.

$\therefore$ $x_2-x_1=\sqrt{(x_2+x_1)^2-4x_1x_2}=\sqrt{4g^2-4c}$, but from the given condition $x_2-x_1=5$, $\therefore$ $4g^2-4c=25 \cdots (3)$

Again, the equation of the y-axis is $x=0$.

$\therefore$ The ordinates of the two points at which the y-axis intersects the circle-(1) will be obtained from the roots of the equation $0+y^2+2g.0+2fy+c=0$, or, $y^2+2fy+c=0$.

Let y_1 and y_2 be the roots of this equation.

Then $y_2+y_1=-2f$ and $y_1y_2=c$.

Again, by the problem $y_2-y_1=2$.

$\therefore\ 2=\sqrt{(y_2+y_1)^2-4y_1y_2}=\sqrt{4f^2-4c}$, $\therefore\ 4f^2=4c=4\cdots(4)$.

Subtracting (4) from (3) we have $4g^2-4f^2=21\cdots(5)$

and from (2) we have $f=2g+6$.

Substituting this value of f in (5) we have

$4g^2-4(2g+6)^2=21$, or, $4g^2+32g+55=0$,

or, $(2g+5)(2g+11)=0$, $\therefore\ g=-\frac{5}{2}$ or $-\frac{11}{2}$.

Putting these values of g in (2) we have $f=1$ or -5.

Again, putting these values of f in (4) we have $c=0$ or 24.

Since g, f and c have two values each, there are two circles satisfying the given conditions.

Hence the equation of one circle is

$x^2+y^2+2(-\frac{5}{2})x+2.1.y+0=0$, or, $x^2+y^2-5x+2y=0$;

and that of the other circle is

$x^2+y^2+2(-\frac{11}{2})x+2(-5)y+24=0$,

or, $x^2+y^2-11x-10y+24=0$.

Ex. 12. Find the equations to the two tangents to the circle $x^2+y^2=3$, which make an angle of 60° with the x-axis. [C. U. (B. Sc.) '55]

Here the equation of the circle is $x^2+y^2=(\sqrt{3})^2$, so the equations of the two tangents to the circle will be

$$y=mx\pm a\sqrt{1+m^2}.$$

Here $m=\tan 60^\circ=\sqrt{3}$ and $a=\sqrt{3}$,

$\therefore\ y=x\sqrt{3}\pm\sqrt{3}.\sqrt{1+3}=\sqrt{3}x\pm\sqrt{3}.\sqrt{4}=\sqrt{3}x\pm2\sqrt{3}$.

$\therefore$ The required equations are $y=\sqrt{3}x\pm2\sqrt{3}$.

Ex. 13. If $y=x \sin\alpha+a\sec\alpha$ be a tangent to the circle $x^2+y^2=a^2$, show that $\cos^2\alpha=1$. [C. U. (B. Sc.) '19]

$\because$ $c^2=a^2(1+m^2)$, if $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$,

$\therefore$ The line $y=x\sin\alpha+a\sec\alpha$ being a tangent to the circle $x^2+y^2=a^2$, we have

$$a^2\sec^2\alpha=a^2(1+\sin^2\alpha),\quad \text{or,}\quad \sec^2\alpha=1+\sin^2\alpha,$$

or, $\sec^2\alpha-1=\sin^2\alpha$, or, $\tan^2\alpha=\sin^2\alpha$,

or, $\dfrac{\sin^2\alpha}{\cos^2\alpha}=\sin^2\alpha$, $\therefore$ $\cos^2\alpha=\dfrac{\sin^2\alpha}{\sin^2\alpha}=1$.

Ex. 14. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch each other and find their point of contact.

The equation of the first circle is $x^2+y^2-4x+6y+8=0$,

or, $(x-2)^2+(y+3)^2=5=(\sqrt{5})^2$.

$\therefore$ Its centre is the pt. $(2, -3)$ and radius $=\sqrt{5}$.

Again, the equation of the second circle is

$$x^2+y^2-10x-6y+14=0,$$

or, $(x-5)^2+(y-3)^2=20=(2\sqrt{5})^2$.

Its centre is the pt. $(5, 3)$ and radius $=2\sqrt{5}$.

Now, the distance between the two centres

$$=\sqrt{(5-2)^2+(3+3)^2}=\sqrt{45}=3\sqrt{5}.$$

The sum of the two radii $=\sqrt{5}+2\sqrt{5}=3\sqrt{5}$

$=$ the line of centres.

$\therefore$ The circles touch each other.

Again, the ratio of the two radii $=\sqrt{5} : 2\sqrt{5}=1 : 2$.

$\therefore$ The st. line joining the centres $(2, -3)$ and $(5, 3)$ is divided in the ratio $1 : 2$ at the point of contact.

$\therefore$ The co-ordinates of the point of contact are

$$\left\{\frac{2\times 2+1\times 5}{2+1},\ \frac{2\times(-3)+1\times 3}{2+1}\right\}\quad \text{or}\quad (3, -1).$$

Ex. 15. Find the equation to the common chord of the two circles $x^2+y^2-6x-10y+9=0$ and $x^2+y^2-4x+6y-12=0$, and show that this line is perpendicular to the line of centres of the circles.

Suppose the given circles $x^2+y^2-6x-10y+9=0 \cdots (1)$ and $x^2+y^2-4x+6y-12=0 \cdots (2)$ cut each other at A and B. AB is their common chord whose equation is to be found. Take any point P (h, k) on AB produced and from P draw the tangents PQ and PR to the circles (1) and (2) respectively.

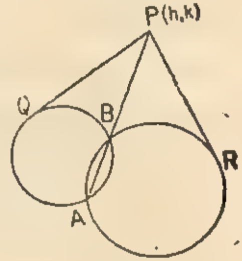


Fig. 9(a)

Now, PQ is a tangent and PBA a secant of the circle-(1),

$$\therefore \quad PQ^2 = PA.PB.$$

Again, PR is a tangent and PBA a secant of the circle-(2).

$$\therefore \quad PR^2 = PA.PB. \quad \therefore \quad PQ^2 = PR^2.$$

Now, $PQ^2 = h^2+k^2-6h-10k+9$ and

$PR^2 = h^2+k^2-4h+6k-12$ [vide Art. 153]

$\therefore \quad h^2+k^2-6h-10k+9 = h^2+k^2-4h+6k-12,$

or, $\quad 2h+16k-21=0.$

The locus of the pt. P is the equation of the chord AB.

Hence the required equation of the common chord is

$$2x+16y-21=0.$$

Again, equn. (1) may be written as $(x-3)^2+(y-5)^2=25$ and equn. (2) may be written as $(x-2)^2+(y+3)^2=25$.

$\therefore$ The gradient of the line of centres of the circles

$= \dfrac{5-(-3)}{3-2} = 8$;

and the gradient of the common chord $= -\frac{2}{16} = -\frac{1}{8}$.

$\therefore$ The product of the gradients $= 8 \times -\frac{1}{8} = -1$.

$\therefore$ The common chord is perpendicular to the line of centres.

Exercise 19

1. Find the equation of the circle which touches the co-ordinate axes at (1, 0) and (0, 1). [C. U.]

2. Find the equation to the circle of radius 3, which touches both the axes.

3. Show that the circles $x^2+y^2-14x-10y+58=0$ and $x^2+y^2-2x+6y-26=0$ touch each other externally.

4. Show that the circles $x^2+y^2-12x-16y-125=0$ and $x^2+y^2+12x-6y+41=0$ touch each other internally.

5. Find the equation of the circle which touches x-axis, passes through the point (1, 1) and whose centre lies in the first quadrant on the line $x+y=3$. [C. U.]

6. Find the equation to the circle which touches the x-axis at a distance $+5$ from the origin and cuts off a chord of length 24 from the y-axis. [H. S.' 68]

7. Find the co-ordinates of the point of intersection of the circle $x^2+y^2=25$ with the straight line $x+y=7$. [H. S. '62(c)]

8. Find where the line $3x+4y+7=0$ cuts the circle $x^2+y^2-4x-6y-12=0$. [C. U.]

9. Prove that the lines $x=7$, $y=8$ both touch the circle $x^2+y^2-4x-6y=12$. Find the points of contact. [C. U]

10. Show that the line $y=x+2$ touches the circle $x^2+y^2=2$ and find the co-ordinates of the point of contact.

11. Show that the line $3y=2x+26$ touches the circle $x^2+y^2-4x+6y=104$ and find the co-ordinates of the point of contact.

12. Prove that the straight line $y=x+\sqrt{2}c$ touches the circle $x^2+y^2=c^2$ and find its point of contact.

13. Show that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches the axes of x and y. [U. P. B. '44]

14. Show that the circle $x^2+y^2=25$ is touched by the straight line $3x+4y=25$. [U. U. '48]

15. Show that the straight line $y-3x=10$ cuts the circle $x^2+y^2=10$ in two coincident points and determine the co-ordinates of this point. [C. U. '43]

16. Find the length of the chord cut off from the straight line

(a) $3x-4y+15=0$ by the circle $x^2+y^2=25$;

(b) $3x+4y-c=0$ by the circle $x^2+y^2=64$. [W.B. H.S. '60]

17. (a) Find the equation to that chord of the circle $x^2+y^2=81$ which is bisected at the point $(-2, 3)$. [C. U.]

(b) Find the equation to the chord of the circle $x^2+y^2=25$, which is bisected at the point $(1, 2)$.

18. Find the equation to the chord of the circle $x^2+y^2-4x+6y-3=0$, which is bisected at the point $(1, -2)$.

19. Write down the equation of the tangent to each of the following circles :—

(i) $x^2+y^2=25$ at the point $(3, -4)$.

(ii) $x^2+y^2+4x+6y-87=0$ at the point $(4, 5)$.

(iii) $x^2+y^2-8x+10y=128$ at the point $(-8, 0)$.

20. Find the equations of the tangents to the circle $x^2+y^2=41$ at the points where $x=4$.

21. Find the equations of the normals to the circles :

(i) $x^2+y^2=29$ at the point $(2, 5)$.

(ii) $x^2+y^2+4x+6y=87$ at the point $(6, 3)$.

22. For what values of k, the straight line $y=kx+13$ touches the circle $x^2+y^2=144$?

23. Find the equations of the tangents to the circle $x^2+y^2=9$, which are parallel to the line $3x+4y=0$.

[C. Pre. U. '63]

24. Find the equations of the tangents to the circle $x^2+y^2=25$, which are perpendicular to the line $4x-3y=12$.

25. Find the equations of the tangents to the circle $x^2+y^2=25$,

(i) which are parallel to the straight line $3x+4y=0$.

(ii) which pass through the point (13, 0). [C. U. '58]

26. Find the equations of the tangents to the circle $x^2+y^2=25$ which are parallel to the straight line $4x+3y=0$. [U. U. '47]

27. Find the equations of the tangents to the circle $x^2+y^2-6x+4y=12$, which are parallel to the line $4x+3y+5=0$. [C. U.]

28. Find the equations of the tangents to the circle $x^2+y^2-6x+4y=7$, which are perpendicular to the line $2x-y+3=0$.

29. Find the equation to the circle which has its centre at $(1, -3)$ and touches $2x-y-4=0$. [U. P. B. '51]

30. Prove that the two circles $x^2+y^2+2ax+c^2=0$ and $x^2+y^2+2by+c^2=0$ touch each other if $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c^2}$.

31. Find the length of the tangent to the circles

(i) $x^2+y^2=13$ from the point (2, 5).

(ii) $x^2+y^2-6x-8y+24=0$ from (6, 5).

(iii) $3x^2+3y^2+4x-7y+25=0$ from $(-1, 2)$.

32. The length of the tangent from (f, g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$, show that $f^2+g^2+4f+4g+2=0$. [C. U.]

33. Find the locus of the mid-points of all chords of the circle $x^2+y^2=25$, which pass through the fixed point (1, 1).

34. Show that the line $x+\sqrt{3}y=8$ touches the circle $x^2+y^2=16$ and find the point of contact. [U. U.]

35. Prove that the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$ and find its point of contact. [C. U. '39]

36. If the line $lx+my=1$ touches the circle $x^2+y^2=a^2$, then (l, m) lies on a certain circle. Find its equation. [U. P. B. '42]

37. Find the equations of the tangents to the circle $x^2+y^2=25$, which are inclined at an angle of 30° to the axis of x. [U. P. B. '46]

38. Find the equations of the tangents from the origin to the circle $x^2+y^2+20(x+y)+20=0$. [U. P. B.]

39. Find the value of c so that $y=mx+c$ may be a tangent to the circle $x^2+y^2=4y$ for all values of m. [U. P. B. '48]

40. Find the equation to the common chord and the line of centres of the two circles $x^2+y^2+6x-3y+4=0$ and $2x^2+2y^2-3x-9y+2=0$ and show that the common chord is perpendicular to the line of centres.

41. Find the equation to the common chord of the two circles $x^2+y^2-4x+6y-36=0$ and $x^2+y^2-5x+8y-43=0$ also find its length.

157. Definition : If a point moves in a plane so that its distances from a fixed point and a fixed straight line on the same plane are always in a costant ratio, then its locus is called a **conic** or **a conic section.**

Parabola : Parabola is the locus of a point which moves in a plane so that it is always equidistant from a fixed point and a fixed straight line.

The fixed point is called the **focus** and the fixed straight line is called the **directrix.**

The straight line passing through the focus and perpendicular to the directrix is called the **axis.**

The point at which the axis intersects the parabola is called the **vertex.** The line through the focus of the parabola perpendicular to the axis is known as the **latus rectum.**

N. B. (*i*) In a conic the ratio of the two distances of the moving point from the fixed point and the fixed straight line is always constant. This *constant ratio* is called the **eccentricity** and is denoted by e. In the case of a parabola this ratio is unity (*i.e.*, $e=1$). So a *parabola is a conic in which the eccentricity e is equal to unity.*

(*ii*) The focus is denoted by S, the vertex by A, the axis by AX and the latus rectum by LL′.

(*iii*) The *focal distance* of any pt. P on the parabola is the distance SP.

158. *The length of the latus rectum of a parabola is equal to four times the focal distance of the vertex.*

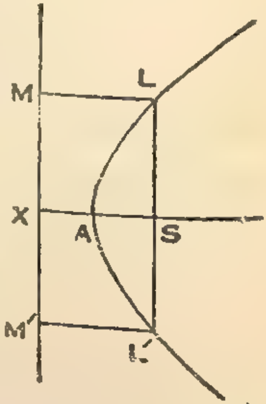


Fig. 10

Let LL′ be the latus rectum, S the focus, MM′ the directrix, SX the axis and A the vertex.

To prove that $LL'=4AS$.

LM, L′M′ are drawn perpendicular to the directrix.

Proof : Here from the definition of a parabola, we have

$$AS=AX,\ LS=LM,\ SL'=L'M'.$$

Now,
$$LL'=SL+SL'=LM+L'M'$$
$$=2SX=2.2AS=4AS.$$

159. Definition : If from any point P on the parabola PN is drawn perpendicular to the axis, then PN is called the *ordinate* and AN the *abscissa* of the point P.

Thus the **latus rectum** of a conic is the **double ordinate** drawn through the focus.

160. *The square on the ordinate of any point on the parabola is equal to the rectangle contained by the abscissa of the point and the latus rectum, i. e.,* $PN^2=4AS.AN$.

Let S, A, MX, SX be respectively the focus, the vertex, the directrix and the axis. Let P be any point on the parabola and PN be drawn perpendicular to the axis.

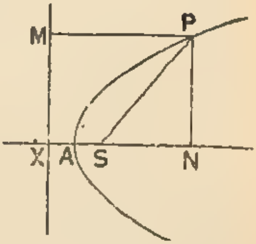


Fig. 11

To prove that $PN^2=4AS.AN$.

Proof : Join SP and draw PM perpendicular to the directrix.

Here $SP=PM$ and $AX=AS$.

Now, in the right-angled $\triangle SPN$,

$$\begin{aligned} PN^2 &= SP^2-SN^2=PM^2-SN^2=XN^2-SN^2 \\ &=(AN+AX)^2-(AN-AS)^2 \\ &=(AN+AS)^2-(AN-AS)^2=4AS.AN. \end{aligned}$$

161 (A). To find the equation to a parabola.

(*i*) *Where vertex is the origin and axis is the axis of x.*

Taking the axis AX as the x-axis and the tangent to the parabola at the vertex A to be the y-axis, the co-ordinates of the point P on the parabola are taken to be (x, y).

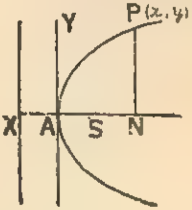


Fig. 12

PN is drawn perpendicular to the axis.

So, $AN=x$ and $PN=y$;

but $PN^2=4AS.AN$.

Suppose $AS=a$. Then the required equation to the parabola is $\boldsymbol{y^2=4ax}$.

Corollary : The latus rectum$=4AS=4a$ (*i.e.,* the coefficient of x). The abscissa and ordinate of the focus S are

(AS, 0), *i.e.*, $(a, 0)$; the equation of the directrix is $x=-a$ or $x+a=0$.

(*ii*) *Where focus is the origin and axis is the axis of x.*

Taking the focus as the origin and the axis as the x-axis, let the co-ordinates of the point P on the parabola be (x, y) and PN be drawn perpendicular to the axis.

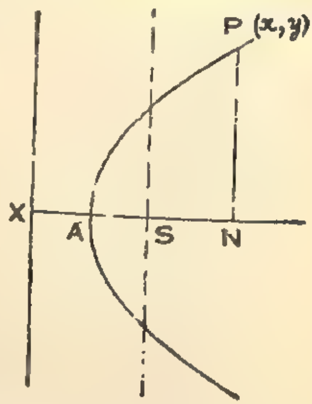


Fig. 13

Then $SN=x$, $PN=y$ and

$AN=SN+AS=x+a$;

but $PN^2=4AS.AN$,

$\therefore$ The equation of the parabola is $\boldsymbol{y^2=4a(x+a)}$.

Corollary : The latus rectum $=4AS=4a$ (*i.e.*, the coefficient of x) and the co-ordinates of the focus S are $(0, 0)$. The equation of the directrix is $x=-2a$ or $x+2a=0$.

(*iii*) *Where axis is the axis of x and the directrix is the axis of y.*

Here the directrix is the y-axis and the axis is the x-axis.

Let the co-ordinates of P be (x, y), and PN be perpendicular to the axis.

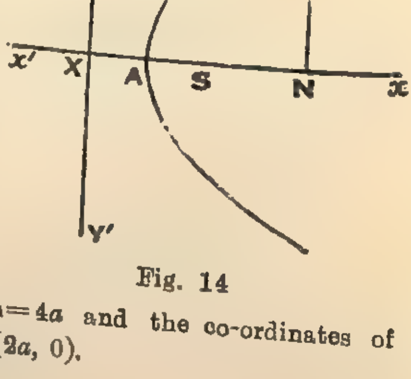


Fig. 14

Then $XN=x$, $PN=y$ and $AN=XN-AX=XN-AS=x-a$;

but $PN^2=4AS.AN$,

$\therefore$ The equation of the parabola is $\boldsymbol{y^2=4a(x-a)}$.

Corollary : The latus rectum $=4a$ and the co-ordinates of the focus $S=(XS, 0)=(2AS, 0)=(2a, 0)$.

Similarly if the axis of the parabola be taken as the y-axis, the equation of the parabola is :

(*i*) $x^2=4ay$, when the vertex A is the origin ;

(*ii*) $x^2=4a(y+a)$, when the focus S is the origin ;

(*iii*) $x^2=4a(y-a)$, when the point of intersection of the axis and the directrix is the origin.

N. B. The equation to the parabola may be of different forms. Such as (1) $y^2=4ax$, (2) $y^2=-4ax$, (3) $x^2=4ay$, (4) $x^2=-4ay$.

(1) Here the equation of the parabola is $y^2=4ax$, the co-ordinates of the vertex are (0, 0) and the x-axis is its axis. The latus rectum is the coefficient of the variable of the first order in the equation (here x) independent of the sign (here $4a$). The co-ordinates of the focus are $(a, 0)$, *i. e.*, ($\frac{1}{4}$ of the coefficient of x, 0), and the equation of the directrix is $x=-a$. Since y^2 is positive for all real values of y, x is always positive. Hence the concavity of the parabola is towards the positive side of the x-axis, *i.e.*, the parabola extends infinitely on the positive side of the x-axis. This is the standard form of the equation to a parabola.

(2) If the equation be $y^2=-4ax$, the parabola will be of the shape shown in fig. 14(a).

The co-ordinates of its vertex are (0, 0), the x-axis is its axis, the latus rectum is $4a$ (the coefficient of x independent of the sign), the co-ordinates of the focus are $(-a, 0)$ and the equation of the directrix is $x=-(-a)=a$.

Here the coefficient of x being negative, the concavity of the parabola is towards the negative side of the x-axis.

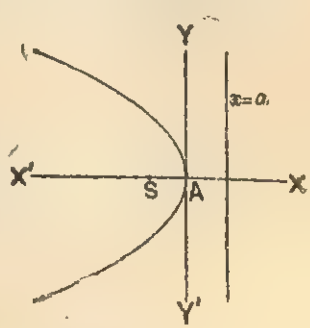


Fig. 14(a)

(3) If the equation be $x^2=4ay$, the shape of the parabola will be as in fig. 14 (b). Here the co-ordinates of the vertex are (0, 0), its axis is the y-axis, the co-ordinates of the focus are (0, a), the latus rectum is $4a$ and $y=-a$ is the equation of the directrix. The concavity of the parabola is to the positive side of the y-axis.

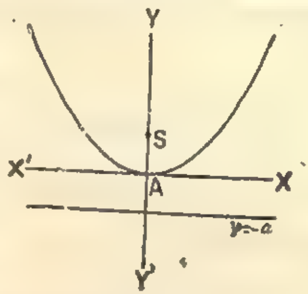


Fig. 14(b)

(4) If the equation be $x^2=-4ay$ (draw the figure), the co-ordinates of the vertex are (0, 0), the y-axis is its axis, the latus rectum is $4a$, the co-ordinates of the focus are (0, $-a$), the equation of the directrix is $y=-(-a)=a$, and its concavity is to the negative side of the y-axis.

161. (B). To find the equation of a parobola.

(i) *When the axis of the parabala is parallel to x-axis.*

Let (α, β) be the co-ordinates of the vertex A and the latus rectum be $4a$.

It is evident that if the axes OX and OY be transferred to the parallel axes AX′ and AY′, and if (X, Y) be the co-ordinates of the point P corresponding to the new axes, then the equation of the parabola becomes $Y^2=4aX\cdots(1)$.

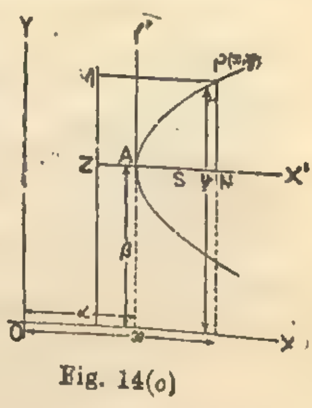


Fig. 14(c)

If (x, y) be the co-ordinates of P corresponding to the original axes, then we find from the figure that $X=x-\alpha$ and $Y=y-\beta$.

Thus the equation-(1) becomes $(y-\beta)^2=4a(x-\alpha)\cdots(2)$ which is the required equation.

[N. B. Here the co-ordinates of the focus are $(a+\alpha, \beta)$, vertex is the point (α, β), $LL'=4a$, the directrix is the line $x=\alpha-a$ and the equation of the latus rectum is $x=\alpha+a$.]

Alternative Proof : Since the co-ordinates of P and S are respectively (x, y) and $(a+\alpha, \beta)$,

$\therefore$ $SP^2=(x-a-\alpha)^2+(y-\beta)^2$.

Again, $PM=ZN=ZA+AN=a+x-\alpha$ [see figure]

From the definition of parabola we have $SP=PM$,

$\therefore$ $SP^2=PM^2$.

$\therefore$ $(x-a-\alpha)^2+(y-\beta)^2=(x+a-\alpha)^2$,

or, $(y-\beta)^2=(x+a-\alpha)^2-(x-a-\alpha)^2=4a(x-\alpha)$.

$\therefore$ The required equation is $(y-\beta)^2=4a(x-\alpha)\cdots(2)$.

(ii) *When the axis of the parabola is parallel to y-axis.*

Drawing a figure and proceeding as in (i) we have $(x-\alpha)^2=4a(y-\beta)\cdots(3)$ as the equation of the parabola.

(iii) Form of the equation of a parabola when its axis is parallel to x-axis or y-axis.

When the axis of the parabola is parallel to the x-axis, the equation of the parabola is $(y-\beta)^2=4a(x-\alpha)$ [From (2)],

or, $4ax=y^2-2\beta y+\beta^2+4a\alpha$, or, $x=\dfrac{1}{4a}y^2-\dfrac{\beta}{2a}y+\dfrac{\beta^2+4a\alpha}{4a}$,

i.e., $x=Ay^2+By+C$.

Again, when the axis of the parabola is parallel to the y-axis, the equation of the parabola is

$$(x-\alpha)^2=4a(y-\beta) \text{ [From (3)]}$$

or, $4ay=x^2-2\alpha x+\alpha^2+4a\beta$, or, $y=\dfrac{1}{4a}x^2-\dfrac{\alpha}{2a}x+\dfrac{\alpha^2+4a\beta}{4a}$,

i.e., $y=Ax^2+Bx+C$.

161 (C). *To determine the vertex of a parabola when its equation is of the forms* $x = Ay^2 + By + C$ *and* $y = Ax^2 + Bx + C$.

It appears from the equation $x = Ay^2 + By + C$ that the axis of the parabola is parallel to x-axis, so it is to be expressed in the form $(y-\beta)^2 = 4a(x-\alpha)$.

Now, $Ay^2 + By + C = x$, or, $y^2 + \frac{B}{A}y = \frac{x}{A} - \frac{C}{A}$,

or, $\left(y + \frac{B}{2A}\right)^2 = \frac{x}{A} + \frac{B^2}{4A^2} - \frac{C}{A} = \frac{x}{A} + \frac{B^2 - 4AC}{4A^2}$,

or, $\left(y + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(x + \frac{B^2 - 4AC}{4A}\right)$......(1)

The co-ordinates of the vertex of $(y-\beta)^2 = 4a(x-\alpha)$ are (α, β).

$\therefore$ Comparing this equation with (1) we find that the co-ordinates of the vertex are $\left(-\frac{B^2 - 4AC}{4A}, \ -\frac{B}{2A}\right)$.

The equation $y = Ax^2 + Bx + C$ can be similarly expressed in the form

$$\left(x + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(y + \frac{B^2 - 4AC}{4A}\right).$$

Since the axis of the parabola in this case is parallel to the y-axis, so comparing this equation with $(x-\alpha)^2 = 4a(y-\beta)$ we find the co-ordinates of the vertex to be

$$\left(-\frac{B}{2A}, \ -\frac{B^2 - 4AC}{4A}\right).$$

161 (D). *To find the equation of a parabola when its focus and the equation of the directrix are given. (General equation).*

Let the co-ordinates of the vertex S be (α, β) and the equation of the directrix be $AX + BY + C = 0$.

Let P (x, y) be any point on the parabola and let PM be perpendicular from P to the directrix.

Now, $SP^2=(x-\alpha)^2+(y-\beta)^2$ and $PM^2=\left\{\dfrac{Ax+By+C}{\sqrt{A^2+B^2}}\right\}^2$.

Since from the definition of the parabola $SP=PM$,

$$\therefore \quad SP^2=PM^2.$$

$$\therefore \quad (x-\alpha)^2+(y-\beta)^2=\left\{\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right\}^2,$$

or, $(A^2+B^2)\{(x-\alpha)^2+(y-\beta)^2\}=(Ax+By+C)^2$, which is the required equation.

This equation can be written as

$$(A^2+B^2)\{(x^2+y^2)-2(\alpha x+\beta y)+(\alpha^2+\beta^2)\} - (Ax+By)^2-2C(Ax+By)-C^2=0,$$

or, $$(A^2+B^2)(x^2+y^2)-(Ax+By)^2-2x\{\alpha(A^2+B^2)+CA\} - 2y\{\beta(A^2+B^2)+BC\}+(A^2+B^2)(\alpha^2+\beta^2)-C^2=0,$$

or, $$(Bx-Ay)^2-2x\{\alpha(A^2+B^2)+CA\}-2y\{\beta(A^2+B^2)+BC\} +(A^2+B^2)(\alpha^2+\beta^2)-C^2=0.$$

Hence the form of the general equation of the parabola will be $(lx+my)^2+2gx+2fy+c=0$.

[**N. B.** It is noticed in the above equation that the second degree terms in x and y are as a perfect square. So if the general form of the quadratic equation in x and y, i.e., $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a parabola, its second degree terms, i.e., $ax^2+2hxy+by^2$, will be a perfect square, and this will be so when $h^2=ab$.

Hence the condition for the general equation of the second degree in x and y representing a parabola is $h^2=ab$.]

Examples (20)

Ex. 1. Find the length of the latus rectum and the co-ordinates of the focus of each of the parabolas :

(*a*) $y^2=4x$, (*b*) $y^2=3x$, (*c*) $2y^2=5x$, (*d*) $y^2+6x=0$,
(*e*) $x^2=2y$, (*f*) $x^2=-4py$, (*g*) $y^2=px+q$.

The equation of a parabola is $y^2=4ax$.

(*a*) Here the equation is $y^2=4x$,

$\therefore \quad 4a=4, \quad \therefore \quad a=1.$

$\therefore$ The co-ordinates of the focus are $(a, 0)$, i.e, $(1, 0)$ and the latus rectum$=4a=4\times1=4$.

(*b*) Here $y^2=3x$; so $4a=3$, $\quad \therefore \quad a=\frac{3}{4}$.

$\therefore$ The co-ordinates of the focus are $(\frac{3}{4}, 0)$ and the latus rectum$=4a=4\times\frac{3}{4}=3$.

(*c*) Here $2y^2=5x$, or $y^2=\frac{5}{2}x$. So $4a=\frac{5}{2}$, $\quad \therefore \quad a=\frac{5}{8}$.

$\therefore$ The co-ordinates of the focus are $(\frac{5}{8}, 0)$ and the latus rectum$=4a=4\times\frac{5}{8}=\frac{5}{2}$.

(*d*) Since the equation of the parabola is $y^2+6x=0$,

or, $y^2=-6x$, so the parabola extends towards the negative side of the x-axis.

Its latus rectum$=4a=6$, and the co-ordinates of the focus are $(-a, 0)$, or, $(-\frac{3}{2}, 0)$.

(*e*) Here the equation of the parabola is $x^2=2y$, so its focus lies on the y-axis and its concavity is towards the positive side of the y-axis.

Here $4a=2$, $\quad \therefore \quad a=\frac{1}{2}$.

$\therefore$ The latus rectum$=4a=4\times\frac{1}{2}=2$, and the co-ordinates of the focus$=(0, a)=(0, \frac{1}{2})$.

(*f*) Here $x^2=-4py$ is the parabola, so its focus lies on the negative portion of the y-axis.

$\therefore$ The co-ordinates of the focus$=(0, -p)$ and the latus rectum$=4p$.

(*g*) Here the parabola is $y^2=px+q$, or, $y^2=p\left(x+\frac{q}{p}\right)$.

Now suppose $y = \text{Y}$ and $x+\frac{q}{p}=\text{X}$, i.e., suppose the origin to be transferred to the point $\left(-\frac{q}{p}, 0\right)$ and the axes to be the lines through this point parallel to the original axes.

Then the given equation becomes $\text{Y}^2=p\text{X}$. So the focus will be determined by $\text{X}=\frac{p}{4}$ and $\text{Y}=0$.

$\therefore$ With respect to the original axes the co-ordinates of the focus are $\left\{\left(\frac{p}{4}-\frac{q}{p}\right), 0\right\}$, or, $\left(\frac{p^2-4q}{4p}, 0\right)$ and the latus rectum $=p$.

Ex. 2. Find the vertex and the directrix of each of the parabolas : (*a*) $y^2=12x$ and (*b*) $y^2-4x-2y-7=0$.

(*a*) Here $4a=12$. $\therefore$ $a=3$.

$\therefore$ The vertex is the point $(0, 0)$ and the equation of the directrix is $x+a=0$, i.e., $x+3=0$.

(*b*) Here the given equation is $y^2-2y-4x-7=0$,

or, $y^2-2y+1=4x+8$, or, $(y-1)^2=4(x+2)$.

Now if we put $\text{Y}=y-1$ and $\text{X}=x+2$, i.e., if we transfer to parallel axes through the point $(-2, 1)$, the equation becomes $\text{Y}^2=4\text{X}$. Then $4a=4$, $\therefore$ $a=1$.

Hence the vertex is given by $\text{X}=0$ and $\text{Y}=0$, and the directrix by $\text{X}+1=0$.

$\therefore$ With respect to the original axes the vertex is the pt. $(-2, 1)$ and the directrix is the line $x+2+1=0$, or, $x+3=0$.

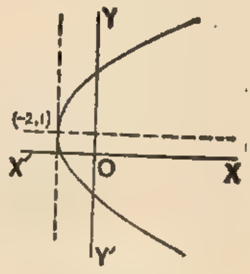


Fig. 14(*d*)

[N. B. In the Ex. 2(*b*) above, $\text{X}=0$, but $\text{X}=x+2$, so $x+2=0$, $\therefore$ $x=-2$. Again, $\text{Y}=0$, but $\text{Y}=y-1$, so $y-1=0$ $\therefore$ $y=1$. $\therefore$ $(-2, 1)$ is the vertex.]

Ex. 3. The parabola $y^2=4px$ passes through the point $(3, -2)$, find the latus rectum and co-ordinates of its focus.

[C. U. '34]

$\because$ The parabola $y^2=4px$ passes through the point $(3, -2)$,

$\therefore$ $(-2)^2=4p\times 3$ or $4=12p$, $\therefore$ $p=\frac{1}{3}$.

$\therefore$ The equation of the parabola becomes $y^2=\frac{4}{3}x$.

$\therefore$ The latus rectum $=\frac{4}{3}$.

Again, here $a=\frac{1}{4}\cdot\frac{4}{3}=\frac{1}{3}$, $\therefore$ the co-ordinates of the focus are $(a, 0)$ or $(\frac{1}{3}, 0)$.

Ex. 4. If $a+b\neq 0$, find the co-ordinates of the focus of the parabola $y^2=2mx$ which passes through the intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$.

[C. U. '52]

Solving the equations $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ we have $x=\frac{ab}{a+b}$ and $y=\frac{ab}{a+b}$; so the co-ordinates of the pt. of intersection of the lines are $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

Since the parabola $y^2=2mx$ passes through that pt. of intersection, we have

$$\left(\frac{ab}{a+b}\right)^2=2m\left(\frac{ab}{a+b}\right), \quad \therefore \quad m=\frac{ab}{2(a+b)}.$$

$\therefore$ The required co-ordinates of the focus are

$$\left\{\frac{ab}{4(a+b)}, 0\right\}.$$

Ex. 5. In the parabola $4(y-1)^2=-7(x-3)$, find (i) the latus rectum, (ii) the co-ordinates of the focus and the vertex.

[C. U. '56]

The given equation is $4(y-1)^2=-7(x-3)$

or, $(y-1)^2=-\frac{7}{4}(x-3)$

[See Art. 161. B (i)] Here the axis of the parabola is parallel to the x-axis and the parabola extends towards the negative side of x-axis.

Here the length of the latus rectum $=\frac{7}{4}$, the co-ordinates of the vertex are $(3, 1)$. Now, $4a=-\frac{7}{4}$, $\therefore$ $a=-\frac{7}{16}$.

$\therefore$ the co-ordinates of the focus$=\{(-\frac{7}{16}+3), 1\}=(\frac{41}{16}, 1)$.

Ex. 6. A double ordinate of a parabola $y^2=4ax$ is of length $8a$. Prove that the lines joining the vertex to its ends are at right angles. [W. B. H. S, 1960]

A 'double ordinate' is a chord parallel to the latus rectum.

Let PQ be such a double ordinate and let it cut the x-axis, *i.e.*, the axis of the parabola at N.

To prove that $\angle$PAQ is a right angle.

A is the vertex of the parabola. The ordinate (y) of the point P$=4a$ ($\because$ PQ is bisected at N).

Substituting $y=4a$ in the equation of the parabola, we have $4ax=16a^2$, $\therefore$ $x=4a$.

Hence we have AN$=$PN.

$\therefore$ ANP is an isosceles right-angled triangle.

$\therefore$ $\angle$PAN$=45^\circ$. Similarly $\angle$QAN$=45^\circ$.

$\therefore$ $\angle$PAQ$=90^\circ=$a right angle.

Otherwise : From the above proof we have the co-ordinates of P$=(4a, 4a)$, those of Q$=(4a, -4a)$ and those of A$=(0, 0)$.

Now, the gradient of PA$=\dfrac{4a}{4a}=1$ and that of

QA$=\dfrac{-4a}{4a}=-1$;

So the product of the gradients$=1\times-1=-1$.

$\therefore$ PA and QA are at right angles to each other, i.e., $\angle$PAQ is a right angle.

Ex. 7. Find the equation of the parabola whose focus is the point (2, 1) and whose directrix is the straight line $4x-3y=1$ and determine the length of the latus rectum.

Let $P(x, y)$ be any point on the parabola. Here the focus S is the pt. (2, 1). $\therefore$ $SP=\sqrt{(x-2)^2+(y-1)^2}$.

Again, if PM be perpendicular from P to the directrix $4x-3y=1$, then $PM=\dfrac{4x-3y-1}{\sqrt{4^2+3^2}}=\dfrac{4x-3y-1}{5}$.

$\because$ $SP^2=PM^2$, $\therefore$ $(x-2)^2+(y-1)^2=\dfrac{(4x-3y-1)^2}{25}$.

On simplification we have

$9x^2+24xy+16y^2-92x-56y+124=0$, which is the required equation of the parabola.

If SZ be the length of the perpendicular from S (2, 1) to the directrix $4x-3y-1=0$,

then $SZ=\dfrac{4\times2-3\times1-1}{\sqrt{4^2+3^2}}=\dfrac{4}{5}$.

The length of the latus rectum $LL'=2SZ$, but here $SZ=\frac{4}{5}$.

$\therefore$ the length of the latus rectum $=2\times\frac{4}{5}=\frac{8}{5}$.

Ex. 8. Find the equation of the parabola whose focus is at the point $(-1, 1)$ and whose directrix is the straight line $x+y+1=0$.

[C. U. (B. Sc.) '32]

Let $P(x, y)$ be any point on the parabola.

Here the focus S is the pt. $(-1, 1)$.

Now, the distance of P from $S=\sqrt{(x+1)^2+(y-1)^2}$.

Again, the distance of P from the directrix $x+y+1=0$ is

$$\frac{x+y+1}{\sqrt{1^2+1^2}} \text{ or } \frac{x+y+1}{\sqrt{2}}.$$

Since from the definition of the parabola these distances are equal,

$$\therefore \quad \sqrt{(x+1)^2+(y-1)^2}=\frac{x+y+1}{\sqrt{2}},$$

or, $(x+1)^2+(y-1)^2=\dfrac{(x+y+1)^2}{2}$,

or, $2(x+1)^2+2(y-1)^2=(x+y+1)^2$,

or, $2x^2+4x+2+2y^2-4y+2=x^2+y^2+1+2xy+2x+2y$,

or, $x^2+y^2+2x-6y-2xy+3=0$,

or, $(x-y)^2+2x-6y+3=0$, this is the required equation.

Ex. 9. If the focus of a parabola be (5, 3) and its vertex be (3, 1), find the equation of the parabola.

Here the co-ordinates of the focus S and the vertex A are (5, 3) and (3, 1) respectively. From SA produced cut off AZ = AS. Let the co-ordinates of Z be (α, β).

$\because$ A is the middle point of ZS,

$\therefore$ $3=\dfrac{5+\alpha}{2}$, or, $\alpha=1$, and $1=\dfrac{3+\beta}{2}$ or $\beta=-1$.

$\therefore$ The co-ordinates of Z are $(1, -1)$.

Now, the equation of any st. line passing through Z is $y+1=m(x-1)\cdots(1)$.

The st. line-(1) will be the directrix when it is perpendicular to AZ at Z.

The gradient of $AZ=\dfrac{1-(-1)}{3-1}=1$.

So the st. line-(1) will be the directrix when $m\times 1=-1$, i.e., when $m=-1$.

$\therefore$ The directrix of the parabola is the line

$y+1=-(x-1)$ or $x+y=0$.

Now take any point P (x, y) on the parabola.

$\because$ $SP^2=PM^2$, $\therefore$ $(x-5)^2+(y-3)^2=\left(\dfrac{x+y}{\sqrt{2}}\right)^2$,

or, $2(x^2+y^2-10x-6y+34)=x^2+2xy+y^2$,

or, $x^2-2xy+y^2-20x-12y+68=0$, this is the required equation of the parabola.

Ex. 10. If the vertex of a parabola be $(3, -2)$, the latus rectum be 4 and its axis is parallel to the x-axis, find its equation.

Here the co-ordinates of A are $(3, -2)$ and $LL' = 4$.

$\because$ The axis of the parabola is parallel to the x-axis,

$\therefore$ the co-ordinates of S are $(3 + \frac{4}{4}, -2)$ or $(4, -2)$ and the equation of the directrix is $x = 3 - 1$ or $x - 2 = 0$

[See the note in Art 161 B (i)]

Now take any pt. P (x, y) on the parabola.

$\because$ $SP^2 = PM^2$, $\therefore$ $(x-4)^2 + (y+2)^2 = (x-2)^2$,

or, $(y+2)^2 = (x-2)^2 - (x-4)^2$

$$= (x - 2 + x - 4)(x - 2 - x + 4) = 4(x - 3),$$

or, $(y+2)^2 = 4(x-3)$, which is the required equation of the parabola.

Ex. 11. Find the equation of the circle on the latus rectum of $y^2 = 4ax$ as diameter and show that it passes through the point where the axis of the parabola meets the directrix.

Here the equation of the parabola is $y^2 = 4ax$.

$\therefore$ The latus rectum $= 4a$.

$\because$ The diameter of the circle $=$ the latus rectum $= 4a$,

$\therefore$ its radius $= 2a$.

Again, since the latus rectum is the diameter, the focus S is the centre of the circle.

$\therefore$ The co-ordinates of the centre are $(a, 0)$.

$\therefore$ The required equation of the circle is

$$(x-a)^2 + (y-0)^2 = (2a)^2, \text{ or, } (x-a)^2 + y^2 = 4a^2.$$

Again, the axis of the parabola intersects the directrix at Z whose co-ordinates are $(-a, 0)$.

Putting the co-ordinates of Z in the equation of the circle, we have $(-a-a)^2 + 0 = 4a^2$, so the equation is satisfied by the co-ordinates.

Hence the circle passes through the point where the axis of the parabola meets the directrix.

Exercise 20

1. Find the latus rectum and the co-ordinates of the focus of each of the parabolas :—

(i) $y^2=14x$ (ii) $y^2=5x$ (iii) $2y^2=7x$

(iv) $y^2+2x=0$ (v) $x^2=4y$ (vi) $x^2+8y=0$

(vii) $2x^2+7y=0$ (viii) $y^2=ax+b$.

2. Find the vertex and the directrix of each of the parabolas (a) $x^2=8y$ (b) $x^2-4x-2y-7=0$.

3. The parabola $y^2=3px$ passes through the point $(2, -1)$, find the latus rectum and the co-ordinates of the focus.

4. Show that the straight line parallel to the axis of the parabola meets the parabola in one point only.

5. Find the equation of the directrix of the parabola $y^2=13x$ and find the co-ordinates of the ends of the latus rectum.

6. Find the vertex, the focus and the length of the latus rectum of the parabola $y=ax^2+bx+c$. [C. U. '12]

7. Find the length of the latus rectum of the parabola $5y^2=7x$, and also the co-ordinates of the focus. [C. U. '36]

8. The parabola $y^2=4px$ goes through the point $(3, -2)$. Obtain the length of the latus rectum and the co-ordinates of the focus. [C. U. '34]

9. Find the latus rectum and the co-ordinates of the focus of the parabola $3y^2=4x$, and determine the points in which it is met by the straight line $2x=3y$. [C. U. '35]

10. Find the vertex, the focus and the latus rectum of the parabola $y^2=4y-4x$. [C. U.]

11. Find the vertex, the focus and the directrix of the parabola $(y+3)^2=2(x+2)$. [U. P. B. '46]

12. Find the point of intersection of the lines $\frac{x}{3}+\frac{y}{2}=1$ and $\frac{x}{2}+\frac{y}{3}=1$, and determine the co-ordinates of the focus of the parabola $y^2=2ax$ which passes through this point. [C. U. '43]

13. Find the equation of the parabola :

(a) Whose focus is at the point $(2, -1)$ and whose directrix is the straight line $2x+3y=6$.

(b) Whose directrix is $3x-4y+5=0$ and whose focus is $(1, 3)$.

(c) Whose vertex is the origin, axis is the y-axis and which passes through the point $(6, 9)$.

(d) Whose vertex is $(-2, 2)$ and focus is $(1, -2)$.

(e) Whose focus is $(0, 0)$ and the tangent at the vertex is the line $y-x=4$.

(f) Whose latus rectum is $10\sqrt{2}$, axis is the line $x=y$ and the equation of the directrix is $x+y+5=0$.

14. Find the equation of a parabola :—

(a) Whose vertex is $(-3, 2)$, latus rectum is 8 and axis is parallel to the x-axis.

(b) Whose vertex is $(1, -1)$, latus rectum is 8 and axis is parallel to the y-axis.

15. Obtain the equation of the parabola whose focus is at the point $(3, -2)$ and whose directrix is the straight line $2x-y+3=0$. [C. U. '58]

16. A parabola opens out along the positive direction of the axis of y. Its focus is the point $(0, 3)$ and the length of the latus rectum is 12. Find its equation. [C. U. '50 (Sp)]

17. Prove that the equation $y^2+2ax+2by+c=0$ represents a parabola whose axis is parallel to the axis of x. Find its vertex. [C. U. '53]

18. Find the equation to the parabola whose axis is parallel to the y-axis and which passes through the points $(0, 4)$, $(1, 9)$ and $(-2, 6)$. Also determine its latus rectum.

19. A parabola opens out along the positive axis of y as the axis. Its focus is the point $(0, 3)$ and the length of its latus rectum is 12. Find its equation. [C. U. '50 Comp.]

20. A double ordinate of the curve $y^2=4px$ is of length $8p$; prove that the lines from the vertex to its ends are at right angles. [C. U. '52]

21. Find the equation of a parabola whose focus is at the origin and whose directrix is the straight line $2x+y-1=0$. Also determine its latus rectum. [C. U. '54]

22. Find the equation of the circle whose centre is at the vertex of the parabola $y^2=8x$ and which passes through the focus of it.

23. Find the radius of the circle passing through the extremities of the latus rectum and the vertex of a parabola. (Give the result in terms of the latus rectum.)

162. *To find the* **length of the chord** *of the parabola $y^2=4ax$ intercepted on the line $y=mx+c$.*

Let the line $y=mx+c\cdots(1)$ cut the parabola $y^2=4ax\cdots(2)$ at P, Q and let their co-ordinates be (x_1, y_1) and (x_2, y_2) respectively. Now putting the value of y from (1) in (2) we have $(mx+c)^2=4ax$,

or $\quad m^2x^2+2(mc-2a)x+c^2=0\cdots(3)$

Evidently the roots of the equation-(3) will be x_1, x_2.

$$\therefore \quad x_1+x_2=\frac{-2(mc-2a)}{m^2} \text{ and } x_1x_2=\frac{c^2}{m^2}.$$

$$\therefore \quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$
$$=\frac{4(mc-2a)^2}{m^4}-\frac{4c^2}{m^2}=\frac{16a(a-mc)}{m^4}.$$

Again, since the pts. P and Q also lie on the st. line-(1),

$\therefore \quad y_1=mx_1+c\cdots(4)$ and $y_2=mx_2+c\cdots(5)$

Subtracting (5) from (4) we have

$$y_1-y_2=m(x_1-x_2)\cdots(6)$$

Now, the length of the chord PQ

$$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$=\sqrt{(x_1-x_2)^2+m^2(x_1-x_2)^2}$$
$$=(x_1-x_2)\sqrt{1+m^2}$$
$$=\frac{4}{m^2}\sqrt{1+m^2}.\sqrt{a(a-mc)}.$$

Corollary : When $PQ=0$, the st. line-(1) will be a tangent to the parabola-(2).

$\therefore$ the condition for the line-(1) being a tangent is

$$\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}=0.$$

Here, neither m nor a is zero, so $a-mc=0$,

$\therefore$ $c=\dfrac{a}{m}$ and this is the condition.

163. *To find the condition that the straight line $y=mx+c$ may touch the parabola $y^2=4ax$.*

Putting the value of y from $y=mx+c$ in $y^2=4ax$,

we have $(mx+c)^2=4ax$,

or, $m^2x^2+2(mc-2a)x+c^2=0\cdots(1)$.

This is a quadratic equation in x. Its two roots will represent the x-co-ordinates (i.e., abscissæ) of the two points of intersection of the given st. line and the parabola. When the st. line will touch the circle, the two points of intersection must coincide and the condition for which is that the roots of the equation-(1) must be equal.

The roots will be equal, if $4(mc-2a)^2-4m^2c^2=0$,

or, $m^2c^2-4amc+4a^2-m^2c^2=0$, or, $4amc=4a^2$,

or, $mc=a$, or, $c=\dfrac{a}{m}$.

$\therefore$ The condition that the st. line $y=mx+c$ may touch the parabola $y^2=4ax$ is $c=\dfrac{a}{m}$.

Corollary : From the above condition we have that $y=mx+\dfrac{a}{m}$ is a tangent to $y^2=4ax$ for any value of m.

164. *When $y=mx+c$ touches the parabola $y^2=4ax$, find the point of contact.*

We know that $y=mx+\dfrac{a}{m}\cdots(1)$ is always a tangent to the parabola $y^2=4ax\cdots(2)$.

Now substituting the value of y from (1) in (2) we have

$\left(mx+\dfrac{a}{m}\right)^2=4ax$, or, $\left(mx+\dfrac{a}{m}\right)^2-4ax=0$,

or, $\left(mx-\dfrac{a}{m}\right)^2=0$, or, $mx=\dfrac{a}{m}, \dfrac{a}{m}$; $\therefore$ $x=\dfrac{a}{m^2}, \dfrac{a}{m^2}$.

Putting these values of x in (1) we have

$$y=m\times\frac{a}{m^2}+\frac{a}{m}=\frac{2a}{m},\quad \therefore\ y=\frac{2a}{m}, \frac{2a}{m}.$$

Hence, the required co-ordinates of the point of contact are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

165. *Two tangents can be drawn to a parabola from an external point.*

The st. line $y=mx+\dfrac{a}{m}$ is always a tangent to the parabola $y^2=4ax$.

If the tangent passes through an external point (x', y'), then $y'=mx'+\dfrac{a}{m}$, or, $m^2x'-my'+a=0$, and it is a quadratic equation in m. So we have two values of m from the equation.

Hence two tangents can be drawn from the external point (x', y') to the parabola.

166. *To find the* ***equation of the tangent*** *to the parabala $y^2=4ax$ at the point (x_1, y_1).*

Let (x_2, y_2) be another point on the parabola.

Now, the equation of the st. line passing through the points (x_1, y_1) and (x_2, y_2) is $y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)\cdots(1)$

Since the two points lie on the parabola, we have $y_1^2=4ax_1$ and $y_2^2=4ax_2$;

$\therefore\ y_1^2-y_2^2=4a(x_1-x_2)$, $\therefore\ \frac{y_1-y_2}{x_1-x_2}=\frac{4a}{y_1+y_2}\cdots(2)$.

Hence from (1) and (2) we have $y-y_1=\frac{4a}{y_1+y_2}(x-x_1)$ as the equation of the chord passing through the points (x_1, y_1) and (x_2, y_2).

This chord will be a tangent when the points (x_1, y_1) and (x_2, y_2) are coincident points. Then $x_2=x_1$ and $y_2=y_1$.

$\therefore$ The equation of the tangent is

$y-y_1=\frac{4a}{2y_1}(x-x_1)$, or, $yy_1-y_1^2=2ax-2ax_1$,

or, $yy_1=2ax+y_1^2-2ax_1$, or, $yy_1=2ax+4ax_1-2ax_1$
$=2ax+2ax_1=2a(x+x_1)$.

$\therefore$ The equation of the tangent at the pt. (x_1, y_1) is

$$\boldsymbol{yy_1=2a(x+x_1)}.$$

[N. B. If we write the equation $y^2=4ax$ of the parabola in the form $y.y=2a(x+x)$, and put x_1, y_1 in place of one x and one y, then we have the equation of the tangent.]

167. *To find the* **equation of the normal** *to the parabola $y^2=4ax$ at the point (x_1, y_1).*

Let the equation of the st. line passing through (x_1, y_1) be $y-y_1=m(x-x_1)$.

$\because$ This st. line being a normal is perpendicular to the tangent $yy_1=2a(x+x_1)$ at the point,

$\therefore\ m\times\frac{2a}{y_1}=-1$ [$\because$ the gradient of the tangent is $\frac{2a}{y_1}$]

$\therefore\ m=-\frac{y_1}{2a}$.

$\therefore$ The equation of the normal to the parabola is

$$y-y_1=-\frac{y_1}{2a}(x-x_1), \quad \text{or,} \quad 2a(y-y_1)+y_1(x-x_1)=0,$$

or, $\mathbf{y_1x+2ay=y_1(x_1+2a)}$.

168. *To find the equation of the normal in terms of its gradient.*

In Art. 167 we have taken $-\frac{y_1}{2a}=m\cdots(1)$

$\therefore$ m is the gradient of the normal.

Now, from (1) we have $y_1=-2am$.

Since (x_1, y_1) is a point on the parabola $y^2=4ax$,

we have $y_1^2=4ax_1$,

$$\therefore \quad x_1=\frac{y_1^2}{4a}=\frac{4a^2m^2}{4a}=am^2.$$

Now, putting the values of x_1 and y_1 in the equation of the normal $y-y_1=-\frac{y_1}{2a}(x-x_1)$, we have $y+2am=m(x-am^2)$,

or, $y=mx-2am-am^3\cdots(2)$, this is the equation of the normal.

Corollary : $(am^2,-2am)$ are the the co-ordinates of the foot of the normal, *i.e.*, of the point on the parabola at which (2) is a normal.

169. *To prove that, in general, three normals can be drawn from a point to a parabola.*

Let (x_1, y_1) be the point from which the normal is to be drawn.

The equation of the normal to the parabola $y^2=4ax$ at the pt. $(am^2, -2am)$ is $y=mx-2am-am^3 \cdots\cdots(1)$, and this equaion-(1) passes through the point (x_1,y_1), if $y_1=mx_1-2am-am^3$, *i.e.*, if $am^3+(2a-x_1)m+y_1=0 \ldots(2)$.

This is a cubic equation in m giving three values of m, generally distinct, for which the normal at $(am^2, -2am)$ passes through the point (x_1, y_1).

Hence three normals can be drawn from any general point.

Corollary : (1) If the three roots of the equation-(2) above be m_1, m_2, m_3, then the co-ordinates of the feet of the three normals are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. These feet are called the **co-normal** points.

(2) Since the coefficient of m^2 in (2) is zero, the sum of the three roots will be zero, *i.e.* $m_1+m_2+m_3=0$.

(3) The sum of the ordinates of the feet of the normals that can be drawn from any point to a parabola is zero, *i. e.*, $-2am_1-2am_2-2am_3=0$,

or, $-2a(m_1+m_2+m_3)=0$ $[\because m_1+m_2+m_3=0]$.

170. *To find the equation of the chord of contact of tangents drawn from the point (x', y') to the parabola $y^2=4ax$).*

The equations of the tangents at the points (x_1, y_1) and (x_2, y_2) are $yy_1=2a(x+x_1)$ and $yy_2=2a(x+x_2)$ respectively.

If the two tangents pass through the point (x', y'), then $y'y_1=2a(x'+x_1)$ and $y'y_2=2a(x'+x_2)$; from this it is obvious that the st. line $yy'=2a(x+x')$ passes through the points (x_1, y_1) and (x_2, y_2), *i. e.*, it passes through the points of contact of the two tangents that can be drawn from the point (x', y') of the parabola.

$\therefore$ The equation to the chord of contact of tangents from the point (x', y') is $\mathbf{yy'=2a(x+x')}$.

171. Subtangent and subnormal.

P is a point on the parabola (fig. 15), PN is perpendicular to the axis, and the tangent PT and the normal PG at the point P intersect the axis of the parabola at T and G respectively.

TN is the *subtangent* and NG is the *subnormal.*

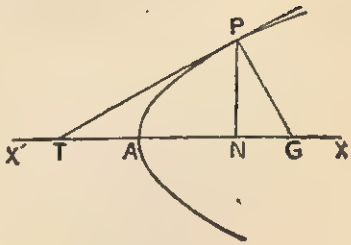


Fig. 15

172. *The subtangent at any point of a parabola is bisected at the vertex.*

Let $P(x', y')$ be a point on the parabola $y^2=4ax$.

Let PN be perpendicular to the axis and let the tangent PT intersect the axis at T.

The equation of the tangent PT is $yy'=2a(x+x')$, but the ordinate of T is 0,

$\therefore \quad 0=2a(x+x'), \quad \therefore \quad x=-x'.$

So the abscissa of the pt. $T=-x'$, *i.e.*, $AT=x'$ (neglecting the negative sign), but $AN=x'$, $\therefore$ $AT=AN$.

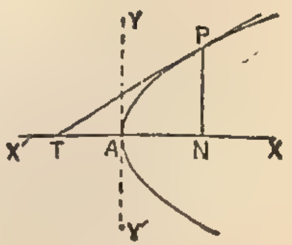


Fig. 16

Hence the subtangent TN is bisected at the vertex A.

173. *The normal to a parabola at any point on it makes equal angles with the focal distance of the point and the axis.*

Let the vertex A be the origin, the axis of the parabola be the x-axis and the latus rectum be $4a$. Then the equation of the parabola is $y^2=4ax$. Let $P(x_1, y_1)$ be a point on it and let the tangent PT and the normal PG cut the axis at T and G respectively.

To prove that $\angle SPG = \angle SGP$.

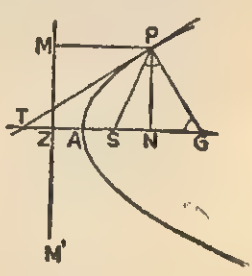


Fig. 17

Draw PM perpendicular to the directrix MM$'$.

$\because$ The subtangent TN is bisected at the vertex A,

$\therefore$ AT$=$AN. $\therefore$ TS$=$TA$+$AS$=$AN$+$AS$=x_1+a$.

Again, SP$=$PM$=$ZN$=$AN$+$ZA$=$AN$+$AS$=x_1+a$.

$\therefore$ SP$=$ST, $\therefore$ $\angle$SPT$=\angle$STP.

Now, $\because$ $\angle$GPT$=$1 right angle,

$\therefore$ $\angle$PTG$+\angle$PGT$=$1 right angle.

$\therefore$ $\angle$SPT$+\angle$SPG$=$1 right angle$=\angle$STP$+\angle$SGP.

But $\angle$SPT$=\angle$STP (proved), $\therefore$ $\angle$SPG$=\angle$SGP.

Corollary : $\because$ $\angle$SPG$=\angle$SGP, $\therefore$ SP$=$SG.

$\therefore$ ST$=$SP$=$SG.

174. *The subnormal at any point on a parabola is equal to the semi-latus rectum.*

[See Fig. 15] Let the normal PG drawn at the point P(x_1, y_1) of the parabola $y^2=4ax$ intersect the axis $(y=0)$ at G, and let PN be perpendicular on the axis.

Then NG is the subnormal and the length of the latus rectum$=4a$.

To prove that the subnormal NG$=$half of the latus rectum $=2a$.

Here AN$=x_1$, PN$=y_1$.

The equation of the normal to the parabola $y^2=4ax$ at P(x_1, y_1) is $y_1x+2ay=y_1(x_1+2a)$, or, $y-y_1=-\dfrac{y_1}{2a}(x-x_1)$.

Since this normal cuts the axis $(y=0)$ at G, the ordinate of G is zero and its abscissa$=$AG$=x$.

So from the above equation we have

$$0-y_1=-\frac{y_1}{2a}(x-x_1), \text{ or, } -2ay_1=-y_1(x-x_1)$$

$\therefore$ $x-x_1=2a$, or, AG$-$AN$=2a$ ($\because$ AG$=x$, AN$=x_1$).

$\therefore$ The subnormal NG$=$AG$-$AN$=2a=$the semi-latusrectum.

[**N. B.** Since the length of the latus rectum is constant, the length of the subnormal also at any point of the parabola is constant.]

175. *The portion of a tangent to a parabola at any point on it, intercepted between the point of contact and the directrix subtends a right angle at the focus.*

Let S be the focus, MM′ the directrix of the parabola $y^2=4ax$ and $P(x_1, y_1)$ be any point on it. Let the tangent to the parabola at the pt. P cut the directrix at R.

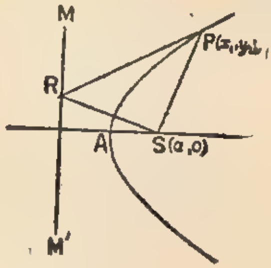


Fig. 18

To prove that $\angle PSR=1$ right angle.

Here the vertex A is the origin, the axis of the parabola is the x-axis and the distance of the focus from the vertex (*i.e.*, AS)$=a$. Then the co-ordinates of S are $(a, 0)$.

Now the equation of the directrix MM′ is $x=-a$ ......(1)

and that of the tangent PR is $yy_1=2a(x+x_1)$......(2).

From (1) and (2) we have $x=-a$ and $y=\dfrac{2a}{y_1}(-a+x_1)$.

$\therefore$ The co-ordinates of $R=\left\{-a, \dfrac{2a}{y_1}\left(-a+x_1\right)\right\}$.

$\because$ The co-ordinates of P and S are (x_1, y_1) and $(a, 0)$ respectively,

$\therefore$ the gradient m_1 of $SP=\dfrac{y_1-0}{x_1-a}=\dfrac{y_1}{x_1-a}$.

Similarly the gradient m_2 of SR

$$=\frac{\frac{2a}{y_1}(-a+x_1)-0}{-a-a}=\frac{2a(-a+x_1)}{-2ay_1}=\frac{(x_1-a)}{-y_1}.$$

$\therefore$ $m_1m_2=\dfrac{y_1}{x_1-a}\times\dfrac{(x_1-a)}{-y_1}=-1$, $\therefore$ $SP\perp SR$.

Hence $\angle PSR$ is a right angle.

180. *The parameter of any diameter of a parabola is four times the focal distance of the vertex of the diameter.*

Let MM' be the directrix of the parabola, S the focus and PSP' be a chord passing through the focus.

Let SK be perpendicular to PSP' and let it cut MM' at K. Draw KV perpendicular to the directrix, and let it cut the parabola at B and PSP' at V. Then KV is the diameter of the chord through the focus, PSP' is the parameter of the diameter KV and B the vertex of KV. Join BS.

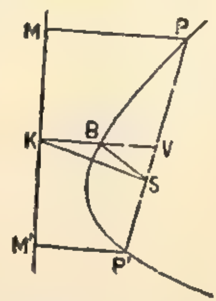


Fig. 20

To prove that $PP'=4BS$.

Draw PM and $P'M'$ perpendicular to the directrix.

Proof : $\because$ PM, KV and $P'M'$ are perpendicular to the same straight line.

$\therefore$ They are parallel to each other.

Again, since K is the mid point of MM', and V is the mid point of PP', $\therefore$ $KV=\frac{1}{2}(PM+P'M')$.

Now, $PP'=PS+P'S=PM+P'M'=2KV$.

Again, $\because$ $BK=BS$, $\therefore$ $\angle BKS=\angle BSK$, and

$\because$ $\angle KSV$ is a right angle, $\therefore$ $\angle BSV=\angle BVS$.

$\therefore$ $BS=BV$, $\therefore$ $BK=BS=BV$, $\therefore$ $KV=2BS$.

Now, $PP'=2KV=2\times 2BS=4BS$.

[Alernative Proof.]

Let PP' be a chord through the focus of the parabola $y^2=4ax\cdots$(i). [See Fig. 20],

and let $y=m(x-a)\cdots$(ii) be the equation of PP'.

The equation of the diameter BV bisecting this chord is $y=\dfrac{2a}{m}$, and the co-ordinates of the vertex B at which the diameter cuts the parabola are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$. [See Art. 183]

Putting the value of y from (ii) in (i) we have

$m^2(x-a)^2=4ax$, or, $m^2x^2-2a(m^2+2)x+a^2m^2=0$.

Let x_1 and x_2 be the roots of this equation. Then x_1 and x_2 are the x-co-ordinates of P and P′, the points of intersection of (i) and (ii).

$$\therefore\quad x_1+x_2=\frac{2a(m^2+2)}{m^2}.$$

Now, $\quad PP'=PS+P'S=(a+x_1)+(a+x_2)=2a+(x_1+x_2)$

$$=2a+\frac{2a(m^2+2)}{m^2}=4a\left(1+\frac{1}{m^2}\right).$$

Again, $\quad BS^2=\left(\frac{a}{m^2}-a\right)^2+\left(\frac{2a}{m}\right)^2=a^2\left\{\left(\frac{1}{m^2}-1\right)^2+\frac{4}{m^2}\right\}$

$$=a^2\left(1+\frac{1}{m^2}\right)^2.$$

$\therefore\quad BS=a\left(1+\frac{1}{m^2}\right).\qquad \therefore\quad PP'=4BS.$

181. *To find the equation of a chord of a parabola which is bisected at the given point* (h, k).

Let $y^2=4ax$ be the parabola and $y=mx+c$ be the chord.

Then by the above theorem (Art. 177) $k=\frac{2a}{m}$, $\therefore m=\frac{2a}{k}$.

Again, $\because$ the point (h, k) is on the chord $y=mx+c$,

$\therefore\quad k=mh+c$.

$\therefore\quad$ The equation of the chord is $y-k=m(x-h)$,

or, $\quad y-k=\frac{2a}{k}(x-h)$, or, $k(y-k)=2a(x-h)$.

182. *To find the locus of the middle points of the chords of a parabola, which pass through the given point* (α, β).

Let (h, k) be the middle point of a chord of the parabola $y^2=4ax$.

Then the equation of this chord is $k(y-k)=2a(x-h)$.

Since this chord passes through the point (α, β),

we have $k(\beta-k)=2a(\alpha-h)$.

$\therefore$ The locus of the middle point (h, k) is

$y(\beta-y)=2a(\alpha-x)$, or, $y^2-\beta y=2a(x-\alpha)$, which is a parabola.

183. *To show that the tangent at the vertex with respect to a diameter is parallel to the system of parallel chords bisected by the diameter.*

Suppose the chords of the parabola $y^2=4ax$ are parallel to the st. line $y=mx+c$.

Then the equation of the diameter is $y=\frac{2a}{m}$.

Hence $x=\frac{y^2}{4a}=\frac{a}{m^2}$ at the point where the diameter intersects the parabola.

$\therefore$ The co-ordinates of the vertex of the diameter are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

$\therefore$ The equation of the tangent at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

is $y.\frac{2a}{m}=2a\left(x+\frac{a}{m^2}\right)$, or, $y=mx+\frac{a}{m}$.

$\because$ the st. line $y=mx+\frac{a}{m}$ is parallel to the st. line $y=mx+c$, $\therefore$ the tangent is parallel to the chords.

184. *If* P *and* P$'$ *be a pair of points on the parabola $y^2=4ax$ such that the normal at* P *is parallel to the tangent at* P$'$, *then prove that the chord* PP$'$ *passes through the focus.*

Let the normal at the point P on the parabola $y^2=4ax$ be parallel to the tangent at the point P$'$.

Then their gradients must be equal.

Let the gradient of both be m.

The equation of the normal at P is $y=mx-2am-am^3$,

$\therefore$ the co-ordinates of P are $(am^2, -2am)$.

The equation of the tangent at P′ is $y=mx+\dfrac{a}{m}$,

$\therefore$ the co-ordinates of P′ are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

So the equation of the chord PP′ is

$$\frac{y+2am}{-2am-\dfrac{2a}{m}}=\frac{x-am^2}{am^2-\dfrac{a}{m^2}}, \text{ or, } y+2am=\frac{2m}{1-m^2}(x-am^2).$$

Now, putting the co-ordinates $(a, 0)$ of the focus S in this equation, we have its left side$=0+2am=2am$, and its right side$=\dfrac{2m}{1-m^2}(a-am^2)=\dfrac{2am}{1-m^2}(1-m^2)=2am$.

$\therefore$ Both the sides are equal, *i.e.*, the equation of the chord PP′ is satisfied by the co-ordinates of the focus.

$\therefore$ The chord PP′ passes through the focus.

Examples (21)

Ex. 1. Find the co-ordinates of the points of intersection of the straight line $x-5y+6=0$ with the parabola $y^2=x$.

$\because$ $x-5y+6=0$, $\therefore$ $x=5y-6$······(1)

Putting this value of x in $y^2=x$, we have $y^2=5y-6$,

or, $y^2-5y+6=0$, or, $(y-2)(y-3)=0$, $\therefore$ $y=2$ or 3.

$\therefore$ from (1) we have $x=4$ or 9.

$\therefore$ The co-ordinates of the two points of intersection are $(4, 2)$ and $(9, 3)$.

Ex. 6. Find the equation of the tangent to the parabola $y^2=4x$, which is parallel to the straight line $x+2y=3$.

The equation of the st. line parallel to the st. line $x+2y=3$ is $x+2y=c$. Then $x=c-2y$.

Putting this value of x in $y^2=4x$, we have $y^2=4c-8y$ or $y^2+8y-4c=0$, The two roots of this quadratic equation will be the y co-ordinates of the two points of intersection of the given parabola and the st. line $x+2y=c$.

So the st. line will be a tangent when the two points of intersection will coincide, *i.e.*, when the roots of the equation

$y^2+8y-4c=0$ are equal. The roots of this equation are equal when $64+16c=0$, or, $c=-4$.

$\therefore$ The required equation of the tangent is

$x+2y=-4$ or $x+2y+4=0$.

Ex. 7. Find the equations of the tangent and the normal to the parabolas :

(*i*) $y^2=4x$ at $(1, 2)$; (*ii*) $y^2+12x=0$ at $(-3, 6)$; (*iii*) $y^2=12x$ at the ends of the latus rectum.

(*i*) The equation of the tangent to the parabola $y^2=4ax$ at (x_1, y_1) is $yy_1=2a(x+x_1)$.

Here $4a=4$, $\therefore$ $a=1$, and $(1, 2)$ is the given point $\therefore$ $x_1=1$, $y_1=2$.

$\therefore$ The required equation of the tangent is

$y.2=2(x+1)$ or $y=x+1$.

$\therefore$ The equation of the normal is $y-y_1=-\frac{y_1}{2a}(x-x_1)$

i. e., $y-2=-\frac{2}{2}(x-1)$, or, $y-2=-x+1$, or, $x+y=3$.

(*ii*) The equation of the tangent to the parabola $y^2=-4ax$ at the point (x_1, y_1) is $yy_1=-2a(x+x_1)$, and the equation of the normal is $y-y_1=\frac{y_1}{2a}(x-x_1)$.

Here $y^2=-12x$, $\therefore$ $a=3$. The given point is $(-3, 6)$,

$\therefore$ $x_1=-3$, $y_1=6$.

$\therefore$ The required equation of the tangent is

$$y.6=-6(x-3), \quad \text{or,} \quad x+y=3.$$

Again, the equation of the normal is $y-6=\frac{6}{6}(x+3)$,

or, $y-6=x+3$, or, $y=x+9$.

(*iii*) Here $a=12\div4=3$. The co-ordinates of the focus are $(3, 0)$ and the length of the latus rectum $=12$.

$\therefore$ The co-ordinates of the ends of the latus rectum $=(3, 6)$ and $(3, -6)$.

$\therefore$ The equation of the tangent at $(3, 6)$ is $y.6=6(x+3)$, or, $y=x+3$.

The equation of the normal at the point is $y-6=-\frac{6}{6}(x-3)$ or $x+y=9$.

Again, the tangent at $(3, -6)$ is the st. line

$y(-6)=6(x+3)$ or $x+y+3=0$,

and the normal at the point is $y+6=-\frac{-6}{6}(x-3)$,

or, $y=x-9$.

Ex. 8. Prove that the line $2x+4y=9$ is a normal to the parabola $y^2=8x$. Find the foot of the normal.

The general equation of the normal to the parabola $y^2=4ax$ is $y=mx-2am-am^3\cdots(1)$ and the co-ordinates of its foot are $(am^2, -2am)\cdots(2)$ [Art. 168].

Here the equation of the parabola is $y^2=8x$,

$$\therefore \quad 4a=8, \qquad \therefore \quad a=2.$$

$\therefore$ The general equation of the normal to the parabola $y^2=8x$ will be $y=mx-4m-2m^3$ and its foot $(2m^2, -4m)$.

If a value of m can be found for which the equation of the parabola will be the same as $2x+4y=9$, then the st. line will be a normal to the parabola.

Now, we have $2x+4y=9$ or $y=-\frac{1}{2}x+\frac{9}{4}$.

Putting $m=-\frac{1}{2}$, the equation of the normal is

$y=-\frac{1}{2}x-4(-\frac{1}{2})-2(-\frac{1}{2})^3$, or, $y=-\frac{1}{2}x+2+\frac{1}{4}$,

or, $y+\frac{1}{2}x=\frac{9}{4}$, or, $2x+4y=9$.

Again, $2m^2=2(-\frac{1}{2})^2=\frac{1}{2}$ and $-4m=-4\times-\frac{1}{2}=2$.

Hence the st. line $2x+4y=9$ is a normal to the parabola $y^2=8x$ and its foot is $(\frac{1}{2}, 2)$.

Ex. 9. Find the point of the parabola $y^2=4ax$ at which the normal is inclined at 30° to the axis. [C. U.]

Here gradient m of the normal$=\tan 30°=\dfrac{1}{\sqrt{3}}$.

The co-ordinates of the foot$=(am^2, -2am)$

$$=\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right).$$

Ex. 10. A tangent to the parabola $y^2=12x$ makes an angle of 60° with the x-axis; find its equation and its point of contact.

Let $y=mx+c$ be the equation of the tangent.

Then $m=\tan 60°=\sqrt{3}$.

$$\therefore\quad c=\frac{a}{m}=\frac{3}{\sqrt{3}}=\sqrt{3}\ [\because \text{ here } a=12\div 4=3]$$

$\therefore$ The equation of the tangent is $y=\sqrt{3}x+\sqrt{3}$,

or, $y=\sqrt{3}(x+1)$.

Now, putting this value of y in $y^2=12x$ we have

$3(x+1)^2=12x$, or, $(x+1)^2-4x=0$, or, $(x-1)^2=0$,

$\therefore$ $x=1, 1$. $\therefore$ $y=\sqrt{3}(x+1)=\sqrt{3}(1+1)=2\sqrt{3}$.

$\therefore$ The co-ordinates of the point of contact are $(1, 2\sqrt{3})$.

Ex. 11. Find the equation of the chord of the parabola $y^2=12x$ which is bisected at the point (2, 3).

Here, $h=2$ and $k=3$, the middle pt. being (2, 3),

and $\because$ $4a=12$ (from the given equation), $\therefore$ $a=3$.

Now from the formula $k(y-k)=2a(x-h)$, the required equation is $3(y-3)=6(x-2)$ or $y=2x-1$.

Ex. 12. Find the locus of the middle points of all chords of the parabola $y^2=8x$, which pass through the point $(1, -2)$.

Let (h, k) be the middle point of one of the chords. Its equation is $k(y-k)=4(x-h)$, and since it passes through the point $(1, -2)$, we have $k(-2-k)=4(1-h)$,

or $k^2+2k=4(h-1)$.

$\therefore$ The locus of the middle point is $y^2+2y=4(x-1)$, which is a parabola.

Ex. 13. Find the equation of the diameter of the parabola $2y^2=3x$, which bisects all chords parallel to the straight line $4x+5y=20$.

The parabola is $2y^2=3x$ or $y^2=\frac{3}{2}x$, $\therefore$ here $a=\frac{3}{8}$.

Again, the equation of the st. line is $y=-\frac{4}{5}x+4$.

$\therefore$ here $m=-\frac{4}{5}$.

$\therefore$ The equation of the diameter is

$$y=\frac{2a}{m}=\frac{2\times\frac{3}{8}}{-\frac{4}{5}}=-\frac{3}{4}\times\frac{5}{4}=-\frac{15}{16}.$$

$\therefore$ The required equation of the diameter is $16y+15=0$.

Ex. 14. Prove that the normal chord of a parabola at the point whose ordinate is equal to the abscissa subtends a right angle at the focus. [C. U. '40]

Let $y^2=4ax$ be the parabola. Here the abscissa and the ordinate of the point are equal. Let each be α.

$$\therefore \quad \alpha^2=4a\alpha, \qquad \therefore \quad \alpha=4a.$$

$\therefore$ $(4a, 4a)$ are the co-ordinates of the point whose abscissa and ordinate are equal.

Let PQ be the normal chord, P being the pt. $(4a, 4a)$.

$\therefore$ The equation of PQ is $y-4a=\dfrac{-4a}{2a}(x-4a)$,

or, $y=-2x+12a\cdots(1)$

From (1) we have $x=\dfrac{12a-y}{2}$, putting this value of x in the equation of the parabola, we have

$$y^2=2a(12a-y), \quad \text{or,} \quad y^2+2ay-24a^2=0,$$

or, $(y+6a)(y-4a)=0$, $\therefore$ $y=-6a$ or $4a$.

$\therefore$ The ordinate of Q is $-6a$.

The abscissa of Q is obtained by putting $y=-6a$ in (1).

So we have $-6a=-2x+12a$, $\therefore$ $x=9a$.

$\therefore$ The co-ordinates of Q and the focus S are $(9a, -6a)$ and $(a, 0)$ respectively.

Now, the gradient of SP $=\dfrac{4a-0}{4a-a}=\dfrac{4}{3}$, and the gradient of SQ $=\dfrac{-6a-0}{9a-a}=-\dfrac{3}{4}$.

$\therefore$ The product of the two gradients $=\frac{4}{3}\times-\frac{3}{4}=-1$.

$\therefore$ SP and SQ are at right angles to each other.

$\therefore$ The normal chord PQ subtends a right angle at the focus S.

Ex. 15. Two equal parabolas have the same vertex and their axes are at right angles ; prove that the common tangent touches each parabola at the end of a latus rectum. [C. U. '35]

[As here the parabolas are equal, their latera recta must be equal. Again because their axes are at right angles, it is evident that if the axis of one parabola be along the x-axis, the axis of the other will be along the y-axis.]

Let $y^2=4ax\cdots(1)$ be the equation of one parabola.

Then, by the given condition, the equation of the other parabola is $x^2=4ay$ ...(2)

The co-ordinates of the ends of the latus rectum of the parabola-(1) are $(a, \pm 2a)$ and those of the parabola-(2) are $(\pm 2a, a)$.

Now, the equation of the tangent to the first parabola at the pt. $(a, -2a)$ is $-2ay=2a(x+a)$,

or, $-y=x+a$, or, $x+y+a=0$.

The equation of the tangent to the second parabola at the pt. $(-2a, a)$ is $-2ax=2a(y+a)$,

or, $-x=y+a$ or $x+y+a=0$.

$\therefore$ Both the tangents have the same equation.

Hence, the two parabolas have a common tangent which touches each parabola at the end of a latus rectum.

Ex. 16. If l and l' be the lengths of the segments of any focal chord of the parabola $y^2=4ax$, prove that $\frac{1}{l}+\frac{1}{l'}=\frac{1}{a}$.

[C. U. (B. Sc.) '51]

Let PSP$'$ be any focal chord of the parabola and the co-ordinates of P and P$'$ be $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ respectively.

$\because$ the chord PSP$'$ passes through the focus,

$\therefore$ $m_1m_2=-1$, $\therefore$ $m_1^2m_2^2=1$.

$\because$ The abscissa of P is am_1^2,

$\therefore$ SP$=a+am_1^2=a(1+m_1^2)$,

$\therefore$ Here $l=a(1+m_1^2)\cdots\cdots(1)$

Similarly, SP$'=a+am_2^2=a(1+m_2^2)$,

$\therefore$ $l'=a(1+m_2^2)\cdots\cdots(2)$.

Now, $\frac{1}{l}+\frac{1}{l'}=\frac{1}{a(1+m_1^2)}+\frac{1}{a(1+m_2^2)}=\frac{1+m_2^2+1+m_1^2}{a(1+m_1^2)(1+m_2^2)}$

$$=\frac{2+m_2^2+m_1^2}{a(1+m_1^2+m_2^2+m_1^2m_2^2)}=\frac{2+m_2^2+m_1^2}{a(1+m_1^2+m_2^2+1)}$$

$$[\because m_1^2m_2^2=1]$$

$$=\frac{2+m_2^2+m_1^2}{a(2+m_2^2+m_1^2)}=\frac{1}{a}.$$

Ex. 17. Show that the straight line $3y=1$ bisects all chords of the parabola $3y^2=4x$, which are parallel to the straight line $y=2x+3$.

The equation of the parabola is $y^2=\frac{4}{3}x$,

$$\therefore \quad a=\frac{4}{3\times 4}=\frac{1}{3}.$$

$\because$ The equation of the st. line is $y=2x+3$, $\therefore$ $m=2$.

The equation of the diameter that bisects the chords parallel to the st. line $y=2x+3$ is $y=\frac{2a}{m}=\frac{2\times\frac{1}{3}}{2}=\frac{1}{3}$, or, $3y=1$.

$\therefore$ The st. line $3y=1$ bisects all chords of the parabola parallel to the st. line $y=2x+3$.

Ex. 18. Find the position of a point (x_1, y_1) with respect to the parabola $y^2=4ax$.

Let $P(x_1, y_1)$ be a point outside the parabola and PN be perpendicular to its axis AX.

Let PN cut the parabola at Q.

$\therefore$ The x-co-ordinate of $Q=x_1$ and since Q lies on the parabola, we have $QN^2=4ax_1$.

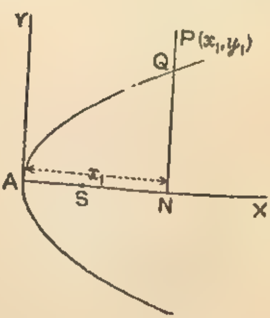


Fig. 21

Evidently, the point P will be outside, on or within the parabola, according as $PN >=$ or $< QN$,

i.e., $PN^2 >=$ or $< QN^2$, i.e., $y_1^2 >=$ or $< 4ax_1$,

i.e., when the value of $y_1^2-4ax_1$ is positive, zero or negative.

Exercise 21

1. Find the points of intersection of the line

(i) $y=x+2$ with the parabola $y^2=9x$.

(ii) $3x-y=1$ with the parabola $y^2=4x$.

2. Show the the straight line

(i) $y=2x+1$ touches the parabola $y^2=8x$.

(ii) $8y=16x+3$ touches the parabola $y^2=3x$.

(iii) $3y=x+3$ touches the parabola $3y^2=4x$, and find the point of contact in each case.

3. Prove that the straight line $4x-2y+3=0$ touches the parabola $y^2=12x$ and find the co-ordinates of the point of contact. [C. U.]

4. Find the condition that the straight line $lx+my+n=0$ may touch the parabola $x^2=4ay$.

5. Find the co-ordinates of the points where the line $x-5y+6=0$ meets the parabola $y^2=x$. [C. U. '44]

6. Find the equations of the tangent and the normal to

(i) $x^2=-12y$ at $(6, -3)$

(ii) $y^2=8x$ at the ends of the latus rectum.

(iii) $x^2+2x+y=4$ at $(-2, 4)$

(iv) $y^2=6x$ at the point whose ordinate is 12.

(v) $y^2=4a(x-a)$ at the ends of the latus rectum.

7. Find the equation of the normal :

(a) to the parabola $y^2=-4x$ at the point $(-4, 4)$

(b) to the parabola $x^2=4y$ at the point $(6, 9)$.

8. Find the equations of the normals to the parabola $y^2=4ax$ at the ends of the latus rectum. [U. P. B. '52]

9. Find the equation of the normal to the parabola $y^2=8x$, which is parallel to the line $x-2y+3=0$ and also find the co-ordinates of its foot.

10. Prove that the normal to the circle $x^2+y^2+4x+2y-8=0$ at the point $(1, 1)$ is a tangent to the parabola $9y^2=8x$.

11. Find the equation of the tangent to the parabola $y^2=7x$ which is parallel to the straight line $4y=x-5$ and find the co-ordinates of the point of contact.

12. Find the equation of the tangent to the parabola $y^2=8x$, which is perpendicular to the line $2x-3y=6$.

13. For the parabola $y^2=8x$ form the equations of two tangents which pass through the point $(-2, \frac{16}{3})$. Also, find the angle between them. [C. U. '57]

14. The normal to the parabola $y^2=4ax$ at $(am_1^2, 2am_1)$ meets it again at $(am_2^2, 2am_2)$. Prove that $m_1^2+m_1m_2+2=0$.

15. Find the length of the normal chord of the parabola $y^2=4x$ at the point whose ordinate is equal to its abscissa.

16. If the line $y=3x+1$ touches the parabola $y^2=4ax$, find the length of the latus rectum. [C. U. '36, D. U. '49]

17. Show that the straight line $4a(y-b)=x$ touches the parabola $ay^2=bx$. [C. U. (B. Sc.) '21]

18. For what value of a will the straight line $y=3x+1$ touch the parabola $y^2=4ax$?

19. For what value of a will be straight line $y=2x+3$ touch the parabola $y^2=4ax$? [D. U. '48]

20. Show that the line $x+my+am^2=0$ touches the parabola $y^2=4ax$. Find also the co-ordinates of the point of contact. [U. P. B. '48]

21. Find the point of the parabola $y^2=8x$ at which the normal is inclined at the angle $60°$ to the axis. [C. U. '44]

22. A tangent to the parabola $y^2=8x$ makes an angle $45°$ with the straight line $y=3x+5$. Find its equation and its point of contact. [C. U. '46]

23. Find the co-ordinates of the particular point on the parabola $y^2=4ax$, the normal at which, terminated by the axis, is equal in length to the latus rectum. [C. U. '56]

24. Find the equation of the common tangent to the two parabolas $y^2=32x$ and $x^2=108y$. [C. U.]

25. A straight line touches both $x^2+y^2=2a^2$ and $y^2=8ax$. Find its equation. [C. U. '55]

26. Find the points on the parabola $y^2=4ax$ at which the tangent is inclined at 30° to the axis.

27. Prove that the normal chord of the parabola $y^2=ax$, which is normal at the point $\left(\frac{1}{2}a, \frac{1}{\sqrt{2}}a\right)$ subtends a right angle at the vertex.

28. Show that the chord $4x+3y+1=0$ of the parabola $y^2=8x$ is bisected at the point $(2, -3)$. [C. U.]

29. Show that the tangents at the extremities of a focal chord of a parabola intersect at right angles to the directrix. [U. P. B. '43]

30. If the straight line $4x-2y+3=0$ touches the parabola $y^2=12x$ at the point $(\frac{3}{4}, 3)$, verify that the subtangent is bisected at the vertex. [C. U. '50]

31. A circle and a parabola intersect in four points. Prove that the algebraic sum of the ordinates of the four points is zero. [C. U. (B. Sc.) '20]

32. Find the equation of the diameter of the parabola $y^2=4x$ which bisects all chords parallel to the straight line $y=2x+3$.

33. Find the equation to the chord of the parabola $y^2=12x$, which is bisected at the point $(3, 2)$.

34. Find the equation to the chord of the parabola $y^2=8x$, which is bisected at the point $(2, -3)$.

35. Prove that the locus of the middle points of all chords of the parabola $y^2=4ax$, which are drawn through the vertex is the parabola $y^2=2ax$. [C. U. (B. Sc.) '46]

36. Show that the straight line $4y+9=0$ bisects all chords of the parabola $y^2=3x$, which are parallel to the straight line $2x+3y=6$.

37. Find the locus of the middle points of all chords of the parabola $4y^2=5x$, which pass through the fixed point $(-2, 3)$.

38. Determine the position of the point $(4, 10)$ with respect to the parabola $y^2=9x$. Try to obtain the equations to the tangents to the same which pass through the above point.

39. If upon a parabola $y^2=4ax$, (x_1, y_1) and (x_2, y_2) be a pair of points so related that the associated tangents are perpendicular to each other, establish the following results :

(i) $x_1x_2=a^2$ (ii) $y_1y_2=-4a^2$ (iii) $4x_1x_2+y_1y_2=0$.

40. If the tangents at the points (α_1, β_1) and (α_2, β_2) to the parabola $y^2=4ax$ intersect at a point (x', y'), then prove that $x'=\dfrac{\beta_1\beta_2}{4a}$.

41. (i) Find the equation of the normal to the parabola $y^2=3x$ which is parallel to the str. line $y=2x+1$, also find the co-ordinates of the foot. What is the length of the subnormal ?

(ii) Find the equation of the tangent to the parabola $y^2=12x$ at the point (lying in the first quadrant) whose focal distance is 9. Determine the length of the subtangent.

42. Show that the length of the chord joining the points of contact of tangents drawn from the point (α, β) is

$$\frac{1}{a}\sqrt{\beta^2+4a^2}\cdot\sqrt{\beta^2-4a\alpha}.$$

ELLIPSE

Definition : *The locus of a point which moves in a plane so that the ratio of its distances from a fixed point and a fixed straight line on the plane is always constant and less than unity is called an* **ellipse.**

Or, The **ellipse** *is a conic in which the eccentricity e is less than unity.*

The fixed point is the **focus**, the fixed st. line is the **directrix** and the ratio is called the *eccentricity* of the ellipse.

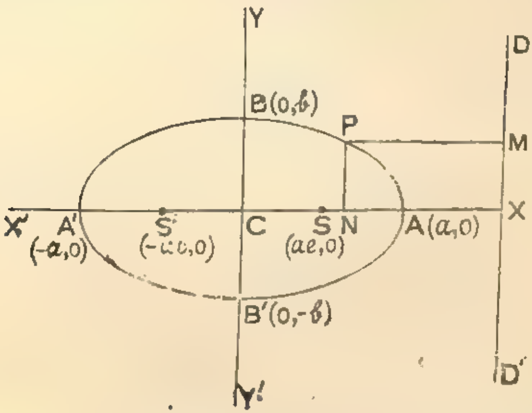


Fig. 22

The eccentricity is denoted by e. If S be the focus and if PM be perpendicular from any point P on the ellipse to its directrix DD′, then $\frac{SP}{PM}=e$ and $e<1$.

If SX be drawn perpendicular from S to the directrix [see diagram], it is termed the **line of major axis.**

The point A at which the line of major axis cuts the ellipse is called its **vertex.** On XS produced take a point A′ such that $\frac{SA'}{A'X}=\frac{SA}{AX}=e$, then A′ will be a point on the ellipse and it is its **second vertex.**

(6) The point on the major axis whose co-ordinates are $(-ae, 0)$ is called the second **focus** and is denoted by S′.

There is another straight line, parallel to the y-axis, whose equation is $ex+a=0$. This straight line is called the second **directrix.**

N. B. From the equation of the ellipse we have $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. Here, if x is greater than a or less than $-a$, then y will be imaginary. So the ellipse does not extend beyond A and A′.

Similarly, $x=\pm\frac{a}{b}\sqrt{b^2-y^2}$, and if $y>b$ or $<-b$, x is imaginary and hence there is no part of the ellipse above B and below B′.

Again, for any value of x the equation gives two equal and opposite values of y and for any value of y we have two equal and opposite values of x. So the curve is symmetrical with respect to the axes of x and y. Hence the ellipse is a closed curve symmetrical about the major as well as the minor axes.

187. *The sum of the focal distances of a point on an ellipse is equal to the major axis.*

Let P be any point on the ellipse, MX and M′X′ be the two directrices, S and S′ the two foci, and MM′ be perpendicular to the directrices through the point P.

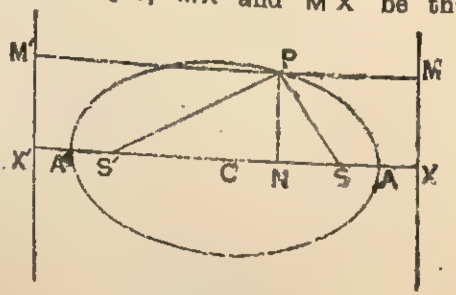


Fig. 24

Now, SP + S′P
$=e\,PM+e.PM'$
$=e(PM+PM')$
$=e.MM'=2e.CX=2a$
= the major axis.

[N. B. The sum of the focal distances of a point on the ellipse is constant, being equal to its major axis.]

188. *To find the length of the latus rectum of an ellipse.*

Let LSL$'$ be the latus rectum.

$\because$ The co-ordinates of S are $(ae, 0)$,

$\therefore$ The abscissa of L$=ae$. Let $(ae,$ SL$)$ be the co-ordinates of L.

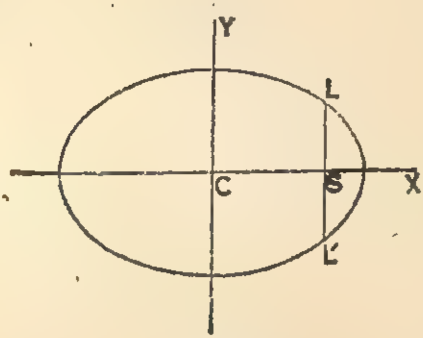


Fig. 25

Since L is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

$\therefore$ this equation is satisfied by the co-ordinates of L.

$\therefore$ $\dfrac{a^2e^2}{a^2}+\dfrac{\text{SL}^2}{b^2}=1$, or, $\dfrac{\text{SL}^2}{b^2}=1-e^2$,

$\therefore$ $\text{SL}^2=b^2(1-e^2)=\dfrac{b^4}{a^2}\left[\because\ e^2=1-\dfrac{b^2}{a^2}\right]$, $\therefore$ $\text{SL}=\dfrac{b^2}{a}$.

$\therefore$ The length of the latus rectum$=$LL$'=2$SL$=\dfrac{2b^2}{a}$.

[N. B. There is a second latus rectum through the second focus and it is also equal to $\dfrac{2b^2}{a}$.]

189. *To prove that* PN^2 : AN.A$'$N : : BC^2 : AC^2.

The equation of the ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

or, $\dfrac{y^2}{b^2}=1-\dfrac{x^2}{a^2}=\dfrac{a^2-x^2}{a^2}=\dfrac{(a+x)(a-x)}{a^2}$,

$\therefore\ \dfrac{PN^2}{b^2}=\dfrac{A'N.AN}{a^2}$, $\therefore\ \dfrac{PN^2}{A'N.AN}=\dfrac{b^2}{a^2}=\dfrac{BC^2}{AC^2}$ [See Fig. 22]

Corollaries : (I) If the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then its major axis is along the x-axis so that its two foci lie on the major axis and the two directrices are parallel to the minor axis. In such an ellipse

(*a*) CS$=$CS$'=ae$, and CX$=$CX$'=\dfrac{a}{e}$.

(*b*) The distance SP of any point P(x, y) on the ellipse from the focus S$=a-ex$ and the distance S$'$P$=a+ex$.

$\therefore$ SB$=a$ ($\because$ for the pt. B, the abscissa $x=0$) and S$'$B$=a$.

(*c*) The co-ordinates of the vertex A are $(a, 0)$ and those of A$'$ are $(-a, 0)$. The co-ordinates of the two ends B and B$'$ of the minor axis are $(0, b)$ and $(0, -b)$ respectively.

(*d*) The co-ordinates of S$'$, X and X$'$ are $(-ae, 0)$, $\left(\dfrac{a}{e}, 0\right)$ and $\left(-\dfrac{a}{e}, 0\right)$ respectively.

(*e*) The equation of the major axis is $y=0$ and that of the minor axis is $x=0$.

(*f*) The equation of the second directrix is $ex=-a$ or $x=-\dfrac{a}{e}$.

(*g*) The equations of the two latera recta are $x=ae$ and $x=-ae$.

(II) If the major axis of the ellipse be along the y-axis. *i.e.*, if the foci lie on the minor axis and the origin lies on the centre as before, then the equation of the ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

[assuming the major axis $=2a$ and the minor axis $=2b$]

In such an ellipse, the co-ordinates of the foci are $(0, ae)$ and $(0, -ae)$, and the equations of the two directrices are $ey=a$ and $ey=-a$.

190. (*a*) *To find the equation of the ellipse taking its major axis along the axis of y.*

Since the major axis is along the y-axis, the co-ordinates of the focus S are $(0, ae)$.

Let $P(x, y)$ be a point on the ellipse and PM be perpendicular to the directrix.

By the definition of ellipse. $SP=e.PM$, $\therefore$ $SP^2=e^2PM^2$.

In fig. 26, we find, $SP^2=(x-0)^2+(y-ac)^2$ and $PM^2=\left(\frac{a}{e}-y\right)^2$

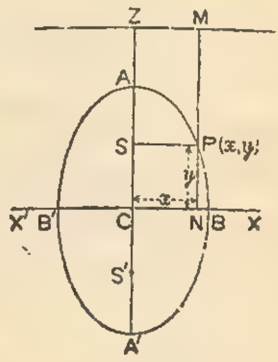


Fig. 26

$\left[\text{Here the directrix is } y=\frac{a}{e}\right]$

$\therefore$ we have the equation $x^2+(y-ae)^2=e^2\left(\frac{a}{e}-y\right)^2$,

or, $\frac{x^2}{a^2(1-e^2)}+\frac{y^2}{a^2}=1$ [on simplification]

Hence the required equation is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \quad [\because\ b^2=a^2(1-e^2)]$$

or, $\dfrac{(y_1-y_2)(y_1+y_2)}{b^2}=-\dfrac{(x_1-x_2)(x_1+x_2)}{a^2}$.

$\therefore\ \dfrac{y_1-y_2}{x_1-x_2}=-\dfrac{b^2}{a^2}\cdot\dfrac{x_1+x_2}{y_1+y_2}$.

$\therefore$ The equation of the chord passing through the points (x_1, y_1) and (x_2, y_2) is

$y-y_1=-\dfrac{b^2}{a^2}\cdot\dfrac{x_1+x_2}{y_1+y_2}(x-x_1)$ and this will be a tangent to the ellipse when the two points coincide, *i,e.,* when $x_2=x_1$ and $y_2=y_1$.

Then $\dfrac{x_1+x_2}{y_1+y_2}=\dfrac{2x_1}{2y_1}=\dfrac{x_1}{y_1}$.

$\therefore$ The equation of the tangent at the point (x_1, y_1) is

$y-y_1=-\dfrac{b^2}{a^2}\cdot\dfrac{x_1}{y_1}(x-x_1)$, or, $\dfrac{yy_1}{b^2}-\dfrac{y_1^2}{b^2}=-\dfrac{xx_1}{a^2}+\dfrac{x_1^2}{a^2}$

or, $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}=1$ [From (ii)]

Hence the required equation is

$$\boldsymbol{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.}$$

[N. B. Here also we find that the equation of the tangent at the point (x_1, y_1) is obtained by putting xx_1 for x^2 and yy_1 for y^2 in the equation of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$]

191 (*b*) *To find the* ***equation of the normal*** *to the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *at the point* (x_1, y_1).

The equation of the st. line passing through the point (x_1, y_1) is $y-y_1=m(x-x_1)$ and the equation of the tangent at that point is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$.

The gradient of the second equation $= -\frac{b^2}{y}.\frac{x_1}{a^2}$.

$\because$ The straight line is perpendicular to the tangent at the point of contact,

$\therefore$ the product of their gradients is -1,

i.e., $m\times\left(-\frac{b^2}{y_1}.\frac{x_1}{a^2}\right)=-1$, $\therefore$ $m=\frac{a^2y_1}{b^2x_1}$.

$\therefore$ The equation of the normal is

$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$, or, $\frac{x-x_1}{b^2x_1}=\frac{y-y_1}{a^2y_1}$,

or, $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$.

192. *To find the condition that the straight line* $y=mx+c$ *may touch the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Putting $mx+c$ for y in the equation of the ellipse we have $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$,

or, $(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0\cdots$(i),

it is a quadratic equation in x.

$\therefore$ It has two roots. So the st. line $y=mx+c$ will cut the ellipse at two points, and it will be a tangent to the ellipse when the two points are coincident, *i. e.*, when the two roots of the equation-(i) are equal.

The condition for the two roots being equal is

$4a^4m^2c^2-4(a^2m^2+b^2)a^2(c^2-b^2)=0$,

or, $a^2m^2c^2-(a^2m^2+b^2)(c^2-b^2)=0$,

or, $a^2m^2b^2+b^4-b^2c^2=0$, or, $c^2=a^2m^2+b^2$,

$\therefore$ $c=\pm\sqrt{a^2m^2+b^2}$.

$\therefore$ The condition for the st. line $y=mx+c$ being a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c=\pm\sqrt{a^2m^2+b^2}$.

Corollary : Each of the st. lines $y=mx\pm\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for any value of m.

193. *Two tangents can be drawn to an ellipse from an external point.*

Let (x', y') be any external point.

The st line $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If this tangent passes through (x', y'),

then $y'=mx'+\sqrt{a^2m^2+b^2}$, or, $(y'-mx')^2=a^2m^2+b^2$.

It is a quadratic equation in m and so it gives two values of m.

Hence two tangents can be drawn to an ellipse from an external point.

194. *To find the locus of the point of intersection of two perpendicular tangents to an ellipse.*

If the two tangents $y=mx\pm\sqrt{a^2m^2+b^2}$ pass through the point (h, k),

then we have $k=mh\pm\sqrt{a^2m^2+b^2}$,

$\therefore$ $(k-mh)^2=a^2m^2+b^2$,

or, $(h^2-a^2)m^2-2mkh+(k^2-b^2)=0$.

Let m_1, m_2 be the roots of this equation.

Then $m_1 m_2 = \dfrac{k^2 - b^2}{h^2 - a^2}$. But since the two tangents are perpendicular to each other, $\therefore \quad m_1 m_2 = -1$.

$$\therefore \quad \frac{k^2 - b^2}{h^2 - a^2} = -1, \qquad \text{or,} \quad h^2 + k^2 = a^2 + b^2.$$

$\therefore$ The locus of the point of intersection (h, k) of two perpendicular tangents to an ellipse is $x^2 + y^2 = a^2 + b^2$, which is a circle, whose centre is the centre of the ellipse and radius $= \sqrt{a^2 + b^2} =$ AB.

This circle is called the **Director circle.**

195. *To find the lengths of the* ***subtangent*** *and the* ***subnormal.***

Let P (x_1, y_1) be a point on the ellipse and let the tangent PT and the normal PG at P meet the major axis at T and G respectively. PN is drawn perpendicular to the axis. Then NT is the subtangent and NG is the subnormal.

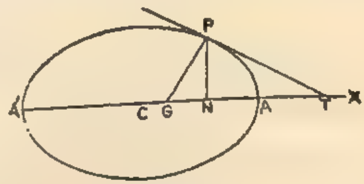


Fig. 28

The equation to the tangent at P is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1 \cdots (1)$.

The equation of the major axis is $y = 0$, and as the tangent intersects that axis at T, the ordinate of T is zero.

The abscissa of T is obtained by putting $y = 0$ in the equation (1).

Thus from (1) we have $\dfrac{xx_1}{a^2} = 1, \qquad \therefore \quad x = \dfrac{a^2}{x_1}$.

$$\therefore \quad \text{CT} = x = \frac{a^2}{x_1}.$$

$$\therefore \quad \text{NT} = \text{CT} - \text{CN} = \frac{a^2}{x_1} - x_1 = \frac{a^2 - x_1^2}{x_1}.$$

$$\therefore \quad \text{The length of the subtangent} = \frac{a^2 - x_1^2}{x_1}.$$

Again, the equation to the normal at the pt. (x_1, y_1) is $\dfrac{x-x_1}{\dfrac{x_1}{a^2}}=\dfrac{y-y_1}{\dfrac{y_1}{b^2}}$......(2).

As this st. line intersects the major axis $(y=0)$ at G the abscissa CG of the point G can be found by putting $y=0$ in the equation-(2).

Thus from (2) we have $\dfrac{x-x_1}{\dfrac{x_1}{a^2}}=\dfrac{-y_1}{\dfrac{y_1}{b^2}}=-b^2,$

or, $x=x_1-\dfrac{b^2}{a^2}x_1$. $\therefore$ $CG=x=x_1\left(1-\dfrac{b^2}{a^2}\right)$.

$\therefore$ The length of the subnormal $NG=CN-CG$

$$=x_1-x_1\left(1-\frac{b^2}{a^2}\right)=\frac{b^2}{a^2}x_1=\frac{a^2(1-e^2)x_1}{a^2}=(1-e^2)x_1.$$

196. *The portion of the tangent at any point of an ellipse intercepted between the point of contact and the directrix subtends a right angle at the focus.*

Let P (x_1, y_1) be a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

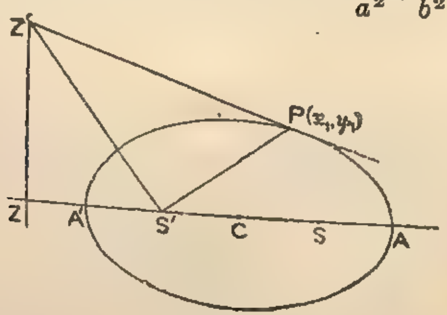


Fig. 29

The equation of the tangent at P is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$......(i).

This tangent cuts the directrix ZZ′, *i.e.*, $x=-\dfrac{a}{e}$...(ii) at Z′ [Fig. 29].

Eliminating x from (i) and (ii) we have

$$-\frac{x_1}{ae}+\frac{yy_1}{b^2}=1, \quad \text{or,} \quad y=\frac{b^2(ae+x_1)}{aey_1},$$

$\therefore$ The co-ordinates of Z' are $\left\{-\frac{a}{e}, \frac{b^2(ae+x_1)}{aey_1}\right\}$

and those of the focus S' are $(-ea, 0)$.

$\therefore$ The gradient of $Z'S'$

$$=\frac{\frac{b^2(ae+x_1)}{aey_1}}{-\frac{a}{e}+ae}=-\frac{ae+x_1}{y_1} \; [\because b^2=a^2(1-e^2)]$$

Again, the gradient of $PS'=\frac{y_1-0}{x_1+ae}=\frac{y_1}{ae+x_1}$.

$\therefore$ The product of the two gradients

$$=-\frac{ae+x_1}{y_1}\times\frac{y_1}{ae+x_1}=-1. \quad \therefore \quad Z'S' \perp PS'.$$

Hence $\angle PS'Z'$ is a right angle.

197. *Prove that the normal at any point of an ellipse bisects the angle between the focal distances of the point.*

Let $P(x_1, y_1)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

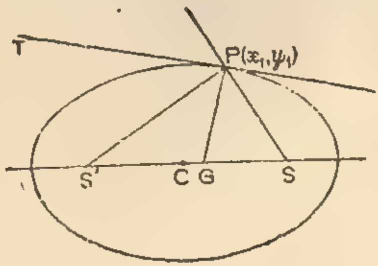


Fig. 30

We know $SP=a-ex_1$ and $S'P=a+ex_1$.

The equation of the normal at P is $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}\ldots$(i).

The normal-(i) has cut the axis $y=0 \cdots$(ii) of the ellipse at G [Fig. 30]. Then from the diagram the x-co-ordinate of G=CG. The value of CG can be obtained by eliminating y from (i) and (ii).

Thus we have

$$CG=x_1-\frac{b^2x_1}{a^2}=\frac{a^2-b^2}{a^2}x_1=e^2x_1\left[\because \quad e^2=\frac{a^2-b^2}{a^2}\right]$$

$$\therefore \quad SG=SC-CG=ae-e^2x_1=e(a-ex_1)$$

and $S'G=CS'+CG=ae+e^2x_1=e(a+ex_1)$

$$\therefore \quad \frac{SG}{S'G}=\frac{e(a-ex_1)}{e(a+ex_1)}=\frac{a-ex_1}{a+ex_1}=\frac{SP}{S'P}.$$

$\therefore$ The normal PG bisects the $\angle$SPS$'$, *i.e.*, the angle between the focal distances.

Corollary : *The tangent at any point of an ellipse makes equal angles with the focal distances of the point.*

Since the tangent PT is perpendicular to the normal PG at the point P [Fig. 30] and PG bisects the $\angle$S$'$PS,

$\therefore$ PT must be the external bisector of $\angle$SPS$'$.

$\therefore$ The tangent at P makes equal angles with the focal distances; PS and PS$'$.

Examples (22)

Ex. 1. Find the latus rectum, eccentricity and the co-ordinates of the focus of the (i) ellipse $4x^2+9y^2=36$, and the (ii) ellipse $5x^2+4y^2=20$.

(i) The equation of the ellipse is $4x^2+9y^2=36$,

$$\text{or} \quad \frac{x^2}{9}+\frac{y^2}{4}=1.$$

$\therefore$ Here $b^2=4$ and $a=3$.

$\therefore$ the length of the latus rectum $=\dfrac{2b^2}{a}=\dfrac{2\times 4}{3}=\dfrac{8}{3}$.

Eccentricity $=e=\sqrt{\dfrac{a^2-b^2}{a^2}}=\sqrt{\dfrac{9-4}{9}}=\dfrac{\sqrt{5}}{3}$.

$\therefore$ the co-ordinates of the two foci are $(\pm ae, 0)$

i.e., $\left(\pm 3\times\dfrac{\sqrt{5}}{3}, 0\right)$, or, $(\pm\sqrt{5}, 0)$.

(ii) The equation of the ellipse is $5x^2+4y^2=20$,

or, $\dfrac{x^2}{4}+\dfrac{y^2}{5}=1$, from which it is evident that BB′ is here the major axis, *i.e.*, the foci are on the y-axis.

$\therefore$ The latus rectum $=\dfrac{2\times 4}{\sqrt{5}}=\dfrac{8\sqrt{5}}{5}$.

Eccentricity $e=\sqrt{\dfrac{5-4}{5}}=\dfrac{1}{\sqrt{5}}$.

$\therefore$ the co-ordinates of the foci are

$$\left(0,\ \pm\sqrt{5}\times\frac{1}{\sqrt{5}}\right)\quad\text{or}\quad(0, \pm 1).$$

Ex. 2. Find the foci, directrices and eccentricity of the ellipse $4x^2+9y^2+16x-9y+12=0$.

The given equation is $4x^2+16x+9y^2-9y+12=0$,

or, $4x^2+16x+16+9y^2-9y+\frac{9}{4}=6\frac{1}{4}$,

or, $4(x+2)^2+9(y-\frac{1}{2})^2=\frac{25}{4}$, or, $\frac{16}{25}(x+2)^2+\frac{36}{25}(y-\frac{1}{2})^2=1$.

Putting $x+2=X$ and $y-\frac{1}{2}=Y$, *i.e.*, transferring the origin to the pt. $(-2, \frac{1}{2})$ and the axes to st. lines through that point parallel to the original axes, the equation becomes

$$\tfrac{16}{25}X^2+\tfrac{36}{25}Y^2=1.$$

Here $a^2=\frac{25}{16}$ and $b^2=\frac{25}{36}$.

$$\therefore\ e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{\frac{25}{16}-\frac{25}{36}}{\frac{25}{16}}}=\frac{\sqrt{5}}{3}.$$

Again the co-ordinates of the foci with respect to the new axes $=(ae, 0)$ and $(-ae, 0)=\left(\frac{5\sqrt{5}}{12}, 0\right)$ and $\left(-\frac{5\sqrt{5}}{12}, 0\right)$,

and the directrices are $X\pm\frac{a}{e}=0$, i.e., $X\pm\frac{3}{4}\sqrt{5}=0$.

$\therefore$ The co-ordinates of the foci with respect to the original axes are $(-2\pm\frac{5}{12}\sqrt{5}, \frac{1}{2})\left[\because\ x+2=X=\pm\frac{5\sqrt{5}}{12}, \therefore x=-2\pm\frac{5\sqrt{5}}{12};\right.$

and $\because\ y-\frac{1}{2}=Y=0,\ \therefore\ y=\frac{1}{2}$]

And the directrices are $x+2\pm\frac{3}{4}\sqrt{5}=0$ [$\because\ X=x+2$].

Ex. 3. Find the eccentricity of the ellipse whose latus rectum is 4 inches and the distance of the vertex from the nearest focus is 1·5 inches.

[C. U. '44]

Here the latus rectum $=4$ in., $\therefore$ SL $=2$ in. and AS $=1·5$ in.

Again, $\because$ SL $=e.$XS $=e($XA$+$AS$)=e.$XA$+e.$AS$=$AS$+e.$AS,

$\therefore\ 2=1·5+1·5e$, or, $1·5e=2-1·5=·5$,

$\therefore\ e=\frac{·5}{1·5}=\frac{1}{3}$.

Ex, 4. Find the equation of the ellipse whose focus is $(1, -2)$, directrix is $y=2x+3$ and eccentricity is $\frac{1}{2}$.

Let S be the focus, P(x, y) be any point on the ellipse and PM be perpendicular to the directrix.

$\therefore$ SP $=\sqrt{(x-1)^2+(y+2)^2}=\sqrt{x^2+y^2-2x+4y+5}$

and the distance of P(x, y) from the directrix $(2x-y+3=0)$

$=\frac{2x-y+3}{\sqrt{2^2+1^2}}$, $\therefore$ PM $=\frac{2x-y+3}{\sqrt{5}}$; but SP $=e.$PM,

$\therefore\ \sqrt{x^2+y^2-2x+4y+5}=\frac{1}{2}\cdot\frac{2x-y+3}{\sqrt{5}}$

$$\therefore\ x^2+y^2-2x+4y+5=\frac{(2x-y+3)^2}{20},$$

or, $20x^2+20y^2-40x+80y+100$
$\qquad =4x^2+y^2+9-4xy+12x-6y,$

or, $16x^2+19y^2+4xy-52x+86y+91=0$, this is the required equation of the ellipse.

Ex. 5. Find the equation of the ellipse passing through the points (2, 2) and (3, 1), whose axes are the axes of the co-ordinates, and find its latus rectum and the co-ordinates of the foci.

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$\because$ It passes through the points (2, 2) and (3, 1),

$\therefore$ It is satisfied by these co-ordinates.

$\therefore\ \left.\begin{aligned}\frac{4}{a^2}+\frac{4}{b^2}=1\\ \frac{9}{a^2}+\frac{1}{b^2}=1\end{aligned}\right\}$ Solving these two equations we have $a^2=\frac{32}{3}$ and $b^2=\frac{32}{5}$.

$\therefore$ The required equation is $\dfrac{x^2}{\frac{32}{3}}+\dfrac{y^2}{\frac{32}{5}}=1,$

$$\text{or, }\frac{3x^2}{32}+\frac{5y^2}{32}=1,\qquad \text{or, }3x^2+5y^2=32.$$

The latus rectum $=\dfrac{2b^2}{a}=\dfrac{2.\frac{32}{5}}{\sqrt{\frac{32}{3}}}=\dfrac{2\sqrt{32}.\sqrt{3}}{5}=\dfrac{8}{5}\sqrt{6}.$

Again, $e=\sqrt{\dfrac{a^2-b^2}{a^2}}=\sqrt{\dfrac{\frac{32}{3}-\frac{32}{5}}{\frac{32}{3}}}=\sqrt{\dfrac{2}{5}}=\dfrac{\sqrt{10}}{5};$

$$\therefore\ ae=\frac{\sqrt{32}}{\sqrt{3}}\times\frac{\sqrt{10}}{5}=\frac{4\times\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{5}\times\sqrt{2}}{5}=\frac{8}{\sqrt{15}}.$$

$\therefore$ The co-ordinates of the foci are $(\pm ae,\ 0)$

or $\left(\pm\dfrac{8}{\sqrt{15}},\ 0\right)$.

Ex. 6. Find the equations of the ellipses, whose axes are the axes of co-ordinates and

(a) whose latus rectum is 5 and eccentricity is $\frac{2}{3}$. [C. U.]

(b) whose foci are the points $(\pm 4, 0)$ and eccentricity is $\frac{1}{3}$.

(a) Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of the ellipse.

$\therefore \quad \frac{2b^2}{a}=$ latus rectum $=5$, and $\frac{a^2-b^2}{a^2}=e^2=\frac{4}{9}$.

$\therefore \quad 2b^2=5a \cdots$(i)
and $5a^2-9b^2=0 \cdots$(ii) $\Bigg\}$ From (i) $b^2=\frac{5a}{2}$. Putting this value of b^2 in (ii) we have

$5a^2-\frac{45a}{2}=0$, or, $5a\left(a-\frac{9}{2}\right)=0$,

but $\because \quad a\neq 0, \quad \therefore \ a-\frac{9}{2}=0, \quad \therefore \quad a=\frac{9}{2}$,

$\therefore \ a^2=\frac{81}{4}$, and $b^2=\frac{5}{2}\times\frac{9}{2}=\frac{45}{4}$.

$\therefore$ The required equation to the ellipse is $\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{45}{4}}=1$

or, $\frac{4x^2}{81}+\frac{4y^2}{45}=1$, or, $20x^2+36y^2=405$.

(b) Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\because$ The foci are $(\pm ae, 0)$, and $e=\frac{1}{3}$,

$\therefore$ Here $ae=4$, or, $\frac{1}{3}.\,a=4$, $\therefore \ a=12$.

Again, $\because \ e^2=\frac{a^2-b^2}{a^2}$, or, $a^2e^2=a^2-b^2$,

$\therefore \quad 16=144-b^2, \qquad \therefore \quad b^2=128$.

$\therefore$ the equation of the ellipse is $\frac{x^2}{144}+\frac{y^2}{128}=1$,

or, $8x^2+9y^2=1152$.

Ex. 7. The foci of an ellipse are the points (0, 1) and (0, −1) and the minor axis is of unit length. Find the equation of the ellipse. Explain how an ellipse can be considered as a circle when its two foci coincide. [C. U. '51]

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (b>a)$. Its foci are the points $(0, be)$ and $(0, -be)$. Here $be=1$ and $a=\frac{1}{2}$;

but $a^2=b^2(1-e^2)$, or, $a^2=b^2-b^2e^2$,

or, $\frac{1}{4}=b^2-1$ [$\because$ $be=1$], $\therefore$ $b^2=1+\frac{1}{4}=\frac{5}{4}$.

$\therefore$ The equation of the ellipse is $\frac{x^2}{\frac{1}{4}}+\frac{y^2}{\frac{5}{4}}=1$,

or, $4x^2+\frac{4y^2}{5}=1$, or, $20x^2+4y^2=5$.

Again, if the two foci coincide, they must coincide at the centre of the ellipse.

$\therefore$ In that case $be=0$, $\therefore$ $e=0$; but $a^2=b^2(1-e^2)$,

$\therefore$ $a^2=b^2$ [$\because$ $e^2=0$].

Hence, the ellipse becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$, or, $x^2+y^2=a^2$, and it is a circle.

Ex. 8. Find the equations to the tangent and the normal

(i) at the point (3, 2) of the ellipse $4x^2+9y^2=72$;

(ii) at the point of the ellipse $7x^2+8y^2=36$, whose abscissa is 2;

(iii) at the end of the latus rectum lying in the first quadrant of the ellipse $3x^2+4y^2=12$.

(i) The equation of the tangent from the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

$\therefore$ The equation to the tangent at the pt. (3, 2) is

$4x.3+9y.2=72$, or, $2x+3y=12$.

Let $y-2=m(x-3)$ be the equation of the normal.

$\because$ it is perpendicular to the tangent $2x+3y=12$,

$\therefore$ $m(-\frac{2}{3})=-1$, $\therefore$ $m=\frac{3}{2}$.

$\therefore$ The equation to the normal is $y-2=\frac{3}{2}(x-3)$,

or, $2y-4=3x-9$, or, $3x-2y=5$.

(ii) $\because$ The abscissa of the point$=2$, *i.e.*, $x=2$,

$\therefore$ from the given equation we have $28+8y^2=36$,

or, $8y^2=8$, $\therefore$ $y=\pm1$.

$\therefore$ The equation of the tangent to the ellipse at the point (2, 1) is $7x.2+8y.1=36$, or, $7x+4y=18$.

Let the equation of the normal at the point be

$$y-1=m(x-2)$$

$\therefore$ $m(-\frac{7}{4})=-1$, $\therefore$ $m=\frac{4}{7}$.

$\therefore$ The equation of the normal is

$y-1=\frac{4}{7}(x-2)$, or, $4x-7y=1$.

Similarly the equation of the tangent at the point $(2,-1)$ is $7x-4y=18$.

If m be the slope of the normal, then

$m\times\frac{7}{4}=-1$, $\therefore$ $m=-\frac{4}{7}$.

$\therefore$ the equation of the normal is

$y+1=-\frac{4}{7}(x-2)$, or, $4x+7y=1$.

(iii) The equation of the ellipse is $3x^2+4y^2=12$,

or, $\frac{x^2}{4}+\frac{y^2}{3}=1$.

Then, $a^2=4$, $b^2=3$.

$\therefore$ The latus rectum $=\frac{2b^2}{a}=\frac{2\times3}{2}=3$.

Eccentricity $e=\sqrt{\dfrac{a^2-b^2}{a^2}}=\sqrt{\dfrac{4-3}{4}}=\dfrac{1}{2}$.

$\therefore$ The co-ordinates of the focus are

$(ae, 0)$ or $(2\times\frac{1}{2}, 0)$ or $(1, 0)$.

$\therefore$ The co-ordinates of the end of the latus rectum are $(1, \frac{3}{2})$.

$\therefore$ The equation of the tangent is

$$3x.1+4y.\tfrac{3}{2}=12, \quad \text{or,} \quad x+2y=4.$$

If the gradient of the normal be m, then

$m\times(-\frac{1}{2})=-1$, $\therefore$ $m=2$.

$\therefore$ The equation to the normal is $y-\frac{3}{2}=2(x-1)$,

or, $2y-3=4x-4$, or, $4x-2y=1$.

Ex. 9. Show that the straight line $3y=4x+11$ touches the ellipse $2x^2+3y^2=11$ and find the points of contact.

From the equation of the st. line we have $y=\dfrac{4x+11}{3}$.

Putting the value of y in the equation of the ellipse we have

$$2x^2+3\times\frac{(4x+11)^2}{9}=11,$$

or, $x^2+4x+4=0$, or, $(x+2)^2=0$,

$\therefore$ $x=-2$ or -2, $\therefore$ $y=1$ or 1.

$\therefore$ The co-ordinates of the two points of intersection are $(-2, 1)$ and $(-2, 1)$,

$\therefore$ They are coincident at $(-2, 1)$. Hence the st. line is a tangent at the point.

$\therefore$ The straight line $3y=4x+11$ touches the ellipse $2x^2+3y^2=11$ and the co-ordinates of the point of contact are $(-2, 1)$.

Ex. 10. For what values of m, the straight line $y=mx-11$ touches the ellipse $2x^2+y^2=22$? Find the point of contact.

Putting the value of y from $y=mx-11$ in $2x^2+y^2=22$,

we have $2x^2+(mx-11)^2=22$,

or, $(m^2+2)x^2-22mx+99=0\cdots\cdots(1)$;

The roots of this equation-(1) give the x-co-ordinates of the two points of intersection of the straight line and the ellipse. So the straight line will touch the ellipse when the two points coincide. This condition is satisfied when the two roots are equal.

$\therefore \quad 22^2m^2-4(m^2+2)\times 99=0,$

or, $11m^2-9(m^2+2)=0$, or, $m^2=9$, $\therefore \quad m=\pm 3$.

Again, when $m=3$, then $11x^2-66x+99=0$,

or, $x^2-6x+9=0$, or, $(x-3)^2=0$, $\therefore \quad x=3, 3$.

$\therefore \quad y=mx-11=3\times 3-11=-2.$

$\therefore$ The co-ordinates of the point of contact of the tangent to the ellipse are $(3, -2)$.

Similarly, when $m=-3$, then $y=-3\times 3-11=-20$.

$\therefore$ The co-ordinates of the point of contact of the tangent $y=-3x-11$ are $(-3, -20)$.

Ex. 11. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is touched by the straight line $5y=3x+25$ and its eccentricity is $\frac{3}{5}$. Find a and b. [G. U. '48]

$\because \quad e=\frac{3}{5}$ and $e=\sqrt{1-\frac{b^2}{a^2}}$,

$\therefore \quad \frac{9}{25}=1-\frac{b^2}{a^2}$, or, $\frac{b^2}{a^2}=1-\frac{9}{25}=\frac{16}{25}$, or, $b^2=\frac{16}{25}a^2\cdots\cdots(1)$

Again, the given st. line is $5y=3x+25$, or, $y=\frac{3}{5}x+5$.

When the st. line $y=mx+c$ touches the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

we have $c^2=m^2a^2+b^2$, and here $c=5$ and $m=\frac{3}{5}$.

$\therefore$ Here $25=\frac{9}{25}a^2+b^2=\frac{9}{25}a^2+\frac{16}{25}a^2=a^2$, $\therefore$ $a=5$ or -5.

$\therefore$ From (1) we have $b=4$ (The negative value -5 of a is ignored).

$\therefore$ $a=5$ and $b=4$.

Ex 12. Find the equations of the tangents to the ellipse $2x^2+y^2=17$, which are parallel to the straight line $4x-3y=12$.

The straight line parallel to the straight line $4x-3y=12$ is $4x-3y=c$, or, $y=\dfrac{4x-c}{3}$.

Putting this value of y in $2x^2+y^2=17$

we have $2x^2+\dfrac{16x^2-8cx+c^2}{9}=17$,

or, $34x^2-8cx+c^2-153=0\cdots\cdots$(i)

The straight line $4x-3y=c$ will touch the ellipse when the two points of intersection of the straight line and the ellipse are coincident. This condition is satisfied, if the roots of the equation-(i) are equal.

$\therefore$ $64c^2-4\times34(c^2-153)=0$, or, $9c^2=17\times153$,

$\therefore$ $c^2=17\times17$, $\therefore$ $c=\pm17$.

$\therefore$ The equations of the tangents are $4x-3y=17$ and $4x-3y+17=0$.

Ex. 13. Find the equations of the tangents to the ellipse $2x^2+y^2=17$, which are perpendicular to the straight line $3x+4y=9$.

The straight line $4x-3y=c$ is always perpendicular to the st. line $3x+4y=9$.

$$\therefore\quad y=\frac{4x-c}{3}.$$

Now proceeding as in Ex. 12, the equations of the tangents are $4x-3y=17$ and $4x-3y+17=0$.

Ex. 14. For what value of p does the ellipse $px^2+4y^2=1$ pass through the points $(\pm 1, 0)$? Find the lengths of its two axes. [C.U. '35]

$\because$ The ellipse passes through the points $(1, 0)$ and $(-1, 0)$,

$\therefore$ its equation is satisfied by those co-ordinates.

Now putting $x=\pm 1$ and $y=0$ in the equation

$px^2+4y^2=1$ we have $p(\pm 1)^2+4\times 0=1$, or, $p=1$.

$\therefore$ The equation of the ellipse is x^2+4y^2+1 ($\because$ $p=1$),

$$\text{or,}\quad x^2+\frac{y^2}{\frac{1}{4}}=1,\quad \text{or,}\quad \frac{x^2}{1^2}+\frac{y^2}{(\frac{1}{2})^2}=1.$$

$\therefore$ Here halves of the axes are 1 and $\frac{1}{2}$.

Hence the lengths of the axes $=2$ and 1.

Ex. 15. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point of intersection of the straight lines $7x+13y-87=0$ and $5x-8y+7=0$ and its latus rectum is $\frac{32}{5}\sqrt{2}$; find a and b.

Solving $7x+13y-87=0$ and $5x-8y+7=0$ we have

$$x=5,\ y=4.$$

$\therefore$ the st. lines intersect at the pt. $(5, 4)$.

Now since the ellipse passes through the point $(5, 4)$,

$$\therefore\quad \frac{25}{a^2}+\frac{16}{b^2}=1\cdots\cdots\text{(i)}$$

Again, $\because$ the latus rectum $=\frac{2b^2}{a}$,

$\therefore$ Here $\frac{2b^2}{a}=\frac{32}{5}\sqrt{2}$, or, $b^2=a.\frac{16}{5}\sqrt{2}$.

Now, from (1) we have $\frac{25}{a^2} = \frac{16}{a.\frac{16}{5}\sqrt{2}} = 1,$

or, $25 + \frac{5a}{\sqrt{2}} = a^2$, or, $a^2 - \frac{5}{\sqrt{2}}a - 25 = 0,$

or, $\left(a - \frac{10}{\sqrt{2}}\right)\left(a + \frac{5}{\sqrt{2}}\right) = 0$, $\therefore\ a = \frac{10}{\sqrt{2}}$ or $-\frac{5}{\sqrt{2}}$.

$\therefore\ b^2 = a.\frac{16}{5}\sqrt{2} = \frac{10}{\sqrt{2}} \times \frac{16}{5}\sqrt{2} = 32$ (ignoring the negative value of a)

$\therefore\ b = \sqrt{32} = 4\sqrt{2}$, $\therefore\ a = \frac{10}{\sqrt{2}}$ and $b = 4\sqrt{2}$.

Ex. 16. Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [H. S. '68]

If possible, suppose the st. line touches the ellipse at the point (x', y'). The equation of the tangent to the ellipse at the point, $(x'\ y')$ is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$......(i).

Again, by assumption, the st. line $lx + my + n = 0$......(ii) is also a tangent to the ellipse at the point.

Now, since st. lines (i) and (ii) are identical,

$\therefore$ we have $\frac{x'}{a^2 l} = \frac{y'}{b^2 m} = \frac{1}{-n},$

or, $\frac{x'}{a} = \frac{al}{-n}$...(iii) and $\frac{y'}{b} = \frac{bm}{-n}$...(iv)

Squaring both sides of (iii) and (iv) and adding the results

we have $\frac{a^2 l^2}{n^2} + \frac{b^2 m^2}{n^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2}.$

$\because$ the point (x', y') lies on the ellipse by assumption,

$\therefore\ \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$

$\therefore$ The required condition is $\dfrac{a^2l^2}{n^2}+\dfrac{b^2m^2}{n^2}=1$,

or, $a^2l^2+b^2m^2=n^2$.

Ex. 17. Find the condition that the line $lx+my=n$ may be a normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. [C. U. '56]

If possible, suppose the st. line $lx+my=n$......(i) to be a normal to the ellipse at the point (x', y').

The equation of the normal to the ellipse at the point (x', y') is

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}}, \text{ or, } \frac{a^2}{x'}.x-\frac{b^2}{y'}.y=a^2-b^2 \text{...(ii)}$$

$\because$ (i) and (ii) are both normals to the ellipse at the same point,

$\therefore$ the st. lines are identical, so $\dfrac{l}{\dfrac{a^2}{x'}}=\dfrac{m}{-\dfrac{b^2}{y'}}=\dfrac{n}{a^2-b^2}$,

or, $\dfrac{x'}{a}=\dfrac{an}{l(a^2-b^2)}$...(iii) and $-\dfrac{y'}{b}=\dfrac{bn}{m(a^2-b^2)}$...(iv)

Now, squaring both sides of (iii) and (iv) and adding the results,

we have $\dfrac{a^2n^2}{l^2(a^2-b^2)^2}+\dfrac{b^2n^2}{m^2(a^2-b^2)^2}=\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}$.

$\because$ the pt. (x', y') lies on the ellipse (by assumption),

$$\therefore \quad \frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$$

$\therefore$ the required condition is

$$\frac{a^2n^2}{l^2(a^2-b^2)^2}+\frac{b^2n^2}{m^2(a^2-b^2)^2}=1, \text{ or, } \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}.$$

Exercise 22

1. Find the eccentricity, the co-ordinates of the foci and the latera recta of the ellipses :—

(i) $9x^2+16y^2=144$. (ii) $3x^2+4y^2=12$.

(iii) $25x^2+16y^2=400$. (iv) $2x^2+3y^2=8$.

(v) $2x^2+3y^2=1$. [U. P. B. '52]

2. Find the foci and directrices of the ellipse :—

(a) $3x^2+4y^2=9$. (b) $4x^2+9y^2-8x-36y+39=0$.

3. Find the lengths of the axes, the eccentricity and the position of the foci of the ellipse $3x^2+4y^2=48$. [C. U. '41]

4. Find the latus rectum, eccentricity and the co-ordinates of the foci of the ellipse $9x^2+5y^2-30y=0$.

5. Find the centre, eccentricity, foci and directrices of the ellipse $4x^2+9y^2-24x-36y+36=0$.

6. The latus rectum is half of the major axis of an ellipse. Find its eccentricity. [U. P. B. '48]

7. Find the eccentricity and the position of the foci of the ellipse $x^2+2y^2=2$. [C. U. '37]

8. Find the eccentricity and the position of the foci of the ellipse $3x^2+4y^2=48$.

9. Taking the axes as the axes of co-ordinates, find the equation of the ellipse :

(i) whose major and minor axes are 8 and 6.

(ii) whose eccentricity is $\dfrac{1}{\sqrt{2}}$ and latus rectum is 3.

(iii) whose major axis is $\dfrac{9}{2}$ and eccentricity is $\dfrac{1}{\sqrt{3}}$.

(iv) which passes through the points (2, 3) and (– 4, 1).

10. Find the equation of the ellipse

(i) whose focus is (2, 1), directrix is $x+2y-1=0$ and eccentricity is $\dfrac{1}{\sqrt{3}}$;

(ii) whose focus is $(-1, 1)$, directrix is $x-y+3=0$ and eccentricity is $\frac{1}{2}$.

[C. U. '45]

11. An ellipse whose axes lie along the co-ordinate axes is of eccentricity $\frac{4}{5}$ and passes through the point $(\frac{10}{3}, \sqrt{5})$. Determine its equation.

[C. U. '42]

12. Find the equation of the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point $(-3, 1)$ and has the eccentricity $\sqrt{\frac{2}{5}}$. [C. U. '48]

13. Find the equation of the ellipse referred to its axes as axes of co-ordinates which passes through the points $(2, 2)$ and $(3, 1)$. Find also its eccentricity. [C. U. '39]

14. Find the equation of the ellipse referred to its axes as the axes of x and y respectively, which passes through the points $(-3, 1)$ and $(2, -2)$. Find also its eccentricity. [C. U. '45]

15. Find the equation of an ellipse whose focus is $(6, 7)$, directrix $x+y+2=0$ and eccentricity $\frac{1}{\sqrt{3}}$.

16. Find the equation of the ellipse, referred to its centre as origin and major axis as x-axis, whose latus rectum is 5 and eccentricity $\frac{2}{3}$.

[C. U.]

17. The distance between the focus and the directrix of an ellipse is 16 inches and its eccentricity is $\frac{3}{5}$. Obtain the lengths of the principal axes.

[C. U. '43]

18. Show that the straight line $y=x+5$ touches the ellipse $9x^2+16y^2=144$ and find the point of contact.

19. Prove that the straight line $y=x+\sqrt{\frac{7}{12}}$ touches the ellipse $3x^2+4y^2=1$ and find the point of contact.

20. Show that the straight line $y=x+\sqrt{\frac{5}{6}}$ is a tangent to the ellipse $2x^2+3y^2=1$.

[C. U. (B. Sc.) '55]

21. Prove that $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and find its point of contact.

22. Find the equations of the tangent and the normal to the ellipse $4x^2+9y^2=20$ at the point $(1, \frac{4}{3})$.

23. Find the equations of the tangent and the normal to the ellipse $5x^2+3y^2=137$ at the point $y=2$.

24. Find the equations of the tangents at the ends of the latera recta of the ellipse $9x^2+16y^2=144$.

25. For what values of m, the straight line $3y=mx+7$ touches the ellipse $2x^2+3y^2=14$? Find the points of contact in those cases.

26. Find the equations of the tangents to the ellipse $4x^2+3y^2=5$, which are parallel to the straight line $y=3x+4$.

27. Find the equations of the tangents to the ellipse $x^2+9y^2=2$, which are perpendicular to the straight line $3x+y=2$ and find the points of contact.

28. Find the equation to the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the st. line $\frac{x}{3}+\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of co-ordinates. Determine the eccentricity and the position of the foci of the ellipse. [C. U. '38]

29. In an ellipse show that $CS.CX=CA^2$ where C is the centre, S is a focus, A is the corresponding vertex and X is the point where the line CS meets the corresponding directrix of the ellipse. Verify this theorem when the ellipse is $x^2+2y^2=2$. [C. U. '50]

30. If any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepts lengths h, k on the axes, prove that $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1$. [C. U. '51]

31. Prove that in an ellipse the sum of the squares of the perpendiculars on any tangent from two points on the minor axis, each distant $\sqrt{a^2-b^2}$ from the centre is $2a^2$. [C.U.(B. Sc.)'53]

32 (i) Find the distance from the origin of the point where the tangent at the extremity (lying in the positive quadrant) of the latus rectum of the ellipse $9x^2+25y^2=225$ intersects the major axis. [H. S. 1960]

(ii) Find the equation to the tangent of the ellipse $9x^2+16y^2=144$, having equal positive intercepts on the axes. [H. S. 1961]

33. If SY and S′Y′ be drawn perpendiculars from the foci S and S′ of an ellipse upon the tangent at any point of it, then prove that $SY.S'Y'=b^2$.

34. Prove that the tangents drawn at the ends of any of the latera recta of the ellipe $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect on the major axis.

35. Prove that the straight line $\frac{ax}{3}+\frac{by}{4}=c$ will be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $5c=a^2e^2$. [C. U. 1958]

36. Prove that the normal at the point (2, 3) to the ellipse $3x^2+4y^2=48$ bisects the angle between the focal distances of the point.

37. Find the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that the tangents at each of them make equal angles with the axes. Prove also, that the length of the perpendicular from the centre on either of these tangents is $\sqrt{\frac{a^2+b^2}{2}}$.

38. For the ellipse $\frac{x^2}{3}+y^2=1$, prove that the tangents at the two points $(\frac{3}{2}, \frac{1}{2})$ and $(-\frac{3}{2}, \frac{1}{2})$ are perpendicular to each other and that they intersect at a point which lies on the circle $x^2+y^2=4$.

198. To find the position of a point with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

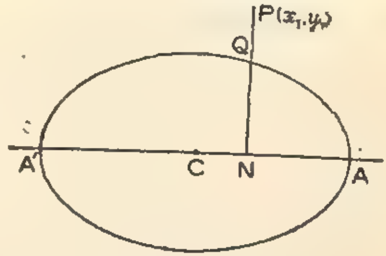


Fig. 31

Let P (x_1, y_1) be a point outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let PN be perpendicular to the major axis AA′.

Then $CN=x_1$ and $PN=y_1$.

Suppose PN cuts the ellipse at Q [Fig. 31]. Then the co-ordinates of Q are (x_1, QN).

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{QN^2}{b^2}=1, \text{ or, } QN^2=b^2\left(1-\frac{x_1^2}{a^2}\right).$$

It is clear from the diagram that the point P will be outside, on, or within the ellipse according as

$$PN >= \text{or} < QN, \text{ or, as } PN^2 >= \text{or} < QN^2$$

$$\text{or, as } y_1^2 >= \text{or} < b^2\left(1-\frac{x_1^2}{a^2}\right),$$

$$\text{or, as } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >= \text{or} < 0.$$

Hence the point (x_1, y_1) will be outside, on or within the ellipse according as the value of $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is positive, zero, or negative.

199. Chord of Contact.

To find the equation of the chord of contact of the tangent drawn from the pt. (x', y') to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_1, y_1) and (x_2, y_2) be the two points of contact of the two tangents drawn from the point (x', y') to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equations of the tangents to the ellipse at the points (x_1, y_1) and (x_2, y_2) are respectively $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \cdots$(i) and

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \cdots \text{(ii)}.$$

$\because$ both the lines (i) and (ii) pass through the point (x', y'),

$$\therefore \quad \frac{x'x_1}{a^2}+\frac{y'y_1}{b^2}=1 \cdots \text{(iii)} \quad \text{and} \quad \frac{x'x_2}{a^2}+\frac{y'y_2}{b^2}=1 \cdots \text{(iv)}$$

It is clear from the equations (iii) and (iv) that the st. line $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ passes through the points of contact (x_1, y_1) and (x_2, y_2).

Hence the required equation of the chord of contact is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$.

200. *To find the* **length of the chord** *intercepted on the line $y=mx+c$ by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

Let P(x_1, y_1) and Q (x_2, y_2) be the points at which the st. line $y=mx+c \cdots$(i) intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \cdots$(ii).

Substituting the value of y from (i) in (ii) we have

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1, \quad \text{or,} \quad b^2x^2+a^2(mx+c)^2=a^2b^2,$$

$$\text{or,} \quad (a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0 \cdots \text{(iii)}$$

Evidently the roots of this quadratic equation-(iii) will be x_1 and x_2.

$$\therefore \quad x_1+x_2=-\frac{2a^2mc}{a^2m^2+b^2} \text{ and } x_1x_2=\frac{a^2(c^2-b^2)}{a^2m^2+b^2}.$$

Now, $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$

$$=\frac{4a^4m^2c^2}{(a^2m^2+b^2)^2}-\frac{4a^2(c^2-b^2)}{a^2m^2+b^2}=\frac{4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)^2}$$

Again, since the points P and Q lie on the st. line-(i),

we have $y_1=mx_1+c$...(iv) and $y_2=mx_2+c$...(v)

Subtracting (v) from (iv) we have $y_1-y_2=m(x_1-x_2)$.

$\therefore$ The length of the chord PQ $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$$=\sqrt{(x_1-x_2)^2+m^2(x_1-x_2)^2}$$

$$=(x_1-x_2)\sqrt{1+m^2}$$

$$=\frac{2ab}{a^2m^2+b^2}\cdot\sqrt{a^2m^2+b^2-c^2}\cdot\sqrt{1+m^2}.$$

Corollary : When the length of the chord PQ will be zero, the two points P and Q will be coincident, *i.e.*, the st. line-(i) will touch the ellipse-(ii).

$\therefore$ The condition so that the st. line-(i) will be a tangent to the ellipse is

$$\frac{2ab}{a^2m^2+b^2}\cdot\sqrt{a^2m^2+b^2-c^2}\cdot\sqrt{1+m^2}=0.$$

$\because$ None of a, b and $\sqrt{1+m^2}$ is zero, $\therefore$ $a^2m^2+b^2-c^2$ will be zero, *i.e.*, $c^2=a^2m^2+b^2$.

$\therefore$ The condition for the st. line-(i) being a tangent to the ellipse is $c=\pm\sqrt{a^2m^2+b^2}$.

201. *The locus of the middle points of a system of parallel chords of an ellipse is a straight line passing through the centre.* [*H. S. 1963*]

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is the origin.

Let the equation of PQ, any of the parallel chords [Fig 32], be $y=mx+c\cdots$(i)

The gradient of each of the parallel chords is m and they are different st. lines for different values of c.

Let (h, k) be the co-ordinates of V, the middle point of the chord-(i), *i.e.*, of PQ.

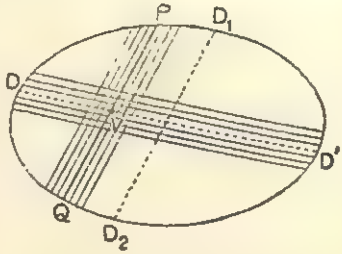


Fig. 32

Then $k=mh+c$, or, $c=k-mh\cdots$(ii)

Now, putting the value of y from (i) in the equation of the ellipse, we have $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$,

or, $b^2x^2+a^2(mx+c)^2=a^2b^2$,

or, $(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0\cdots$(iii)

Let x_1 and x_2 be the two roots of the quadratic equation-(iii). Then x_1 and x_2 are the x-co-ordinates of P and Q, the points of intersection of the st. line-(i) and the ellipse.

$$\therefore\quad x_1+x_2=-\frac{2a^2mc}{a^2m^2+b^2};$$

But $h=\dfrac{x_1+x_2}{2}=-\dfrac{a^2mc}{a^2m^2+b^2}$

$=-\dfrac{a^2m}{a^2m^2+b^2}(k-mh)$ [putting the value of c from (ii)]

$\therefore\quad k=-\dfrac{b^2}{a^2m}h$, and as it is independent of c, it is true for each of the parallel chords.

$\therefore$ The equation of the locus of the middle points is

$$y=-\frac{b^2}{a^2m}x.$$

Since this is a linear equation in x and y, the locus is a straight line and as this equahion is satisfied by the co-ordinates (0, 0), the st. line passes through the origin, *i.e.*, through the centre of the ellipse in this case.

202. Diameter and Conjugate diameters.

Diameter : The locus of the middle points of parallel chords of an ellipse is called a *diameter* of the ellipse.

The chords are called its double ordinates.

In fig. 32, DD′ is a diameter and D_1D_2 is another diameter.

The equation of the diameter DD′ has been found to be $y=-\frac{b^2}{a^2m}x \cdots\cdots$(i).

The value of m will be different for different diameters of the same ellipse.

Suppose the equation-(i) is $y=m'x$, then

$$m'=-\frac{b^2}{a^2m} \quad \text{or,} \quad mm'=-\frac{b^2}{a^2}\cdots\cdots\text{(ii)}$$

Now, if $y=m'x$ be the equation of one diameter, $y=mx$ will be the equation of another diameter $\left[\because m=-\frac{b^2}{a^2m'}\right]$.

Hence we see that if $mm'=-\frac{b^2}{a^2}$, then the st. line $y=m'x$ is the bisector of the chords parallel to another st. line $y=mx$.

Conjugate diameters : If two diameters be such that each bisects all chords parallel to the other, they are called *conjugate diameters.*

From equation-(ii) we find that two diameters having gradients m and m' are conjugate when $mm'=-\frac{b^2}{a^2}$.

203. *The tangent to an ellipse at the extremity of any diameter is parallel to the system of chords bisected by the diameter.*

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\cdots$(i) be the ellipse, $y=mx+c\cdots$(ii) be any of its chords and $y=-\frac{b^2}{a^2m}x\cdots$ (iii) be the diameter that birects the chord.

Let (x_1, y_1) be the co-ordinates of any one of the two points of intersection of the ellipse-(i) and the diameter-(iii).

Then $y_1 = -\dfrac{b^2}{a^2m}x_1$, or, $m = -\dfrac{b^2}{a^2}.\dfrac{x_1}{y_1}\cdots$(iv)

Now, the equation of the tangent to the ellipse-(i) at the point (x_1, y_1) is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$, or, $y=-\dfrac{b^2x_1}{a^2y_1}x+\dfrac{b^2}{y_1}$

$=mx+\dfrac{b^2}{y_1}$ [Putting the value of $-\dfrac{b^2x_1}{a^2y_1}$ from (iv)]

$\therefore$ The tangent $y=mx+\dfrac{b^2}{y_1}$ is parallel to the chord-(ii).

Hence the tangent at any extremity of a diameter of an ellipse is parallel to the chords bisected by the diameter.

204. *To find the equation to the chord of an ellipse which is bisected at the point* (h, k).

Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\cdots$(i) be the ellipse and $y=mx+c\cdots$(ii) be one of its chords.

Putting the value of y from (ii) in (i) we have

$\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$, or, $b^2x^2+a^2(mx+c)^2=a^2b^2$,

or, $(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0\cdots$(iii)

If x_1 and x_2 be the roots of this quadratic equation in x, the x-co-ordinates of the point of intersection of (i) and (ii) and x_1 and x_2.

$\therefore$ $x_1+x_2=-\dfrac{2a^2mc}{a^2m^2+b^2}$.

$\because$ (h, k) is the middle point of the chord-(ii),

$\therefore$ $h=\dfrac{x_1+x_2}{2}=-\dfrac{mca^2}{a^2m^2+b^2}\cdots$(iv)

Since (h, k) lies on st. line-(ii),

$\therefore \quad k=mh+c$, or, $c=k-mh$.

Substituting this value of c in (iv) we have

$$h=-\frac{a^2m(k-mh)}{a^2m^2+b^2}, \text{ or, } m=-\frac{b^2h}{a^2k} \cdots \text{(v)}$$

The equation of any st. line passing through the pt. (h, k) is $y-k=m'(x-h)$. This will be the equation of the chord of the ellipse-(i) bisected at (h, k), when $m'=m=-\dfrac{b^2h}{a^2k}$.

$\therefore$ The required equation of the chord is

$$y-k=-\frac{b^2h}{a^2k}(x-h), \text{ or, } \frac{h}{a^2}(x-h)+\frac{k}{b^2}(y-k)=0.$$

205. *To find the locus of the middle points of all chords of an ellipse, which pass through a fixed point.*

Let (α, β) be the fixed point and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be the ellipse.

If (h, k) be the co-ordinates of the middle point of any chord passing through the fixed point (α, β), then the equation of the chord is

$$\frac{h}{a^2}(x-h)+\frac{k}{b^2}(y-k)=0 \quad \text{[Art. 204]}$$

$\because$ This chord passes through the pt. (α, β),

$$\therefore \quad \frac{h}{a^2}(\alpha-h)+\frac{k}{b^2}(\beta-k)=0.$$

Hence, the equation of the locus of the middle points is

$$\frac{x}{a^2}(\alpha-x)+\frac{y}{b^2}(\beta-y)=0,$$

$$\text{or, } \quad \frac{x(x-\alpha)}{a^2}+\frac{y(y-\beta)}{b^2}=0, \text{ which is an ellipse.}$$

206. The Auxiliary Circle

The circle described on the major axis of an ellipse as diameter is called the *auxiliary circle* of the ellipse.

In the diagram [fig. 33], the circle AQA' is drawn on the major axis AA' of the ellipse ABA' as diameter. This circle is the auxiliary circle of the ellipse.

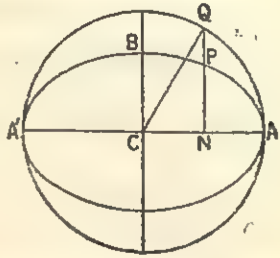


Fig. 33

207. *The ordinates of any point on the ellipse and the corresponding point on the auxiliary circle are in the ratio of the semi-minor axis to the semi-major axis of the ellipse.*

Let P be any point on the ellipse in Fig. 33 and let PN be perpendicular to the major axis AA'. Then NP is the ordinate of the point P. Let NP produced meet the auxiliary circle at Q. Then Q is called the corresponding point to P. The ordinate of the point Q is QN. AC and BC are respectively the semi-major axis and the semi-minor axis.

Now, $\because$ P is a point on the ellipse,

$$\therefore \quad \frac{PN^2}{AN.A'N}=\frac{BC^2}{CA^2} \quad \cdots\cdots(1)$$

Again, AA' is the diameter of the auxiliary circle and Q being a point on the circumference, the $\angle AQA'$ in the semi-circle is a right angle.

Then, $\because$ $QN \perp A'N$, $\therefore$ $QN^2=AN.A'N \quad \cdots\cdots(2)$.

Now, from (1) and (2) we have $\dfrac{PN^2}{QN^2}=\dfrac{BC^2}{CA^2}$,

or, $\dfrac{PN}{QN}=\dfrac{BC}{CA}=\dfrac{\text{semi-minor axis}}{\text{semi-major axis}}$.

[N. B. (1) Here, if $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse, we have $\frac{PN}{QN}=\frac{b}{a}$ for any point P on the ellipse.

(2) If the ordinate of any point P on the ellipse be produced towards P, then the point in which the ordinate meets the auxiliary circle is called the corresponding point to P.

(3) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse, then the equation of its auxiliary circle is $x^2+y^2=a^2$.]

Examples (23)

Ex. 1. Verify that the point (5, 2) is outside the ellipse $2x^2+7y^2=14$. Then proceed to find the equations of the two tangents to the ellipse $2x^2+7y^2=14$ from the point (5, 2).

Here, the co-ordinates of the given point are (5, 2).
$\therefore$ Putting $x=5$, $y=2$ in $2x^2+7y^2-14$ we have the expression $=2.(5)^2+7.(2)^2-14=50+28-14=64$ (positive)

$\therefore$ The point (5, 2) is outside the ellipse $2x^2+7y^2=14$.

Second Part : [*First method*]

The equation of the chord of contact of the two tangents that can be drawn from the pt. (5, 2) to the ellipse $2x^2+7y^2=14\cdots$(i) is $2x.5+7y.2=14$, or, $5x+7y=7\cdots$(ii).

The two points of intersection of (i) and (ii) will be the points of contact of the two tangents.

Eliminating y from (i) and (ii) we have

$$2x^2+7.\left(\frac{7-5x}{7}\right)^2=14, \quad \text{or,} \quad 39x^2-70x-49=0 \text{ ; and}$$

the two roots of this quadratic equation will be the x-co-ordinates of the two points of intersection of (i) and (ii).

$\therefore$ The equation of the chord passing through the pt. (h, k) is $y - k = -\frac{8h}{9k}(x-h)$, but it passes through the given fixed point $(3, -2)$,

$\therefore \quad -2 - k = -\frac{8h}{9k}(3-h)$, or, $8h^2 + 9k^2 - 24h + 18k = 0$.

Hence the equation of the locus of the middle points of the chords passing through the point $(3, -2)$ is

$8x^2 + 9y^2 - 24x + 18y = 0$, which is an ellipse.

Ex. 7. Find the equation of the diameter of the ellipse $2x^2 + 3y^2 = 6$, which bisects all chords parallel to $3x + 4y = 5$.

The equation of the ellipse is $2x^2 + 3y^2 = 6$,

or, $\frac{x^2}{3} + \frac{y^2}{2} = 1 \ldots$(i), $\therefore \quad a^2 = 3$ and $b^2 = 2$.

The equation of the st. line is $3x + 4y = 5$,

or $y = -\frac{3}{4}x + \frac{5}{4} \ldots$(ii), $\therefore$ its gradient $m = -\frac{3}{4}$.

Let $y = m'x$ be the equation of the diameter.

$\because \quad mm' = -\frac{b^2}{a^2}, \quad \therefore \quad -\frac{3}{4}m' = -\frac{b^2}{a^2} = -\frac{2}{3}$, or, $m' = \frac{8}{9}$.

Hence the required equation of the diameter is

$$y = \tfrac{8}{9}x \text{ or } 9y = 8x.$$

Ex. 8. Find the equation of the diameter of the ellipse $4x^2 + 5y^2 = 20$, which is conjugate to the diameter $y = 3x$.

The equation of the ellipse is $4x^2 + 5y^2 = 20$,

or, $\frac{x^2}{5} + \frac{y^2}{4} = 1. \quad \therefore \quad a^2 = 5$ and $b^2 = 4$.

The given equation of the diameter is $y = 3x$, $\therefore \quad m = 3$.

Let the equation of the conjugate diameter be $y = m'x$.

Then $m' \times m = -\frac{b^2}{a^2}$, or, $m' \times 3 = -\frac{4}{5}$, $\therefore \quad m' = -\frac{4}{15}$.

Hence, the required equation of the conjugate diameter is $y = -\frac{4}{15}x$ or $4x + 15y = 0$.

Ex. 9. Show that the diameters, whose equations are $y+3x=0$ and $4y-x=0$, are conjugate diameters of the ellipse $3x^2+4y^2=5$.

Here the equation of the ellipse is $3x^2+4y^2=5$,

or, $\dfrac{x^2}{\frac{5}{3}}+\dfrac{y^2}{\frac{5}{4}}=1$. $\therefore$ $a^2=\frac{5}{3}$ and $b^2=\frac{5}{4}$.

The equations of the two diameters are $y+3x=0$ and $4y-x=0$, i.e., $y=-3x$ and $y=\frac{1}{4}x$ respectively.

$\therefore$ $mm'=-3\times\frac{1}{4}=-\frac{3}{4}$, and $-\dfrac{b^2}{a^2}=\dfrac{-\frac{5}{4}}{\frac{5}{3}}=-\frac{3}{4}$.

$\therefore$ $mm'=-\dfrac{b^2}{a^2}$ [$\because$ each $=-\frac{3}{4}$], and this is the condition for the diameters being conjugate. Hence the given diameters of the ellipse are conjugate diameters.

Ex. 10. For the ellipse $5x^2+6y^2=15$, find a pair of conjugate diameters which are inclined to each other at an angle $\tan^{-1}11$.

From the equation of the ellipse we have $\dfrac{x^2}{3}+\dfrac{y^2}{\frac{5}{2}}=1$,

$\therefore$ $a^2=3$ and $b^2=\frac{5}{2}$.

Let $y=m_1x$ and $y=m_2x$ be the conjugate diameters. Then $m_1m_2=-\dfrac{b^2}{a^2}=-\dfrac{5}{6}\cdots$(i)

Again, since the angle between the diameters is $\tan^{-1}11$,

$$\therefore \quad \frac{m_1-m_2}{1+m_1m_2}=11, \quad \text{or,} \quad \frac{m_1-m_2}{1-\frac{5}{6}}=11,$$

$$\therefore \quad m_1-m_2=\tfrac{11}{6}\cdots\text{(ii)}$$

Now solving (i) and (ii) we have

$$\left.\begin{array}{l} m_1=\frac{5}{6} \\ m_2=-1 \end{array}\right\} \text{ and } \left.\begin{array}{l} m_1=1 \\ m_2=-\frac{5}{6} \end{array}\right\}$$

Hence the required equations of the conjugate diameters are $y=\frac{5}{6}x$ and $y=-x$, and also $y=x$ and $y=-\frac{5}{6}x$, *i.e.*, $5x-6y=0$ and $x+y=0$, and also $x-y=0$ and $5x+6y=0$.

Ex. 11. Prove that the tangents at the ends of any chord of an ellipse meet on the diameter that bisects the chord.

Let $y=mx+c\cdots$(i) be the equation of any chord PQ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If the two tangents to the ellipse at the points P and Q meet each other at the point (x', y'), then the chord of contact of tangents from that point is PQ.

$$\therefore \quad \text{The equation of PQ is } \frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

$$\text{or,} \quad y=-\frac{b^2x'}{a^2y'}\cdot x+\frac{b^2}{y'}\cdots\text{(ii)}$$

Now since both (i) and (ii) are the equations of the st. line PQ,

$$\therefore \quad m=-\frac{b^2x'}{a^2y'}, \quad \text{or,} \quad y'=-\frac{b^2}{a^2m}\cdot x'.$$

$\therefore$ The equation of the locus of the point (x', y') is $y=-\frac{b^2}{a^2m}x$, this is also the equation of the diameter that bisects the chords parallel to the chord-(i).

Hence the tangents at the ends of any chord of an ellipse meet on the diameter that bisects that chord.

Ex. 12. Prove that the tangents at the extremities of a focal chord of an ellipse intersect on the directrix corresponding to that focus.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of the ellipse.

The co-ordinates of one focus are $(ae, 0)$ and the equation of the directrix on the side of the focus is $x=\frac{a}{e}$.

Suppose the tangents drawn at two points P and Q on the ellipse intersect at the point (h, k). So the line PQ is the chord of contact of the tangents drawn from the point (h, k).

$\therefore$ The equation of the chord PQ is $\frac{hx}{a^2}+\frac{ky}{b^2}=1 \cdots$(i)

But the st. line-(i) passes through the focus $(ae, 0)$,

$$\therefore \quad \frac{h}{a^2}\times ae+\frac{k}{b^2}\times 0=1\ ;\ \text{or,}\ h=\frac{a}{e}.$$

$\therefore$ The equation of the locus of the point of intersection (h, k) is $x=\frac{a}{e}$. This is the equation of the directrix on the side of the above focus.

Hence, the tangents at the two ends of any focal chord of an ellipse intersect on the directrix corresponding to that focus.

Exercise 23

1. Find the position of the following points with respect to the ellipse $4x^2+9y^2=20$:

(*i*) $(1, \frac{4}{3})$, (*ii*) $(\frac{1}{2}, 1)$, (*iii*) $(-3, 2)$.

2. (*a*) Find the equation of the chord of contact of the tangents drawn from $(1, -3)$ to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(*b*) Tangents are drawn through the points of intersection of the line $10x-3y=15$ with the ellipse $\frac{x^2}{3}+\frac{y^2}{5}=1$. Find the co-ordinates of the point of intersection of those two tangents.

3. (*a*) Find the points of intersection of the ellipse $7x^2+5y^2=52$ with the straight line $x+y=2$.

(*b*) Find the middle point of the chord intercepted on the line $2x+3y=6$ by the ellipse $4x^2+9y^2=36$.

4. (*a*) Find the length of the chord intercepted by the ellipse $3x^2+5y^2=32$ on the line $x+y=4$.

(*b*) Find the length of the chord intercepted by the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ on the line $x+y=3$. What are the co-ordinates of its middle point ?

5. In the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, construct the equation of that particular chord which is bisected at the point $(2, -1)$.

[C. U. '56]

6. Find the equation of the tangents drawn from $(-15, -7)$ to the ellipse $9x^2+25y^2=225$ and their points of contact.

7. Find the points of contact of the tangents drawn from the point $(0, 5)$ to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

8. The straight line $\frac{x}{3}-\frac{y}{2}=1$ intersects the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ at two points ; find the point of intersection of the tangents drawn through those two points.

9. Find the equation of the diameter to the ellipse $x^2+2y^2=2$, which bisects all chords parallel to $y=5x+3$.

10. Find the equation of the chord of the ellipse $9x^2+10y^2=90$, which has the point $(1, -1)$ as its middle point.

11. Find the equation of the diameter of the ellipse $3x^2+5y^2=15$, which is conjugate to the diameter $4x+3y=0$.

12. Find the locus of the middle points of all chords of the ellipse $5x^2+6y^2=3$, parallel to the line $2x+3y=1$.

13. Show that the diameters $2x+3y=0$ and $y=x$ are conjugate diameters of the ellipse $6x^2+9y^2=2$.

14. Find the equation of the chord of the ellipse $12x^2+15y^2=180$ which passes through $(2, \frac{1}{3})$ and is bisected at that point.

15. Find the locus of the middle points of all chords of the ellipse $4x^2+5y^2=20$, which pass through the point (*i*) (1, 1), (*ii*) (4, -5).

16. Show that $4x-3y+8=0$ and $x+3y-1=0$ are parallel to conjugate diameters of the ellipse $4x^2+9y^2=39$. [C. U. '49]

17. Show that the diameters whose equations are $y+3x=0$ and $4y-x=0$ are conjugate diameters of the ellipse $3x^2+4y^2=5$.

[C. U.]

18. For the ellipse $8x^2+12y^2=96$, find a pair of conjugate diameters which are inclined to each other at angle $\tan^{-1}7$.

[C. U.]

19. For the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$, determine the pair of conjugate diameters which are inclined to each other at an angle of 135°. [C. U. '58]

[*Hints* : Here $m_1m_2=-\frac{3}{6}=-\frac{1}{2}$; $\tan 135°=\frac{m_1-m_2}{1+m_1m_2}$

or, $-1=\frac{m_1-m_2}{1-\frac{1}{2}}$, or, $m_2-m_1=\frac{1}{2}$.]

20. Verify that in the ellipse $\frac{x^2}{20}+\frac{y^2}{10}=1$, the two chords, which have the points (4, 1) and (1, -1) for their respective middle points, are perpendicular to each other.

208. Definition : *The locus of a point which moves in a plane so that the ratio of its distances from a fixed point and a fixed straight line on the plane is always constant and greater than unity is called a* **hyperbola.**

Or, The hyperbola is a conic section in which the eccentricity e is greater than unity.

The fixed point is the *focus*, the fixed straight line is the *directrix*, the constant ratio is the eccentricity of the hyperbola.

S and e represent the focus and the eccentricity respectively.

209. To find the equation to a hyperbola.

Let S be the focus, KZ be the directrix, and SZ be perpendicular to the directrix [Fig. 34].

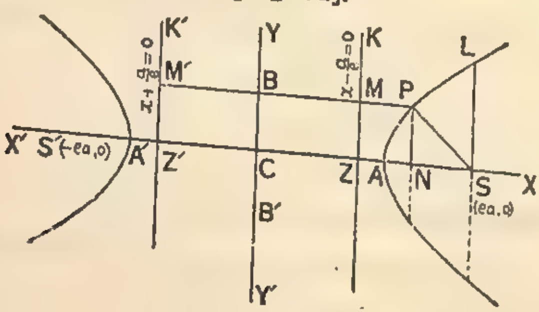


Fig. 34

Let A be a point on SZ such that $\frac{SA}{SZ}=e$ (where $e>1$), *i.e.,* $SA=e.AZ\cdots(1)$

Since e is greater than 1, there will be another point A′ on SZ produced such that $\frac{SA'}{A'Z}=e$, *i.e.,* $SA'=e.A'Z\cdots(2)$

Then A and A′ lie on the hyperbola.

Let C be the middle point of AA$'$ and let the length of AA$'$ be $2a$.

Subtracting (1) from (2) we have

$$SA' - SA = eA'Z - eAZ,$$

or, $AA' = e(CA' + CZ) - e(CA - CZ) = e.2CZ,$

or, $2a = 2e.CZ, \quad \therefore \quad CZ = \frac{a}{e}.$

Again, adding (1) and (2) we have

$$eAZ + eA'Z = SA + SA' = CS - CA + CS + CA' = 2CS,$$

or, $e(AZ + A'Z) = 2CS$, or, $eAA' = 2CS,$

or, $2ea = 2CS, \quad \therefore \quad CS = ea.$

Let YCY$'$ be drawn perpendicular to AA$'$ at the point C.

Let CSX and YCY$'$ be the axes of x and y respectively. On the hyperbola take any point P whose co-ordinates are (x, y) with respect to those axes.

Draw PM and PN perpendicular respectively to the directrix ZK and the axis ZS.

$\because$ CS $= ea$, $\therefore$ the co-ordinates of S are $(ea, 0)$.

$SP = e.PM = e.ZN, \quad \therefore \quad SP^2 = e^2 ZN^2 = e^2(CN - CZ)^2 \cdots (A)$

Again, $SP^2 = NS^2 + PN^2 = (CS - CN)^2 + PN^2 = (ea - x)^2 + y^2$

$\therefore \quad (ea - x)^2 + y^2 = e^2 \left(x - \frac{a}{e}\right)^2 \cdots$ [From (A)]

or, $x^2 - 2aex + a^2e^2 + y^2 = ex^2 - 2aex + a^2,$

or, $x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$, or, $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$

Since $e > 1$, $\therefore$ $a^2 (e^2 - 1)$ is positive.

Let $a^2 (e^2 - 1)$ be called b^2.

Hence the equation of the hyperbola is $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$.

Corollaries : (1) $\because\ b^2=a^2e^2-a^2,\ \therefore\ a^2e^2=a^2+b^2$

$$\therefore\quad e=\sqrt{\frac{a^2+b^2}{a^2}} \text{ and } CS^2=(ea)^2=a^2+b^2.$$

(2) $$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1=\frac{x^2-a^2}{a^2}=\frac{(x-a)(x+a)}{a^2}$$

$$\therefore\quad \frac{PN^2}{b^2}=\frac{AN.A'N}{a^2},\quad \therefore\quad \frac{PN^2}{AN.A'N}=\frac{b^2}{a^2}.$$

Definitions : (i) The point at which the straight line drawn through the focus perpendicular to the directrix intersects the hyperbola is called the **vertex** of the hyperbola. In figure 34, the points A and A′ are the two vertices and C is the centre of the hyperbola.

(ii) The portion AA′ of the x-axis is called the **transverse axis** of the hyperbola. $AA'=2a$.

(iii) The portion of the y-axis equal to $2b$ and bisected at the origin is called the **conjugate axis.** In figure 34, BB′ is the conjugate axis. $BB'=2b$.

(iv) The chord that passes through the focus and is parallel to the directrix is called the **latus rectum** of the hyperbola.

210. *There exist a second focus and a second directrix of a hyperbola.*

Take two points S′ and Z′ on SC produced so that $CS'=CS=ae$ and $CZ'=CZ=\frac{a}{e}$. Let Z′M′ be drawn perpendicular to AA′ and let PM produced intersect Z′M′ at M′.

The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1, \text{ or, } x^2+a^2e^2+y^2=e^2x^2+a^2,$$

or, $$x^2+2aex+a^2e^2+y^2=e^2x^2+2aex+a^2,$$

or, $(x+ae)^2+y^2=e^2\left(\frac{a}{e}+x\right)^2$,

i.e., $S'N^2+PN^2=e^2(CZ'+CN)^2$,

i.e., $S'P^2=e^2.PM'^2$, $\therefore$ $S'P=e.PM'$.

Thus we find that any point P on the curve is such that its distance from S′ is e times its distance from Z′M′. So the same curve is obtained if we start with S′ as focus and Z′M′ as the directrix.

Hence S′ is the second focus and Z′M′ is the second directrix of the hyperbola.

[**N. B.** (i) The co-ordinates of A and A′ are $(a, 0)$ and $(-a, 0)$ respectively.

(ii) The co-ordinates of S and S′ are $(ea, 0)$ and $(-ea, 0)$ respectively.

(iii) The equation of the directrix ZM is $ex-a=0$ or $x=\frac{a}{e}$, and that of the directrix Z′M′ is $ex+a=0$ or $x=-\frac{a}{e}$.]

211. *The difference of the focal distances of any point on the hyperbola is equal to the transverse axis.*

[See Fig. 34] Let S and S′ be the foci and P be any pt. on the hyperbola.

$\because$ $S'P=ePM'$ and $SP=ePM$.

$\therefore$ $SP'-SP=e(PM'-PM)=eMM'=eZZ'=2eCZ$
$=2a=AA'$, i.e., the transverse axis.

[**N. B.** The distance (SP) of any point P on the hyperbola from its focus$=e.PM=e.NZ=e(CN-CZ)=e\left(x-\frac{a}{e}\right)=ex-a$.]

212. Length of the latus rectum

The co-ordinates of the focus S are $(ae, 0)$, let the co-ordinates of the end L of the latus rectum be (ae, SL).

$$\therefore \quad \frac{a^2e^2}{a^2}-\frac{SL^2}{b^2}=1, \text{ or, } e^2-\frac{SL^2}{b^2}=1,$$

or, $\dfrac{SL^2}{b^2}=e^2-1=\dfrac{a^2+b^2}{a^2}-1=\dfrac{b^2}{a^2}$,

$\therefore\quad SL^2=\dfrac{b^4}{a^2},\quad \therefore\quad SL=\dfrac{b^2}{a}$.

$\therefore$ The required length of the latus rectum

$$=LL'=2SL=\frac{2b^2}{a}.$$

212. (A) *To find the* **length of the chord** *of the hyperbola* $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ *on the line* $y=mx+c$.

Let P (x_1, y_1) and Q (x_2, y_2) be the points of intersection of the stragiht line $y=mx+c\cdots(1)$ and the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\cdots(2)$$

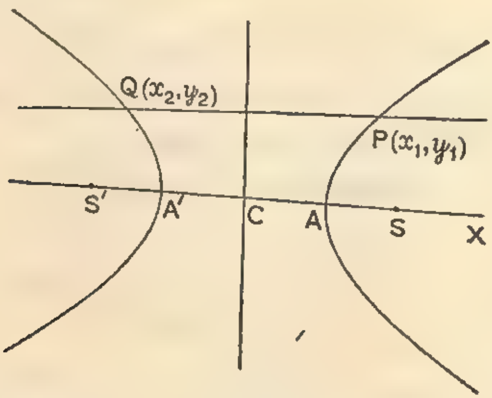


Fig. 35

Then evidiently both the equations (1) and (2) will be satisfied by the co-ordinates (x_1, y_1) and (x_2, y_2).

Putting $mx+c$ for y in (2) we have $\dfrac{x^2}{a^2}-\dfrac{(mx+c)^2}{b^2}=1$,

or, $(a^2m^2-b^2)x^2+2mca^2x+a^2(b^2+c^2)=0\cdots(3)$,

which is a quadratic equation in x.

This equation gives only two values of x. So there are only two points of intersection of the given st. line and the hyperbola.

Now, x_1 and x_2 are the roots of equation (3).

$$\therefore \quad x_1+x_2=-\frac{2mca^2}{a^2m^2-b^2}, \text{ and } x_1x_2=\frac{a^2(b^2+c^2)}{a^2m^2-b^2}.$$

$$\therefore \quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{4m^2c^2a^4}{(a^2m^2-b^2)^2}-\frac{4a^2(b^2+c^2)}{a^2m^2-b^2}$$

$$=\frac{4m^2c^2a^4-4a^2(b^2+c^2)(a^2m^2-b^2)}{(a^2m^2-b^2)^2}$$

$$=\frac{4a^2\{m^2c^2a^2-(b^2+c^2)(a^2m^2-b^2)\}}{(a^2m^2-b^2)^2}$$

$$=\frac{4a^2b^2(b^2-a^2m^2+c^2)}{(a^2m^2-b^2)^2}.$$

Again, $\because$ P(x_1, y_1) and Q(x_2, y_2) lie on $y=mx+c$,

$$\therefore \quad y_1=mx_1+c \text{ and } y_2=mx_2+c,$$

$$\therefore \quad y_1-y_2=m(x_1-x_2).$$

$\therefore$ The length of the chord PQ

$$=\sqrt{(x_1-x_2)^2(y_1-y_2)^2}=\sqrt{(x_1-x_2)^2+m^2(x_1-x_2)^2}$$

$$=\sqrt{(x_1-x_2)^2(1+m^2)}$$

$$=\sqrt{\frac{4a^2b^2(b^2-a^2m^2+c^2)(1+m^2)}{(a^2m^2-b^2)^2}}$$

$$=\frac{\boldsymbol{2ab\sqrt{b^2-a^2m^2+c^2}.\sqrt{1+m^2}}}{\boldsymbol{a^2m^2-b^2}}.$$

[N. B. Since the roots of a quadratic may be real, equal or imaginary, a straight line may intersect a hyperbola in *two distinct points*, or in two *coincident points*, or in two *imaginary points*, i. e., may not intersect the hyperbola at all.]

212. (B) *To find the* **equation of the tangent** *to the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *at the point* (x_1, y_1) *on it.*

Let P be the point (x_1, y_1) on the given hyperbola and let Q be another point (x_2, y_2) on it close to P.

Then the equation of the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \cdots(1)$$

Since P and Q are points on the hyperbola,

$$\therefore \quad \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1\cdots(2) \text{ and } \frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=1\cdots(3)$$

Subtracting (2) from (3) we have

$$\frac{x_2^2-x_1^2}{a^2}-\frac{y_2^2-y_1^2}{b^2}=0, \text{ or, } \frac{y_2^2-y_1^2}{x_2^2-x_1^2}=\frac{b^2}{a^2},$$

$$\text{or, } \frac{y_2-y_1}{x_2-x_1}=\frac{b^2}{a^2}\times\frac{x_2+x_1}{y_2+y_1} \quad \cdots(4)$$

$\therefore$ From (1) we have $y-y_1=\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1)\cdots(5)$

Now, equation (5) will be a tangent to the hyperbola at (x_1, y_1) when Q coincides with P. The condition for this is $x_1=x_2$ and $y_1=y_2$ in (5).

Hence the equation of the required tangent is

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{2x_1}{2y_1}(x-x_1)=\frac{b^2x_1}{a^2y_1}(x-x_1),$$

$$\text{or, } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \quad [\text{ From } (2)]$$

$$\text{i. e., } \boldsymbol{\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1}.$$

212. (C) *To find the* **equation of the normal** *to the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *at the point* (x_1, y_1) *on it.*

The equation of the tangent to the given hyperbola at the point (x_1, y_1) on it is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad [\text{ vide art. 212 (B) }].$$

This may be written as $y=\frac{b^2x_1}{a^2y_1}.x-\frac{b^2}{y_1}$

$\therefore$ The gradient m of this tangent $=\frac{b^2x_1}{a^2y_1}$.

Since the normal at the pt. (x_1, y_1) is perpendicular to the tangent passing through that point,

$\therefore$ the gradient m' of the normal$=-\frac{a^2y_1}{b^2x_1}$.

$\therefore$ The required equation of the normal is

$$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1)$$

or, $$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{-\frac{y_1}{b^2}}.$$

N. B. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}+\frac{y^2}{-b^2}=1$.

Hence we see that, the equation of the hyperbola is obtained by putting $-b^2$ for b^2 in the equation of the ellipse.

Taking this as a formula we find that some propositions for the hyperbola are derived from those for the ellipse.

(1) If $c^2>a^2m^2-b^2$, the straight line $y=mx+c$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at two *distinct real* points.

Again, if $c^2<a^2m^2-b^2$, then the above st. line meets the hyperbola at two *imaginary* points, i.e., does not meet.

(2) The st. line $y=mx+c$ is a *tangent* to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, when $c=\pm\sqrt{a^2m^2-b^2}$.

In this case we may say that the st. line meets the hyperbola at two coincident points.

(3) $y=mx\pm\sqrt{a^2m^2-b^2}$ are always (i. e., for any value of m) tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

[N. B. The co-ordinates of the point of contact of the tangent $y=mx+\sqrt{a^2m^2-b^2}$ are $\left(\frac{-a^2m}{\sqrt{a^2m^2-b^2}}, \frac{-b^2}{\sqrt{a^2m^2-b^2}}\right)$ and those of the point of contact of the tangent $y=mx-\sqrt{a^2m^2-b^2}$ are $\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.]

(4) The equation of the tangent to the hyperbola $\frac{x^2}{y^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(5) The equation of the **normal** to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{-b^2}}$,

$$\text{or,}\quad a^2.\frac{x}{x_1}+b^2.\frac{y}{y_1}=a^2+b^2.$$

(6) $\frac{h}{a^2}(x-h)-\frac{k}{b^2}(y-k)=0$ is the equation of a chord whose middle point is (h, k).

213. Director Circle : The locus of the intersection of two tangents to a hyperbola, which are at right angles, is a circle. This circle is called the *director circle*.

This is the circle $x^2+y^2=a^2-b^2$, i.e., a circle whose centre is the origin and radius is $\sqrt{a^2-b^2}$.

(i) Here, if $b^2<a^2$, this circle is real.

(ii) If $b^2=a^2$, the radius of the circle becomes zero and therefore the circle reduces to a *point circle* at the origin.

Thus the centre is the only point from which tangents at right angles can be drawn to the hyperbola.

(iii) If $b^2>a^2$, the radius of the circle becomes imaginary and so there can be no such circle and therefore in this case no tangents at right angles can be drawn.

214. Definition : The locus of the middle points of parallel chords of a hyperbola is a straight line which is called a **diameter** of the hyperbola.

215. *The equation to the diameter of a hyperbola.*

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola and suppose the diameter bisects the chords parallel to the st. line $y=mx+c$.

The equation of this diameter can be had by substituting $-b^2$ for b^2 in the equation of the diameter of an ellipse.

$\therefore$ The equation of the diameter is $y=\frac{b^2}{a^2m}.x$,

or, $y=m'x$ where $m'=\frac{b^2}{a^2m}$, i.e., $mm'=\frac{b^2}{a^2}$.

216. If two diameters of a hyperbola be such that each bisects all chords parallel to the other, they are called **conjugate** diameters.

217. Conjugate Hyperbolas.

Two hyperbolas are said to be conjugate when the transverse axis and the conjugate axis of one are respectively the conjugate axis and the transverse axis of the other.

Thus the equation of the hyperbola conjugate to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$.

218. *Any pair of conjugate diameters with respect to a hyperbola are also conjugate diameters with respect to the conjugate hyperbola.*

The equations of a hyperbola and its conjugate hyperbola are respectively $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The condition for $y=mx$ and $y=m'x$ being conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $mm'=\frac{b^2}{a^2}$ and also for the same condition they are conjugate diameters of the hyperbola $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Hence the proposition is proved.

219. Rectangular Hyperbola.

The hyperbola whose transverse and conjugate axes are of equal lengths is called a *rectangular* or *equilateral* hyperbola.

Since here $a=b$, the equation to a rectangular hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$, or, $x^2-y^2=a^2$, or, $(x+y)(x-y)=a^2$.

Its eccentricity $e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{2a^2}{a^2}}=\sqrt{2}$.

220. Asymptote.

Definition : An asymptote is a straight line, which meets a conic in two points both situated at an infinite distance, but which is itself not altogether at infinity.

221. Asymptotes of the hyperbola.

The equation of the hyperbola can be written in the from $y=\pm\frac{b}{a}x\sqrt{1-\frac{a^2}{x^2}}$, from which we find that the numerical value of y for any point of the hyperbola must be less than $\pm\frac{b}{a}x$. So the limiting value of y is $\pm\frac{b}{a}x$, and y tends to this

value as x tends to ∞. Hence, as the value of x increases, the hyperbola approaches nearer and nearer to the straight lines $y = \pm\frac{b}{a}x$. So these straight lines are the asymptotes of the hyperbola.

$\therefore$ There are two asymptotes of a hyperbola.

[N. B. (1) The equation to the asymptotes of the hyperbola is $y = \pm\frac{b}{a}x$, which can be written as $\frac{x}{a} \pm \frac{y}{b} = 0$.

The combined form of these two equations is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

(2) From the equations of the two asymptotes it is evident that both of them pass through the centre and are equally inclined to the axis of x (the gradients of both being $\frac{b}{a}$.)

(3) Two conjugate hyperbolas have the same asymptotes.]

222. *The tangent at any point of a hyperbola bisects the angle between the focal distances of the point.*

Let P (x_1, y_1) be any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

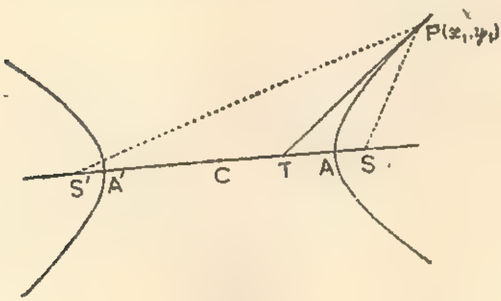


Fig. 36

The equation of the tangent at P is $\frac{xy_1}{a^2} - \frac{yy_1}{b^2} = \cdots$(i).

This tangent cuts the transverse axis $(y=0)$ at the point T [Fig. 35]

$\therefore$ The co-ordinates of T are $\left(\frac{a^2}{x_1}, 0\right)$.

Now, $ST = CS - CT = ae - \frac{a^2}{x_1} = \frac{a(ex_1 - a)}{x_1}$,

and $S'T = CS' + CT = ae + \frac{a^2}{x_1} = \frac{a(ex_1 + a)}{x_1}$.

We know $SP = ex_1 - a$ and $S'P = ex_1 + a$,

$\therefore \frac{SP}{S'P} = \frac{ex_1 - a}{ex_1 + a}$ and $\frac{ST}{S'T} = \frac{a(ex_1 - a)/x_1}{a(ex_1 + a)/x_1} = \frac{ex_1 - a}{ex_1 + a}$,

$$\therefore \quad \frac{SP}{S'P} = \frac{ST}{S'T}.$$

Thus PT divides the base SS′ of the $\triangle SPS'$ in the ratio of its sides SP and S′P. $\therefore$ PT bisects the $\angle SPS'$.

223. *The portion of the tangent to a hyperbola between the point of contact and the directrix subtends a right angle at the focus.*

Suppose P (x_1, y_1) to be a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation of the tangent to the hyperbola at P is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \ldots\ldots \text{(i)}$$

The equation of the directrix is $x = \frac{a}{e} \ldots\ldots$(ii)

If Z′ be the point of intersection of (i) and (ii), the x-co-ordinate of Z′ is $\frac{a}{e}$.

Eliminating x from (i) and (ii) its y-co-ordinate is found to be $\frac{b^2(x_1 - ae)}{aey_1}$.

$\therefore$ The co-ordinates of Z' are $\left\{\frac{a}{e}, \frac{b^2(x_1 - ae)}{aey_1}\right\}$ and those of the focus S are $(ae, 0)$.

$\therefore$ The gradient of SZ'

$$=\frac{\frac{b^2(x_1-ae)}{aey_1}}{\frac{a}{e}-ea}=\frac{b^2(x_1-ae)}{aey_1}\times\frac{e}{a(1-e^2)}$$

$$=\frac{b^2(x_1-ac)}{y_1a^2(1-e^2)}=\frac{ae-x_1}{y_1}\ [\because \quad b^2=a^2(e^2-1)]$$

Again, the gradient of $SP=\frac{y_1-0}{x_1-ae}=-\frac{y_1}{ae-x_1}$.

$\therefore$ The product of the two gradients

$$=\frac{ae-x_1}{y_1}\times-\frac{y_1}{ae-x_1}=-1. \quad \therefore \quad SZ'\perp SP.$$

$\therefore$ $\angle PSZ'$ is a right angle.

Examples (24)

Ex. 1. Find the equation to the hyperbola, referred to its axes as the axes of co-ordinates,

(i) whose transverse and conjugate axes are 6 and 4 ;

(ii) whose conjugate axis is 6 and the distance between whose foci $=10$;

(iii) which passes through the points $(1, 1)$ and $(2, -3)$;

(iv) whose eccentricity is $\sqrt{\frac{3}{2}}$ and whose one of the foci is $(2\sqrt{6}, 0)$.

(i) Here $a=3$ and $b=2$.

$\therefore$ The equation of the hyperbola is

$$\frac{x^2}{9}-\frac{y^2}{4}=1, \quad \text{or,} \quad 4x^2-9y^2=36.$$

(ii) Here $b=3$, and the distance between the foci

$$=2ae=10, \quad \therefore \quad ae=5.$$

But $ae=\sqrt{a^2+b^2}$, $\therefore$ $5=\sqrt{a^2+9}$

or, $25=a^2+9$, $\therefore$ $a^2=16$.

$\therefore$ The equation of the hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{9}=1, \quad \text{or,} \quad 9x^2-16y^2=144.$$

(iii) Let $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ be the hyperbola.

$\because$ It passes through the points (1, 1) and (2, -3),

$$\left.\begin{aligned} \therefore \quad & \frac{1}{a^2}-\frac{1}{b^2}=1 \\ \text{and} \quad & \frac{4}{a^2}-\frac{9}{b^2}=1 \end{aligned}\right\} \quad \text{Solving these equations we have } a^2=\tfrac{5}{2},\ b^2=\tfrac{5}{3}.$$

$\therefore$ The required equation of the hyperbola is

$$\frac{x^2}{\frac{5}{2}}-\frac{y^2}{\frac{5}{3}}=1, \quad \text{or,} \quad \frac{2x^2}{5}-\frac{3y^2}{5}=1, \quad \text{or,} \quad 2x^2-3y^2=5.$$

(iv) Here the focus is $(2\sqrt{6}, 0)$, so $ae=2\sqrt{6}$,

$\therefore$ $a.\sqrt{\frac{3}{2}}=2\sqrt{6}$, $\therefore$ $a=4$.

Again, $a^2e^2=a^2+b^2$, $\therefore$ $16\times\frac{3}{2}=16+b^2$,

or, $24=16+b^2$, $\therefore$ $b^2=8$.

$\therefore$ The equation of the hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{8}=1, \quad \text{or,} \quad x^2-2y^2=16.$$

Ex. 2. Find the latus rectum, eccentricity and the co-ordinates of the foci of the hyperbola $9x^2-16y^2=144$.

Here the hyperbola is $9x^2-16y^2=144$,

or, $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$, so $a=4$ and $b^2=9$.

$\therefore$ The latus rectum $=\dfrac{2b^2}{a}=\dfrac{2\times 9}{4}=\dfrac{9}{2}$,

and eccentricity $e=\sqrt{\dfrac{a^2+b^2}{a^2}}=\sqrt{\dfrac{16+9}{16}}=\sqrt{\dfrac{25}{16}}=\dfrac{5}{4}$ and the co-ordinates of the foci $=(\pm ae, 0)=(\pm 5, 0)$.

Ex. 3. Find the foci, directrices and eccentricities of the hyperbolæ (i) $4x^2-9y^2=36$ and (ii) $x^2-4y^2-6x-16y-23=0$.

(i) From the given equation we have $\frac{x^2}{9}-\frac{y^2}{4}=1$,

so here $a=3$, $b=2$.

$\therefore\quad e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{9+4}{9}}=\frac{\sqrt{13}}{3}$

$\therefore\quad ae=3\times\frac{\sqrt{13}}{3}=\sqrt{13}$.

$\therefore$ The co-ordinates of the foci $=(\pm ae,\ 0)=(\pm\sqrt{13},\ 0)$.

Now, from the formula $x\pm\frac{a}{e}=0$, the required directrices are $x\pm\frac{9}{\sqrt{13}}=0$ $\left(\because\ \frac{a}{e}=\frac{9}{\sqrt{13}}\right)$.

(ii) The given equation can be written as
$(x^2-6x+9)-4(y^2+4y+4)=16$, or, $(x-3)^2-4(y+2)^2=16$,

or, $\frac{(x-3)^2}{16}-\frac{(y+2)^2}{4}=1$.

Now, if we put $x-3=X$ and $y+2=Y$, *i.e.*, if the axes be transferred to parallel axes at the point $(3,\ -2)$, the equation becomes $\frac{X^2}{16}-\frac{Y^2}{4}=1$. Here $a^2=16$ and $b^2=4$.

$\therefore$ the required $e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{16+4}{16}}=\sqrt{\frac{5}{4}}=\frac{1}{2}\sqrt{5}$.

$\therefore\quad ae=4\times\frac{1}{2}\sqrt{5}=2\sqrt{5}$ and $\frac{a}{e}=\frac{4}{\frac{1}{2}\sqrt{5}}=\frac{8}{\sqrt{5}}$.

$\therefore$ Corresponding to the new axes the foci are $(\pm2\sqrt{5},\ 0)$ and the directrices are $X\pm\frac{8}{\sqrt{5}}=0$. Hence with respect to the original axes, the co-ordinates of the foci are $(3\pm2\sqrt{5},\ 0)$ and the directrices are $x-3\pm\frac{8}{\sqrt{5}}=0$.

[N.B. With respect to the new axes, the foci are $(\pm 2\sqrt{5}, 0)$. Here $X = \pm 2\sqrt{5}$, but with respect to the original axes $X = x - 3$, so $x - 3 = \pm 2\sqrt{5}$, $\therefore$ $x = 3 \pm 2\sqrt{5}$. Again, $\because$ $Y = y + 2$, $\therefore$ here $y + 2 = 0$, $\therefore$ $y = -2$. $\therefore$ The required foci are $(3 \pm 2\sqrt{5}, -2)$. Similarly, putting $x - 3$ for X, the required directrices are $x - 3 \pm \frac{8}{\sqrt{5}} = 0$.]

Ex. 4. Find the points of intersection of the hyperbola $2x^2 - 3y^2 = 5$ with the straight line $3y = x + 1$.

From the equation of the st. line we have $x = 3y - 1 \cdots (1)$. Putting this value of x in the equation of the hyperbola we have $2(3y-1)^2 - 3y^2 - 5 = 0$, or, $15y^2 - 12y - 3 = 0$,

or, $5y^2 - 4y - 1 = 0$, or, $(y-1)(5y+1) = 0$,

$\therefore$ $y = 1$ or $-\frac{1}{5}$.

Now, from (1) we have $x = 2$ or $-1\frac{3}{5}$.

$\therefore$ The co-ordinates of the two points of intersection are $(2, 1)$ and $(-1\frac{3}{5}, -\frac{1}{5})$.

Ex. 5. Find the equation to the hyperbola whose focus is $(-1, 2)$, eccentricity 3 and the directrix $x + y = 7$.

Let P (x, y) be any point on the hyperbola.

Now, the distance of the pt. (x, y) from the focus $(-1, 2)$

$$= \sqrt{(x+1)^2 + (y-2)^2} = \sqrt{x^2 + y^2 + 2x - 4y + 5}.$$

Again, the distance of (x, y) from the directrix $(x + y = 7)$.

$$= \frac{x + y - 7}{\sqrt{1^2 + 1^2}} = \frac{x + y - 7}{\sqrt{2}}.$$

$$\because \quad \frac{\text{The distance of P from the focus}}{\text{The distance of P from the directrix}} = e,$$

$$\therefore \quad \text{Here } \frac{\sqrt{(x^2 + y^2 + 2x - 4y + 5)}}{\frac{x + y - 7}{\sqrt{2}}} = e = 3,$$

$$\therefore \quad \sqrt{(x^2 + y^2 + 2x - 4y + 5)} = \frac{3}{\sqrt{2}}(x + y - 7).$$

Now squaring both sides we have on simplification

$7x^2+7y^2+18xy-130x-118y+431=0$, this is the required equation.

Ex. 6. Show that the equation $3x^2-4y^2-12x-8y-4=0$ represents a hyperbola and find its centre and axes.

From the given equation we have

$$(3x^2-12x+12)-(4y^2+8y+4)=12,$$

or, $3(x-2)^2-4(y+1)^2=12$, or, $\dfrac{(x-2)^2}{4}-\dfrac{(y+1)^2}{3}=1.$

If the origin be transferred to the point $(2, -1)$, the equation is reduced to $\dfrac{X^2}{4}-\dfrac{Y^2}{3}=1$ and this is the general form of the equation of a hyperbola whose centre is the origin and axes are the axes of x and y.

Now with respect to the original axes the centre is the point $(2, -1)$ and the axes of the hyperbola are $x-2=0$ and $y+1=0$.

Ex. 7. Find the length of the chord of the hyperbola $x^2-4y^2=25$ intercepted on the st. line $x+3y+5=0$ and also find its middle point.

From the equation of the st. line $x+3y+5=0 \cdots$ (i) we have $x=-(3y+5)$. Substituting this value of x in the equation of the hyperbola, we have

$\{-(3y+5)\}^2-4y^2=25$, or, $9y^2+25+30y-4y^2=25$,

or, $5y^2+30y=0$, or $y^2+6y=0$, or $y(y+6)=0$,

$\therefore$ $y=0$ or -6.

Putting these values of y in (i) we have $x=-5$ or 13.

$\therefore$ the co-ordinates of the two ends of the chord are $(-5, 0)$ and $(13, -6)$.

$\therefore$ The required length of the chord $=\sqrt{(-5-13)^2+(0+6)^2}$

$$=\sqrt{360}=6\sqrt{10} \text{ units};$$

and the co-ordinates of its middle point are

$\{\frac{1}{2}(-5+13), \frac{1}{2}(0, -6)\}$, i.e., $(4, -3)$.

Ex. 8. Show that the straight line $9x-8y=11$ touches the hyperbola $3x^2-4y^2=11$, and find the point of contact.

Here $8y=9x-11$, or, $y=\dfrac{9x-11}{8}$......(1). Putting this value of y in the equation of the hyperbola we have

$$3x^2-4\times\frac{(9x-11)^2}{64}=11,$$

or, $x^2-6x+9=0$, or, $(x-3)^2=0$, $\therefore$ $x=3, 3$.

Since the roots of the equation are equal, the st. line will meet the hyperbola at two coincident points.

$\therefore$ The st. line $9x-8y=11$ touches the hyperbola $3x^2-4y^2=11$.

Again, from (1) we have $y=\dfrac{9\times3-11}{8}=2$,

$\therefore$ The co-ordinates of the point of contact are (3, 2).

Ex. 9. Find the tangent and the normal to the hyperbola $3x^2-4y^2=8$ at the point (2, 1).

Since the tangent touches the hyperbola at the point (2, 1),

$\therefore$ the equation of the tangent is $3x.2-4y.1=8$,

or, $6x-4y=8$, or, $3x-2y=4$.

$\therefore$ the gradient of the tangent is $\frac{3}{2}$,

$\therefore$ The equation of the normal is

$$y-1=-\tfrac{2}{3}(x-2), \quad \text{or,} \quad 2x+3y=7.$$

Ex. 10. Find the equations of the tangents to the hyperbola $x^2-4y^2=4$, which are parallel to the straight line $y=2x+3$.

The equation of any st. line parallel to the st. line $y=2x+3$ is $y=2x+k\cdots$(i).

If possible, suppose the st. line-(i) touches the hyperbola $x^2-4y^2=4\cdots$(ii) at the point (x', y').

The equation of the tangent to the hyperbola-(ii) at the pt. (x', y') is $xx'-4yy'=4$, or, $4y'y=x'x-4\cdots$(iii)

Now, (i) and (iii) are tangents to the hyperbola-(ii) at the same point,

$$\therefore \quad \frac{4y'}{1}=\frac{x'}{2}=\frac{-4}{k}, \quad \text{or,} \quad x'=-\frac{8}{k} \text{ and } y'=-\frac{1}{k}.$$

Since (x', y') is a point on the hyperbola-(ii),

$$\therefore \quad \left(-\frac{8}{k}\right)^2-4\left(-\frac{1}{k}\right)^2=4, \text{ or, } \frac{60}{k^2}=4, \ \therefore \ k=\pm\sqrt{15}.$$

$\therefore$ The required equations of the tangent are $y=2x\pm\sqrt{15}$.

Ex. 11. Show that the line $y=3x$ bisects all chords of the hyperbola $4x^2-9y^2=36$, which are parallel to the straight line $27y=4x+5$.

Let the st. line $y=mx$ be the bisector of all chords parallel to the st. line $27y=4x+5$.

The gradient m of $27y=4x+5$ is $\frac{4}{27}$.

Now, from the equation of the hyperbola we have

$$\frac{x^2}{9}-\frac{y^2}{4}=1,$$

then $a^2=9$, $b^2=4$. $\therefore$ $m\times\frac{4}{27}=\frac{4}{9}$, $\therefore$ $m=3$.

Hence the st. line $y=3x$ bisects all chords parallel to $27y=4x+5$.

Ex. 12. Show that the straight lines $y=2x$ and $5y=2x$ are conjugate diameters of the hyperbola $4x^2-5y^2=20$.

The equation of the hyperbola can be written as

$$\frac{x^2}{5}-\frac{y^2}{4}=1.$$

So $a^2=5$, $b^2=4$.

Now, $\because$ $y=2x$, $\therefore$ $m=2$, and $\because$ $5y=2x$, $\therefore$ $m'=\frac{2}{5}$.

$\therefore$ $mm'=2\times\frac{2}{5}=\frac{4}{5}$.

Again, $\dfrac{b^2}{a^2}=\dfrac{4}{5}$, $\therefore$ $mm'=\dfrac{b^2}{a^2}$.

$\therefore$ $y=2x$ and $5y=2x$ are conjugate diameters.

Ex. 13. Find the equation to the hyperbola referred to its axes as axes of co-ordinates whose conjugate axis is 6 and the distance between the foci is 10.

Here the conjugate axis$=6$, $\therefore$ $b=\frac{6}{2}=3$, and the distance between the foci, i.e., $2ae=10$, or, $ae=5$, $\therefore$ $a=\dfrac{5}{e}$.

Again, $e^2=1+\dfrac{b^2}{a^2}$, or, $e^2=1+\dfrac{9}{\frac{25}{e^2}}=1+\dfrac{9e^2}{25}$,

or, $e^2-\frac{9}{25}e^2=1$, or, $\frac{16}{25}e^2=1$, or, $e^2=\frac{25}{16}$, $\therefore$ $e=\frac{5}{4}$.

$$\therefore \quad a=\frac{5}{e}=\frac{5}{\frac{5}{4}}=4.$$

Hence the required equation is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, or, $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$,

or, $9x^2-16y^2=144$.

Ex. 14. If e denotes the eccentricity of a hyperbola and e' that of its conjugate, show that $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$.

[C. U. (B. Sc.) '41]

Let e be the eccentricity of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

and e' that of the conjugate hyperbola $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$\therefore$ $b^2=a^2(e^2-1)\cdots\cdots(1)$ and $a^2=b^2(e'^2-1)\cdots\cdots(2)$.

Multiplying (1) and (2) we have $a^2b^2=a^2b^2(e^2-1)(e'^2-1)$,

or, $(e^2-1)(e'^2-1)=1$, or, $e^2+e'^2=e^2e'^2$,

or, $\dfrac{e^2+e'^2}{e^2e'^2}=1$, $\therefore$ $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$.

Exercise 24

1. Find the equation to the hyperbola, referred to its axes as the axes of co-ordinates,

(i) whose transverse and conjugate axes are 4 and 3 respectively ;

(ii) whose conjugate axis is 5 and co-ordinates of whose foci are $(\pm 6{\cdot}5, 0)$;

(iii) which passes through the point $(3, -2)$ and whose conjugate axis is 7 ;

(iv) whose eccentricity is $\sqrt{2}$ and whose transverse axis is $8\sqrt{2}$.

2. (a) Find the eccentricity, latus rectum and co-ordinates of the foci of the hyperbola $4x^2 - 9y^2 = 36$.

(b) Calculate the eccentricity and the positions of the two foci of the hyperbola $\frac{x^2}{12^2} - \frac{y^2}{5^2} = 1$. [C. U. '57]

3. Find the foci, directrices and eccentricities of the hyperbola (i) $12x^2 - 4y^2 = 3$ and (ii) $3x^2 - 4y^2 + 18x + 16y + 2 = 0$.

4. (i) Find the equation of the tangent to the hyperbola $x^2 - 3y^2 = 12$ at the point $(-6, 2\sqrt{2})$.

(ii) Find the equation of the normal to the hyperbola $\frac{x^2}{5} - \frac{y^2}{4} = 1$ at that end of the latus rectum the co-ordinates of which are both positive.

5. (i) Find the equations of the tangents to the hyperbola $4x^2 - 9y^2 = 1$, which are parallel to the line $4y = 5x + 3$.

(ii) Find the equations of the tangents to the hyperbola $x^2 - 5y^2 = 40$, which are perpendicular to the line $2x - y + 3 = 0$.

6. Find the condition that the following straight lines will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) $y = mx + c$

(b) $lx + my + n = 0$.

7. (a) Show that the line $y = 2x + 3$ touches the hyperbola $7x^2 - 4y^2 = 28$, and find the point of contact.

(b) Show that the line $y=mx-\sqrt{a^2m^2-b^2}$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all values of m, and find the point of contact.

8. Find the co-ordinates of the foci of the hyperbola $x^2-y^2=9$. Also find the distance from the origin of the point where the tangent to the above hyperbola at (5, 4) meets the x-axis.

[H. S. 1960 (Compl.)]

9. Show that the tangent to the hyperbola $4x^2-3y^2=1$ at the point (1, 1) bisects the angle between the focal distances of the point.

10. Find the equation of the hyperbola having eccentricity 3, one focus $(-1, 3)$ and the equation of the corresponding directrix $x+2y+1=0$.

11. Find the equation to the hyperbola referred to its axes as axes of co-ordinates, whose conjugate axis is 12 and the distance between the foci is 13.

12. Find the co-ordinates of the points of intersection of the hyperbola $25x^2-9y^2=225$ with the straight line $25x+12y=45$.

13. Find the middle point of the chord of the hyperbola $4x^2-9y^2=27$ intercepted by the straight line $2x=5y+1$.

14. Find the co-ordinates of the points common to the hyperbola $bx^2-ay^2=a^2b-ab^2$ and the straight line $\frac{x}{a}+\frac{y}{b}=2$.

15. Find the equation of the chord of the hyperbola $x^2-3y^2=1$ which is bisected at the point (3, 2).

16. Find the length of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, intercepted by the straight line $y=mx+c$.

17. (a) Obtain the length of the chord of the hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$, passing through the origin and making equal angles with the axes.

[H. S. 1960 (Compl.)]

(*b*) Find the length of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ along the line $y=mx$. [H. S. 1961 (Compl.)]

18. Show that the straight line $2x+3y=0$ bisects all chords of the hyperbola $2x^2-3y^2=6$, parallel to the straight line $x+y=6$.

19. Find the locus of the middle points of parallel chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

20. For the hyperbola $16x^2-9y^2=144$, find the equation to the diameter which is conjugate to the diameter $x=2y$.

[C. U. '46]

21. Show that the straight line $3y=4x$ and $8y=5x$ are conjugate diameters of the hyperbola $5x^2-6y^2=30$.

22. (*a*) In the hyperbola $\frac{x^2}{10}-\frac{y^2}{8}=1$, prove that the two chords, which are bisected respectively at the points $(\frac{3}{2}, \frac{5}{2})$ and $(-3, -5)$, are parallel to each other.

(*b*) In the rectangular hyperbola $x^2-y^2=5$, show that the two chords, which are bisected respectively at the points $(7, -2)$ and $(2, 7)$ are perpendicular to each other.

23. Show that $3x+4y-10=0$ is a normal to the hyperbola $2x^2-3y^2=5$; also find its foot.

24. (*a*) Find the locus of the point of intersection of any two tangents which are perpendicular to each other.

(*b*) Find the locus of the foot of the perpendicular from either focus on any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

25. Find the equations of the asymptotes of the following hyperbola :—

(i) $\frac{x^2}{4}-\frac{y^2}{9}=1$; (ii) $a^2y^2-b^2x^2=a^2b^2$.

26. If a pair of diameters be conjugate with respect to a hyperbola, they will be conjugate with respect to its conjugate hyperbola.

LOCUS

There is a discussion about locus in the second part of this book (for class X). Here the methods of finding locus relating to the use of the particular properties of st. lines, circles, parabolas, ellipses and hyperbolas will be shown in the following examples.

Examples (25)

Ex. 1. A straight line passes through a fixed point (h, k); find the locus of the foot of the perpendiculars drawn to it from the origin.

The equation of any st. line passing through the pt. (h, k) is $y-k=m(x-h)\cdots$(i),

[The different values of m will give different st. lines passing through (h, k)]

The equation of the perpendicular on (i) passing through the origin is $y=-\frac{1}{m}x\cdots$(ii)

Let (α, β) be the co-ordinates of the point of intersection of (i) and (ii), *i.e.*, of the foot of the perpendicular-(ii) on (i).

Then both the equations will be satisfied by (α, β).

$\therefore$ $\beta-k=m(\alpha-h)\cdots$(iii) and $\beta=-\frac{1}{m}\alpha\cdots$(iv).

Now eliminating m from (iii) and (iv) we have

$\beta-k=-\frac{\alpha}{\beta}(\alpha-h)$, or, $\beta^2-k\beta=-\alpha^2+h\alpha$,

$$\text{or,}\quad \alpha^2+\beta^2=h\alpha+k\beta.$$

$\therefore$ The required equation of the locus is $x^2+y^2=hx+ky$, which is a circle.

Ex. 2. A straight line passes through a fixed point. Find the locus of the middle point of the portion of the line intercepted between the axes.

Let (x', y') be the fixed point and (h, k) be the co-ordinates of the middle point of the portion of the st. line intercepted between the axes.

Then the equation to that st. line is $\frac{x}{2h}+\frac{y}{2k}=1$

[See Ex. 16. of Examples (3) in Part II of this book]

Since this st. line passes through the fixed point (x', y'),

$$\therefore \quad \frac{x'}{2h}+\frac{y'}{2k}=1.$$

$\therefore$ The required equation of the locus is $\frac{x'}{2x}+\frac{y'}{2y}=1.$

Ex. 3. B and C are fixed points having co-ordinates (3, 0) and (− 3, 0) respectively. If the vertical angle BAC be 90°, show that the locus of the centroid of the triangle ABC is a circle whose equation you are to determine. [W. B. H. S., 1961]

Let the co-ordinates of the vertex A of $\triangle$BAC and its centroid G be (x', y') and (h, k) respectively.

Then $h=\frac{1}{3}(x'+3-3)$

or $x'=3h$,

and $k=\frac{1}{3}(y'+0+0)$

or $y'=3k$.

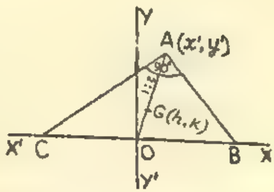


Fig. 37

$\because$ $\angle$BAC$=90°$, $\therefore$ the product of the gradients of AB and AC must be -1.

Now, the gradient of

AB$=\frac{3k}{3h-3}=\frac{k}{h-1}$ and that of AC$=\frac{3k}{3h+3}=\frac{k}{h+1}$.

$\therefore \quad \frac{k}{h-1}\times\frac{k}{h+1}=-1$, or, $\frac{k^2}{h^2-1}=-1$,

or, $k^2=-h^2+1$, or, $h^2+k^2=1$.

$\therefore$ The equation of the locus of the centroid is $x^2+y^2=1$, which is evidently a circle.

Ex. 4. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant; prove that its locus is a circle. What can you say about the centre of the circle? [C. U. '52]

Let the vertices of the triangle be A (α, β), B $(-a, 0)$ and C $(a, 0)$ and let the co-ordinates of any position of the moving pt. P be (h, k).

Then $PA^2=(h-\alpha)^2+(k-\beta)^2$, $PB^2=(h+a)^2+k^2$, and $PC^2=(h-a)^2+k^2$.

By the condition of the problem $PA^2+PB^2+PC^2=$constant.

$\therefore \quad (h-\alpha)^2+(k-\beta)^2+(h+a)^2+k^2+(h-a)^2+k^2=c^2$,

where c^2 is a constant,

or, $3(h^2+k^2)-2h\alpha-2k\beta=c^2-2a^2-\alpha^2-\beta^2$

$\therefore$ The required locus is

$3(x^2+y^2)-2\alpha x-2\beta y=c^2-2a^2-\alpha^2-\beta^2$, which represents a circle, since the coefficient of $x^2=3=$ coefficient of y^2 and the coefficient of $xy=0$ here.

Again, the centre of this circle is the point $\left(\frac{\alpha}{3}, \frac{\beta}{3}\right)$.

The co-ordinates of the centroid of the triangle

$=\{\frac{1}{3}(\alpha-a+a),\ \frac{1}{3}(\beta+0+0)\}=\left(\frac{\alpha}{3}, \frac{\beta}{3}\right)$.

Hence the centroid of the triangle is the centre of the circle.

Ex. 5. A chord of the circle $x^2+y^2=a^2$ always subtends a right angle at the centre, find the locus of its middle point.

[First method] Let (h, k) be the co-ordinates of the mid point of the chord in any position.

$\therefore$ The distance of the mid point of the chord from the centre of the circle $= \sqrt{h^2+k^2}$,

and the length of the chord $= 2\sqrt{a^2-(h^2+k^2)}$.

Since the chord subtends a right angle at the centre, the perpendicular distance of the chord from the centre is half of the chord.

$\therefore$ $\sqrt{h^2+k^2} = \sqrt{a^2-(h^2+k^2)}$, or, $h^2+k^2 = a^2-(h^2+k^2)$,

or, $2(h^2+k^2)=a^2$, or, $h^2+k^2=\dfrac{a^2}{2}$.

$\therefore$ The required locus of the mid point is $x^2+y^2=\dfrac{a^2}{2}$, which is a circle concentric with the given circle.

[**Second method**] Let the extremities of the chord be A$(a \cos \alpha,\ a \sin \alpha)$ and B$(a \cos \beta,\ a \sin \beta)$, and (h, k) be the co-ordinates of the mid point of the chord in any position.

Then, $h=\dfrac{a}{2}(\cos \alpha+\cos \beta)$, or, $\dfrac{2h}{a}=\cos \alpha+\cos \beta \cdots$ (i)

and $k=\dfrac{a}{2}(\sin \alpha+\sin \beta)$, or, $\dfrac{2k}{a}=\sin \alpha+\sin \beta \cdots$ (ii)

Now squaring both sides of (i) and (ii) and then adding the results we have

$$\frac{4}{a^2}(h^2+k^2)=2+2\ (\cos \alpha \cos \beta+\sin \alpha \sin \beta) \cdots \text{(iii)}$$

[$\because$ $\sin^2 \alpha+\cos^2 \alpha=1$ and $\sin^2\beta+\cos^2 \beta=1$]

$\because$ The centre of the given circle lies on the origin O,

$\therefore$ the gradient of AO

$=\dfrac{a \sin \alpha}{a \cos \alpha}=\dfrac{\sin \alpha}{\cos \alpha}$ and that of BO $=\dfrac{a \sin \beta}{a \cos \beta}=\dfrac{\sin \beta}{\cos \beta}$.

$\because$ $\angle$AOB is a right angle,

$\therefore$ $\dfrac{\sin \alpha}{\cos \alpha}\times\dfrac{\sin \beta}{\cos \beta}=-1$, or, $\cos \alpha \cos \beta+\sin \alpha \sin \beta=0$.

Putting this value in (iii) we have

$$\frac{4}{a^2}(h^2+k^2)=2, \quad \text{or,} \quad h^2+k^2=\frac{a^2}{2}.$$

$\therefore$ The required locus is $x^2+y^2=\frac{a^2}{2}$, which is a concentric circle.

Ex. 6. A circle touches the axis of y and cuts off a constant length $2l$ from the axis of x; find the locus of its centre.

Let (h, k) be the co-ordinates of the centre. Then in fig. 38, $OD=h$, $PD=k=OC$, and $AB=2l$.

$\because$ $AD=\frac{1}{2}AB$,

$\therefore$ $AD=l=BD$.

Now, $OA=OD-AD=h-l$ and $OB=OD+BD=h+l$.

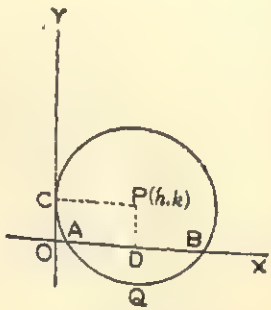


Fig. 38

Again, since OC is a tangent to the circle and OB is a secant,

$\therefore$ $OA.OB=OC^2$,

$\therefore$ $(h-l)(h+l)=k^2$, or, $h^2-k^2=l^2$.

Hence, the required locus is $x^2-y^2=l^2$, which is a rectangular hyperbola.

Ex. 7. Find the locus of a point which moves so that its distance from the point $(a, 0)$ always exceeds its distance from the y-axis by a.

Let $P(h, k)$ be the moving point. Then, the distance of the point P from the y-axis$=h$, and its distance from the point $(a, 0)=\sqrt{(h-a)^2+(k-0)^2}$.

$\therefore$ From the given condition $\sqrt{(h-a)^2+(k-0)^2}=h+a$,

or, $(h-a)^2+k^2=(h+a)^2$,

or, $k^2=(h+a)^2-(h-a)^2=4ah$,

$\therefore$ The equation of the locus is $y^2=4ax$, which is a parabola.

Ex. 8. Find the locus of the point of intersection of two tangents to the parabola $y^2=4ax$ which make an angle α with each other.

Let the equations of the two tangents PT, QT to the parabola $y^2=4ax$ be $y=m_1x+\dfrac{a}{m_1}\cdots$(i) and $y=m_2x+\dfrac{a}{m_2}\cdots$(ii) and let α be the angle between them and (h, k) be the co-ordinates of their intersection.

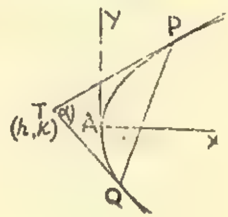


Fig. 39

$\because$ (i) and (ii) both pass through the pt. (h, k),

$$\therefore \quad k=m_1h+\frac{a}{m_1}$$

and $k=m_2h+\dfrac{a}{m_2}$.

Eliminating k from the two equations, we have

$$(m_1-m_2)h=\frac{a}{m_2}-\frac{a}{m_1}=\frac{(m_1-m_2)a}{m_1m_2}.$$

$$\because \quad m_1\neq m_2, \quad \therefore \quad a=m_1m_2h \text{ and } m_1m_2=\frac{a}{h}.$$

Now, $k=m_1h+\dfrac{a}{m_1}=m_1h+m_2h \quad [\because \quad a=m_1m_2h]$

$$\therefore \quad m_1+m_2=\frac{k}{h}.$$

Again, $m_1-m_2=\sqrt{(m_1+m_2)^2-4m_1m_2}$

$$=\sqrt{\frac{k^2}{h^2}-\frac{4a}{h}}=\sqrt{\frac{k^2-4ah}{h^2}}.$$

The angle between the tangents $=\alpha$,

$$\therefore \quad \tan\alpha=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{k^2-4ah}}{h}\div\left(1+\frac{a}{h}\right)=\frac{\sqrt{k^2-4ah}}{a+h},$$

or, $k^2-4ah=(h+a)^2\tan^2\alpha$.

Hence, the required equation of the locus is

$$y^2-4ax=(x+a)^2\tan^2\alpha.$$

Ex. 9. Find the locus of the middle points of the normal chords of the parabola $y^2=4x$.

Let (h, k) be the middle point of the normal chord. Now, the equation of the chord having (h, k) as its middle point is $y-k=\frac{2}{k}(x-h)$ [$\because$ $a=4\div 4=1$ here]

or, $ky-k^2=2x-2h$, or, $ky=2x+k^2-2h \cdots$(i)

The equation of the normal to the parabola is

$y=mx-2m-m^3 \cdots$(ii) [$\because$ $a=1$]

$\therefore$ (i) and (ii) are the same straight line,

$\therefore$ $\frac{k}{1}=\frac{2}{m}=\frac{k^2-2h}{-2m-m^3}$,

$\therefore$ $m=\frac{2}{k} \cdots$(iii) and $k=\frac{k^2-2h}{-m(2+m^2)} \cdots\cdots$(iv)

Now, eliminating m from (iii) and (iv) we have

$k^2-2h=-k\times\frac{2}{k}\left(2+\frac{4}{k^2}\right)=\frac{-4k^2-8}{k^2}$,

or, $k^2(k^2-2h)+4k^2=-8$. or, $k^2(k^2-2h+4)+8=0$,

$\therefore$ The required locus is $y^2(y^2-2x+4)+8=0$.

Ex. 10. A point moves in such a way that the sum of its distances from two fixed points is constant. Prove that the locus of the point is an ellipse.

Let S and S′ be the two fixed points. Join SS′ and produce it both ways.

Let this st. line be taken as the x-axis, the middle point O of SS′ as the origin and the perpendicular on SS′ at O as the y-axis.

Let $(c, 0)$ and $(-c, 0)$ be the co-ordinates of S and S′ and (h, k) be the co-ordinates of any position of the moving point.

By the condition of the problem we have

$$SP + S'P = \text{constant} = 2a \text{ (say)}$$

Now, $SP = \sqrt{(h-c)^2+(k-0)^2}$

and $S'P = \sqrt{(h+c)^2+(k-0)^2}$.

$\therefore \quad \sqrt{(h-c)^2+k^2} + \sqrt{(h+c)^2+k^2} = 2a,$

or, $\sqrt{(h-c)^2+k^2} - 2a = -\sqrt{(h+c)^2+k^2}$.

Squaring both sides we have

$(h-c)^2+k^2-4a\sqrt{(h-c)^2+k^2}+4a^2=(h+c)^2+k^2$

or, $-4a\sqrt{(h-c)^2+k^2}=(h+c)^2-(h-c)^2-4a^2=4(ch-a^2)$

or, $-a\sqrt{(h-c)^2+k^2}=ch-a^2$.

Squaring both sides we have

$$a^2(h-c)^2+a^2k^2=c^2h^2-2a^2ch+a^4,$$

or, $h^2(a^2-c^2)+k^2a^2=a^4-a^2c^2=a^2(a^2-c^2)$,

or, $\dfrac{h^2}{a^2}+\dfrac{k^2}{a^2-c^2}=1 \cdots\cdots$(i).

$\because \quad SP+S'P>SS', \quad \therefore \quad 2a>2c$, or, $a>c$, or, $a^2-c^2>0$.

Now if $a^2-c^2=b^2$ (say), then $b^2<a^2$.

$\therefore$ The equation-(i) is reduced to the form $\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}=1$.

Hence, the locus of the point is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, which is an ellipse.

Ex. 11. Find the locus of the foot of the perpendicular drawn from either focus upon any tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of any tangent PT to the given ellipse ABA′ is $y=mx+\sqrt{a^2m^2+b^2} \cdots$(i)

The equation of any st. line passing through the focus S $(ae, 0)$ of the ellipse and perpendicular to (i) is

$$y=-\frac{1}{m}(x-ae)\cdots\text{(ii)}$$

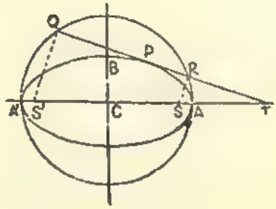


Fig. 40

Let (h, k) be the co-ordinates of the point of intersection R of (i) and (ii), i. e., of the foot of the perpendicular.

Then the equations (i) and (ii) will be satisfied by (h, k),

$\therefore$ from (i) we have $k=mh+\sqrt{a^2m^2+b^2}$.

$\therefore$ $(k-mh)^2=a^2m^2+b^2\cdots$(iii) ;

and from (ii) we have $k=-\frac{1}{m}(h-ae)$, or, $mk+h=ae$,

$\therefore$ $(mk+h)^2=a^2e^2=a^2-b^2\cdots$(iv)

Adding (iii) and (iv) we have

$(k-mh)^2+(mk+h)^2=a^2m^2+a^2$,

or, $(h^2+k^2)(1+m^2)=a^2(1+m^2)$,

or, $h^2+k^2=a^2$ [$\because$ $1+m^2\neq 0$, for m becomes imaginary if $1+m^2=0$].

Since $h^2+k^2=a^2$ is independent of m, this relation is true for any position of the tangent-(i)

$\therefore$ The required equation of the locus is $x^2+y^2=a^2$, which is the auxiliary circle of the ellipse.

The centre of the circle is C and its radius is $\frac{1}{2}$ AA$'$ or a [Art. 206].

Ex. 12. Find the locus of a point which moves so that the difference of its distances from the points $(\pm 5, 0)$ is 6.

Let A and B be the two points whose co-ordinates are $(-5, 0)$ and $(5, 0)$ respectively.

Join AB and produce it both ways, and let it be the x-axis.

Then the middle point of AB is (0, 0) and that is the origin.

Suppose (h, k) to be the co-ordinates of any position of the moving point P.

$\therefore$ $PA=\sqrt{(h+5)^2+k^2}$ and $PB=\sqrt{(h-5)^2+k^2}$.

From the given condition $PA-PB=6$,

or, $\sqrt{(h+5)^2+k^2}-\sqrt{(h-5)^2+k^2}=6$,

or, $\sqrt{(h+5)^2+k^2}=6+\sqrt{(h-5)^2+k^2}$.

Squaring both sides we have

$(h+5)^2+k^2=36+(h-5)^2+k^2+12\sqrt{(h-5)^2+k^2}$,

or, $(h+5)^2-(h-5)^2-36=12\sqrt{(h-5)^2+k^2}$,

or, $5h-9=3\sqrt{(h-5)^2+k^2}$.

or, $25h^2+81-90h=9h^2-90h+225+9k^2$ [squaring].

or, $16h^2-9k^2=144$.

Hence, the required equation of the locus is $16x^2-9y^2=144$, or, $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$, which is a hyperbola.

Exercise 25

1. Find the equation to the locus of a point which moves so that the area of the triangle, formed by joining it to the points $(a, 0)$, $(-a, 0)$ and by joining these points, is constant and equal in magnitude to c^2.

2. Two vertices of a triangle are the points (1, 2), (3, 4). If the third vertex moves on the line $3x+4y+1=0$, prove that the centroid moves on the line $9x+12y-35=0$.

3. A variable line passes through a fixed point (α, β) and meets the axes in A and B. If the rectangle OAPB is completed, find the locus of P.

4. From any point A on the line PQ, which cuts off equal intercepts $2c$ from the axes, perpendiculars AB and AC are drawn on the axes. Find the locus of the middle point of BC.

5. A point P moves so that the perpendiculars from it to the lines $3x+4y+4=0$ and $5x+12y+4=0$ are equal. Find the locus of P.

6. A point P moves in a plane. The perpendicular bisector of the line joining P to the origin, intersects at Q the perpendicular bisector of the line joining P to the point (4, 4). Find the locus of Q.

7. A and B are two fixed points on a plane, and a point P moves on the plane in such a way that PA=2PB always. Prove analytically that the locus of P is a circle. [H. S. 1961]

[**Hints** : Take the line AB as x-axis, its middle point as origin and the length AB$=2a$.]

8. Show that the locus of the feet of the perpendiculars from a fixed point (α, β) on the circle $x^2+y^2=a^2$ upon its diameter is another circle, whose centre and radius you are to determine.

9. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the point $(c, 0)$.

10. Find the equation to the locus of a point which moves in such a way that it is twice as far from the origin as from the point (2, 3).

11. Find the locus of a point whose distance from the point (3, 6) is equal to its distance from the x-axis.

12. Find the equation to the locus of a point which moves on the plane (xy) so that its distance from the y-axis is twice its distance from the point (2, 2). [C. U. '39]

13. On the line joining the origin to any point P on the circle $(x-\alpha)^2+(y-\beta)^2=a^2$, a point Q is taken such that OP : OQ is constant ; show that the locus of Q is a circle.

14. Prove that the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle, is a circle. [C. U. '36]

15. Find the locus of a point which moves in such a way that the sum of its distances from the two points $(\pm 3, 0)$ is 10.

16. Show that the locus of a point, which moves such that the difference of its distances from two fixed points is constant, is a hyperbola.

17. If a circle be drawn so as always to touch a given line and also a given circle, prove that the locus of its centre is a parabola. [C. U. '40]

18. Find the locus of the point of intersection of two tangents to the parabola $y^2=4ax$, when the tangents meet at an angle of 45°.

19. Show that the locus of the middle points of all chords of a hyperbola, all passing through a fixed point, is a hyperbola.

20. Find the locus of the point of intersection of tangents to an ellipse which meet at a given angle θ.

21. Find the locus of the foot of the perpendicular from the centre upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

22. The base of a triangle is fixed. Find the locus of the vertex, which moves in such a way that the vertical angle always remains constant.

23. Prove that the locus of the middle points of the normal chords of the parabola $y^2=4ax$ is $\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a$. [Raj. '51]

24. Find the locus of the middle points of the portions of tangents to an ellipse included between the axes.

25. Two rods of lengths $2a$ and $2b$ slide along the axes of co-ordinates in such a way that in every position of the rods, their extremities lie on a circle. Find the locus of the centre of the circle.

ANSWERS

ALGEBRA

Exercise 1 (A)

1. (*a*) It is an identity, as it is satisfied by all the three values 2, 4, 5 of x.

(*b*) An identity, as it is satisfied by the three values a, b, c of x.

(*c*) An identity, as it is satisfied by the three values a, b, c of x.

2. (i) $\dfrac{\pm\sqrt{b^2-4ac}}{a}$ (ii) $-\dfrac{bc}{a^2}$.

(iii) $\dfrac{b^2-2ac}{c^2}$. (iv) $\dfrac{3abc-b^3}{a^3}$. (v) $\dfrac{3abc-b^3}{a^2c}$

(vi) $\dfrac{b^2(b^2-4ac)}{a^2c^2}$. (vii) $\dfrac{b^2-2ac}{a^4c^2}(b^4-4ab^2c+a^2c^2)$.

(viii) $\dfrac{-3b}{2b^2+ac}$. (ix) $\dfrac{b}{ac}$.

3. (i) $\dfrac{3pq-p^3}{q^3}$. (ii) $\dfrac{3pq-p^3}{q^2}$. (iii) $\dfrac{p^4-4p^2q+2q^2}{q}$.

(iv) $(p^2-q)(p^2-3q)$. (v) $\dfrac{p(p^2-4q)(q-p^2)}{q}$.

(vi) $pq^4(3q-p^2)$. (vii) $(p^4-4p^2q+2q^2)/q^4$.

4. (i) $x^2-7x+12=0$, (ii) $x^2+2x-35=0$.
(iii) $x^2-2x-2=0$, (iv) $x^2-2ax+a^2-b^2=0$.
(v) $x^2-2(a^2+1)x+a^4+a^2+1=0$.

5. (*a*) $x^2-5x+4=0$.
(*b*) (i) $4x^2+3x+1=0$. (ii) $x^2-x+1=0$.
(*c*) $2x^2+(b-2\sqrt{c})x-b\sqrt{c}=0$.

6. $ax^2+2bx+4c=0$.

7. (i) $a^2x^2-(b^2-2ac)x+c^2=0$. (ii) $ac(x+1)^2=bx$.
(iii) $a^4x^2-a^2(2b^2-4ac)x+b^2(b^2-4ac)=0$.
(iv) $acx^2+b(a+c)x+(a+c)^2=0$. (v) $acx^2-bx+1=0$.

8. $x^2 \pm 56x + 768 = 0$.

12. (i) -3. (ii) 8. (iii) 5. (iv) -1.

13. $x^2 - 8x + 11 = 0$. 18. (i) $x^2 - px + q = 0$.

(ii) $qx^2 - (p+q^2)x + pq = 0$. (iii) $x^2 - px + 9q - 2p^2 = 0$.

19. 8. 20. $9261x^2 - 903x - 170 = 0$.

21. $4x^2 - 12x + 5 = 0$. 22. $x^2 + x + 1 = 0$.

23. $x^2 + x + 1 = 0$. 24. $cx^2 + bx + a = 0$.

26. $cx^2 + bx + a = 0$. 28. $a^2 - 2ab - 3b^2 + 16ac = 0$.

32. $5b^2 = 36ac$. 34. $\dfrac{\gamma + \delta \pm \sqrt{(\gamma+\delta)^2 - 4\alpha\beta}}{2}$

35. $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ (taking α, β as the roots of 2nd equation)

37. 2 and 12. 38. (i) $b = 0$, (ii) a, b, c should be of like sign.

(iii) a and c are of like sign, but b is opposite to them in sign.

39. (i) $q = 0$. (ii) $p = q = 0$. 40. (i) $x^2 - 8x + 15 = 0$.

(ii) $x^2 - 12x + 32 = 0$.

45. $a^2a_1^2x^2 - aa_1bb_1x + a_1c_1b^2 + acb_1^2 - 4aa_1cc_1 = 0$.

49. $6x^2 - 12x - 19 = 0$.

Exercise 1 (B)

1. (*a*) Real, rational and unequal.
   (*b*) Real, rational and equal but opposite in sign.
   (*c*) Real, rational and equal.
   (*d*) Imaginary and unequal.
   (*e*) Real, irrational and unequal.

5. $\frac{49}{4}$. 6. 9. 7. (*a*) 2 or $-\frac{10}{9}$. (*b*) $\frac{1}{3}$. 15. $\dfrac{a}{m}$.

20. Is not rational, as the coefficient of x is irrational though the discriminant is a perfect square.

21. (i) The roots are $\sqrt{3} \pm 5$ (ii) the roots are $5 \pm \sqrt{3}$, for answer of the second part see Ex. 10 in Examples 1 (B).

31. $c = \frac{a}{m}$. 34. $p^2 = q$ when the roots are equal.

35. (a) (i) If $ac < 0$, the roots are real, equal and opposite in sign ; if $ac > 0$, the roots are imaginary ;

(ii) One root is 0 and the other is $-\frac{b}{a}$, so they are unequal and real ; (iii) both the roots are zero ;

(iv) $-\frac{c}{b}$ is the only root. (c) least value of $f(x) = 8$.

Exercise 2

1. Positive 2. Negative 3. Positive
4. Any value not between 2 and $3\frac{1}{2}$. 5. $\frac{25}{12}$
8. 11, 3 9. 5 and $\frac{1}{5}$ 15. 4, -5
18. $(3x-2y+1)(x+3y-2)$ 19. ± 7 20. 0, 12
23. 3, 5 24. $(aa'-bb')^2+4(ha'+h'b)(hb'+h'a)=0$.

Exercise 3

1. 40320 ; 210 ; 5040 ; 360 ; 120 2. 5 or 4 3. 4
8. 30 9. 720 10. 60 11. 336 12. 380 13. 870
14. 4320 15. 389188800, 59875200 16. 20160, 5040
17. (i) 10079 (ii) 719 (iii) 359 19. 576
20. 60 21. 60 22. 36 23. 2160 24. 120
25. 24 26. 154 27. $(n-2)\,\underline{|n-1}$
28. (i) 330 (ii) 990 29. 576 30. 4096 31. $\underline{|20} \times {}^{21}P_{16}$
32. 3600 33. 900 34. 720, 600, 96 35. 81
36. $8\underline{|9}$ 37. $\underline{|9}$ 38. 5040 39. 2520 40. 12

41. 20160 (for positions relative to the table), 2520 (for positions relative to each other)

42. 1036800 (relative to the table), 86400 (relative to each other) ; 43200 (without distinction to direction)

43. 28800, 2880, 1440 (relative to table, relative to each other, when no distinction of direction is made).

44. $\dfrac{\underline{|16}\,\underline{|15}}{\underline{|4}}$ 45. (i) 240, (ii) 480 46. 2880

47. $\dfrac{\underline{|41}}{\underline{|4}\,\underline{|5}\,\underline{|5}\,\underline{|7}\,\underline{|7}\,\underline{|7}}$ 48. $\dfrac{\underline{|39}}{\underline{|5}(\underline{|4})^2(\underline{|6})^3}$

49. 126 50. 3456 52. 2520 53. 39600.

Exercise 4

1. (i) 15 (ii) 120 (iii) 435 (iv) 51 2. 14 3. (i) 8, (ii) 1771
4. $n-1$ 5. 56 6. 840 7. 246 8. 868224
9. 816000 10. 36 11. 220 12. (i) 210 (ii) 371
13. 200 14. 25 15. 20, 9 16. 120 17. (i) 40 (ii) 116
18. (i) $\frac{1}{2}\{n(n-1)-m(m-1)+2\}$
(ii) $\frac{1}{6}\{n(n-1)(n-2)-m(m-1)(m-2)\}$ 19. 990
20. $n=6, r=3$ 21. 63 22. 344 22. (a) 1260 23. (i) 100
(ii) 451 24. 15 25. 68 26. $\dfrac{\underline{|22}}{2\,\underline{|11}\,\underline{|11}}$ 27. 5 28. 255

29. 369600 30. $\dfrac{\underline{|p+1}}{\underline{|n}\,\underline{|p-n+1}}$ 31. 31 32. 2520

33. 55 34. 4 36. (i) $\dfrac{\underline{|68}}{(\underline{|17})^4\underline{|4}}$, (ii) $\dfrac{\underline{|68}}{(\underline{|17})^4}$

37. 1260 38. 15400 39. (i) 113, (ii) 2190 40. 7776000
42. (i) 160, (ii) 2724 45. $\dfrac{\underline{|51}}{\underline{|12}\,\underline{|16}\,\underline{|23}}$ 46. 59

47. $\dfrac{\underline{|pr}}{(\underline{|p})^r\underline{|r}}$ 48. 315 49. 30 50. 129360.

Exercise 5

1. $a^5+15a^4+90a^3+270a^2+405a+243$
2. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$
3. $32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5$
4. $x^7-\dfrac{21x^6}{y}+\dfrac{189x^5}{y^2}-\dfrac{945x^4}{y^3}+\dfrac{2835x^3}{y^4}-\dfrac{5103x^2}{y^5}+\dfrac{5103x}{y^6}$
$-\dfrac{2187}{y^7}$ 5. $240\sqrt{3}$ 6. $32-40a^2+10a^4$

7. $a^6-3a^5-3a^4+11a^3+6a^2-12a-8$ 8. ${}^{17}C_7 . \frac{1}{x^7}$

9. $43750x^4y^4$ 10. ${}^{2n}C_{n-1}a^2$

10. (a) $\frac{n(n-1)(n-2)\cdots(n-p+2)}{\underline{|p-1}} . 3^{p-1}x^{p-1}$ and

$\frac{n(n-1)(n-2)\cdots(n-p+2)}{\underline{|p-1}} . 3^{n-p+1}\ x^{n-p+1}$

11. 924 12. $126x$, $-\frac{126}{x}$ 13. -252 14. $\frac{2835}{8}$

15. (i) $(-1)^n \frac{\underline{|2n}}{\underline{|n}\underline{|n}}$, or $\frac{1.3.5\cdots(2n-1)}{\underline{|n}}(-2)^n$; (ii) 252;

17. -252 18. 13440 19. -364 20. 4433

21. 340 ; 21 22. $(-1)^{n-r}\frac{\underline{|2n+1}}{\underline{|n-r}\,\underline{|n+r+1}}$ 24. 495

25. $199\frac{1}{9}$ 26. $\frac{\underline{|2n}}{\underline{|n}\underline{|n}}$ 27. -70 ; 27(a) $\frac{7}{18}$; 28. 252

29. $(n+1)$th. term, $\frac{\underline{|m+n}}{\underline{|n}\,\underline{|m}}$ 30. $\frac{28672}{729}$ 31. t_6

32. $t_4=t_5=\frac{224}{9}$ 33. $\frac{2816}{27}$ 34. -1 35. -1

37. $\frac{\underline{|2n}}{\underline{|n}\underline{|n}}$ 38. $x=1$, $a=2$, $n=7$ 39. 8 40. 11

43. 9 44. 1 47. 970299, ·96

60. (i) 0 (ii) $\frac{\underline{|2n}}{\underline{|n+1}\,\underline{|n-1}}$ (iii) $\frac{1}{2}n(n+1)$

61. If n is even, $\frac{\underline{|n}}{\underline{|\tfrac{1}{2}n}\,\underline{|\tfrac{1}{2}n}}x^{\frac{1}{2}n}$;

If n is odd, $\frac{\underline{|n}}{\underline{|\tfrac{1}{2}(n-1)}\,\underline{|\tfrac{1}{2}(n+1)}}x^{\frac{1}{2}(n-1)}$

and $\frac{\underline{|n}}{\underline{|\tfrac{1}{2}(n+1)}\,\underline{|\tfrac{1}{2}(n-1)}}x^{\frac{1}{2}(n+1)}$

62. The sum of coefficients $=2^6$ 63. $a=2$, $x=3$, $n=5$.

Exercise 6

1. $1-4x+10x^2-20x^3+\cdots$
2. $1+3x+6x^2+10x^3+\cdots$
3. $1-10x+60x^2-280x^3+\cdots$
4. $1-\frac{3}{5}x-\frac{3}{25}x^2-\frac{7}{125}x^3-\cdots$
5. $1-\frac{3}{2}x+\frac{27}{8}x^2-\frac{135}{16}x^3+\cdots$
6. $8+9x+\frac{27}{16}x^2-\frac{27}{128}x^3+\cdots$
7. $\frac{1}{8}-\frac{3}{16}a+\frac{3}{16}a^2-\frac{5}{32}a^3+\cdots$
8. $1-\frac{1}{3}x-\frac{1}{3^2}x^2-\frac{5}{3^4}x^3-\cdots$
9. $a^{-\frac{5}{2}}\left(1+\frac{5x}{a}+\frac{35x^2}{2a^2}+\frac{105x^3}{2a^3}+\cdots\right)$
10. $1+2x+5x^2+\frac{40}{3}x^3+\cdots$
11. $\frac{1}{\sqrt[5]{16}}\left(1+\frac{6}{5}x^2+\frac{81}{50}x^4+\frac{567}{250}x^6+\cdots\right)$
12. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots$
13. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\cdots$
14. $-4a^3,\ (-1)^r(r+1)a^r$
15. $\frac{2}{3}x^3$
16. $36x^7,\ \frac{(r+1)(r+2)}{2}x^r$
17. $\frac{b^r}{a^{r+1}}x^r$
18. $\frac{(n+1)(2n+1)(3n+1)\cdots\{(r-2)n+1\}}{n^{r-1}\,\underline{|r-1}}\cdot\frac{x^{r-1}}{a^{\frac{1}{n}+r-1}}$
19. $3\times\frac{1.3.5\cdots(2r-7)}{2^{r-1}\,\underline{|r-1}}x^{r-1}$
20. $(-1)^r\frac{(r+1)(r+2)}{2}a^{2r}$
21. $(-1)^{r-1}\ \frac{1.3.5\cdots(2r-3)}{2^r\,\underline{|r}}x^r$
22. (i) $(-1)^r\ \frac{1.3.5\cdots(2r-1)}{\underline{|r}}a^r$ (ii) $\frac{1.4.7\cdots(3r-2)}{\underline{|r}}(x)^r$
23. $1+2x+3x^2+4x^3+\cdots;\ (r+1)x^r.$
24. $1+4x+10x^2+20x^3+\cdots;\ \frac{(r+1)(r+2)(r+3)}{6}x^r$
25. $1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\cdots;\ \frac{1.3.5\cdots(2r-1)}{\underline{|r}}x^r$
26. $-\frac{27}{128}$
27. 101
28. 121
29. 2
30. $-\frac{1}{4}\left(3+\frac{5}{3^8}\right)$

31. $4n$ 32. $-\frac{r+9}{2^{r+2}}, -\frac{7}{64}$ 33. $\frac{(r+1)(r+2)(r+3)}{3.2}$

34. $(-1)^n$ 35. 7th. term 36. 3rd. term

37. 8th. and 9th. terms 38. $\frac{7.7}{2}.3^{\frac{3}{4}}$ 39. (i) 10th. term,

(ii) 2nd. term (iii) 9th. term 40. (i) $-\frac{5}{8}a^4$, (ii) $-\frac{1}{2x^3}$.

41. $\frac{1.3.5\cdots(2r-1)}{\lfloor r}$ 42. 9th. term.

Exercise 7

1. 1·2167 2. 4·9920 3. 9·9990 4. 9·9499

5. ·1459 6. 1·0007 7. 1·2599 8. ·1996

9. 4·89898 10. ·9996 11. (a) ·0638

11. (b) 1·0003 12. 1·00199

13. $1+\frac{13}{6}a$ 14. $1-\frac{51}{20}a$ 17. $1+x+x^2+\cdots\cdots$ to ∞

18. $\frac{3}{4}$ 18. $\frac{3\sqrt{3}}{2}$ 20. $\frac{2}{3}$ 21. $\frac{1+x}{(1-x)^2}$

27. $\frac{1-3x}{(1+x)^2}$ 23. $\sqrt{2}$ 24. $\sqrt{\frac{2}{3}}$ 25. $\sqrt{\frac{3}{2}}$

26. $\sqrt[4]{\frac{2}{3}}$ 27. $\sqrt[3]{2}$ 28. 2 29. $(\frac{2}{4})^{\frac{1}{3}}$

30. $4(2)^{\frac{1}{3}}-2$ 31. $(\frac{3}{2})^{-\frac{3}{2}}$ 32. $\frac{19}{8}$ 33. $3\sqrt{3}$ 34. $\frac{1}{24}$

35. $(1+\frac{1}{2})^{-\frac{1}{2}}$, $\sqrt{\frac{2}{3}}$ 38. $\frac{1}{11}$ 39. $\frac{5}{99}$ 40. $\frac{9}{110}$

41. $1\frac{5}{9}$ 43. $\frac{(r+1)(r+2)(r+3)}{1.2.3}$ 44. $\frac{(p+1)(p+2)\cdots(p+r)}{\lfloor r}$.

Exercise 8

1. $n\pi+(-1)^n\frac{\pi}{3}$ 2. $n\pi+\frac{3\pi}{4}$. 3. $2n\pi+\frac{7\pi}{6}$

4. $2n\pi+\frac{\pi}{2}$ or $2n\pi-\frac{\pi}{6}$ 5. $n\pi\pm\frac{\pi}{3}$ 6. $n\pi+\frac{\pi}{2}$ or $n\pi\pm\frac{\pi}{4}$

7. $\frac{n\pi}{2}+\left(-1\right)^n\frac{\pi}{12}$ 8. $2n\pi+\frac{7\pi}{12}$ or $2n\pi-\frac{\pi}{12}$ 9. $2n\pi+\frac{\pi}{3}$

10. $2n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}$ 11. $n\pi+\frac{\pi}{4}$, or, $n\pi+\cot^{-1}\frac{1}{2}$.

12. $\frac{r\pi}{m+(-1)^r n}$ where r is any integer or zero

13. $\left(2n+1\right)\frac{\pi}{5}$ or $\frac{2n\pi}{3}$ 14. $\left(2n+1\right)\frac{\pi}{12}$ 14. (a) $x=\frac{2n+1}{p+q}.\frac{\pi}{2}$

15. $\left(2n+1\right)\frac{\pi}{2}$ or $n\pi+(-1)^n\frac{\pi}{6}$ 16. $2n\pi\pm\frac{2\pi}{3}$

17. $2n\pi+\frac{5\pi}{4}$ 18. $2n\pi$ or $2n\pi+\frac{\pi}{4}$ 19. $\frac{n\pi}{3}$ or $\frac{1}{12}\pi(6n\pm1)$

20. $\frac{n\pi}{4}$ or $\left(2n+1\right)\frac{\pi}{24}$ 21. $n\pi$ or $\frac{1}{4}n\pi+(-1)^n\frac{\pi}{24}$

22. $\frac{n\pi}{2}\left(\text{or } n\pi\right)$ 23. $n\pi+\frac{\pi}{4}$ or $n\pi+\frac{\pi}{3}$ 24. $\frac{1}{3}\left(n\pi+\frac{\pi}{4}\right)$

25. $\frac{1}{3}n\pi$ or $n\pi$ or $\frac{1}{2}n\pi$ 26. $n\pi\pm\frac{\pi}{4}$

27. $2n\pi$ or $2n\pi-\frac{\pi}{2}$ or $n\pi-\frac{\pi}{4}$ 28. $\frac{1}{4}n\pi+(-1)^n\frac{\pi}{40}$

29. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}\pm\frac{\alpha}{3}$ 30. $n\pi\pm\frac{\pi}{6}$ 31. $2n\pi$ or $\frac{1}{6}(4n+1)\pi$

32. $\frac{1}{3}n\pi$ or $n\pi\pm\tan^{-1}\frac{1}{\sqrt{2}}$ 33. $2n\pi-\alpha$ or $(4n-1)\frac{\pi}{2}+\alpha$

34. $\left(2n+1\right)\frac{\pi}{8}$ or $\left(2n+1\right)\frac{\pi}{4}$ or $\left(2n+1\right)\frac{\pi}{2}$

35. $2n\pi+\alpha$ or $(2n+1)\frac{\pi}{3}-\frac{\alpha}{3}$ 36. $2n\pi+\frac{\pi}{2}$ or $2n\pi-\frac{611\pi}{2700}$

37. $2m\pi$ or $\frac{4m\pi}{n\pm1}$ 38. $90°, 450°, 810°$ 39. $\frac{n\pi}{6}$

40. $2n\pi+\frac{\pi}{2}$ or $\sin^{-1}(-\frac{3}{5})$ 42. $30°, 120°, 150°$ 43. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

44. $\frac{\pi}{12}, \frac{-7\pi}{12}$ 45. $\frac{\pi}{3}$ 46. $\frac{1}{8}\pi, \frac{5}{8}\pi, \frac{9}{8}\pi, \frac{13}{8}\pi$

47. $75°$ or $345°$ 48. $x=y=45°$

49. $2n\pi+\frac{\pi}{2}$ or, $2n\pi+\frac{58}{225}\pi$.

Exercise 9

1. $\frac{\pi}{4}$ 2. $-\frac{\pi}{3}$ 3. $\frac{\pi}{6}$ 4. $\frac{\pi}{2}$ 5. $\frac{\pi}{2}$ 6. $\frac{\pi}{2}$

7. (i) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x}$

(ii) $\tan^{-1}x=\sin^{-1}\frac{\pm x}{\sqrt{1+x^2}}=\cos^{-1}\frac{\pm1}{\sqrt{1+x^2}}=\cot^{-1}\frac{1}{x}$

$=\sec^{-1}(\pm\sqrt{1+x^2})=\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{\pm x}$

33. ±1 34. $\pm\frac{1}{2}\sqrt{\frac{3}{7}}$ 35. $-\frac{1}{6}$ or 1 36. 3

37. 0 or $\frac{1}{2}$ 38. 13 39. 1 40. $\pm\frac{1}{2}$ 41. $\frac{-4\pm\sqrt{61}}{9}$

42. $\pm\frac{25}{24}$ 43. $\pm\frac{1}{\sqrt{2}}$ 44. $\frac{a+b}{1-ab}$ 45. $\pm\frac{\sqrt{5}}{3}$

46. 2 47. $x=\frac{1}{2}, y=1$ 48. No value 49. ∞.

Exercise 10

1. $A=30°, B=120°, C=30°$ 2. $60°$ or $120°$ 3. 43

4. $\frac{4}{7}$ 5. $\frac{4}{5}$ 6. $60°$ 7. 84 8. 24 9. 864 sq. yds.

10. $B=90°, A=30°, C=60°$ 11. $120°$.

Exercise 11

1. $r=4, r_1=10\frac{1}{2}$ 2. $R=\dfrac{abc}{4\triangle}$ 23. 7 : 2.

Exercise 12

1. ·70671 2. ·90482 3. ·71857 4. ·26095
5. 1·24884 6. 1·29752 7. 9·84758 8. 9·61614
9. 9·81911 10. 10·28229 11. 68°57′ 12. 3·68372
13. ·0353441 14. ·61837 15. ·30317175 16. 2·64114
17. 35°24′37″ 18. ·66553 19. 58°18′12″
20. 76°21′29″ 21. 10·74013 22. 9·7867587
23. 79°51′47·2″ 24. θ=27°39′55″ 25. ·66843
26. 10·613296 27. 9·8505931 29. ·29974
30. 1·776 31. ·2394
32. 36°52′7″ 33. 16°41′15″.

Exercise 13

1. A=41°23′14″, B=47°46′50″, C=90°49′56″ 2. 9·6733937
3. 38°56′33″, 47°41′7″, 93°22′20″ 4. 58°59′33″
5. 78°27′46″·8 (App.) 6. 88°59′41″ (App.)
7. 48°11′23″, 58°24′43″, 73°23′54″ 8. 55°46′16″·14 (App.)
9. 104°28′39″ (App.) 10. 71°42′ (App.) 11. 53°7′48″
12. 60°, 45°, 75° 13. A=120°, B=C=30° 14. 120°
16. 30°, 105° ; 2 : $\sqrt{2}$: ($\sqrt{3}+1$) 17. 2 : ($\sqrt{3}+1$)
18. 1 : $\sqrt{3}$: 2 21. 12 : 5 : 13 22. 2 : ($\sqrt{3}+1$)
23. $\sqrt{2}$: 2 : ($\sqrt{3}+1$) 24. 14·35948 ft.

Exercise 14

1. a=1, B=120°, C=30° 2. c=2 A=75°, B=60°
3. 16°3′·5, 123°56′·5 4. B=78°17′39·6″, C=49°36′20·4″
5. A=38°12′47″·5, B=21°47′12″·5 6. 119°16′51″, 5°43′9″
7. A=94°42′54″, B=25°17′6″
8. A=71°44′29·5″, B=48°15′30·5″

9. $B=76°47'2{\cdot}2''$, $C=49°12'57''{\cdot}8$
10. $A=116°33'54''$, $B=26°33'54''$
11. $B=19°38'3''$, $C=109°39'57''$, $a=559{\cdot}63$
12. $70°53'36''$, $49°6'24''$ 14. $27{\cdot}0375$
15. $7{\cdot}698622$ 16. $B=30°$, $a=c=2(\sqrt{3}+1)$
17. Each side $=\sqrt{5}+1$; $72°$, $72°$, $36°$
18. $C=70°30'$, $b=18{\cdot}33$, $c=37{\cdot}05$ 19. $769{\cdot}8622$
20. $172{\cdot}6436$.

Exercise 15

1. $B=75°$, $C=45°$
3. $B_1=105°$, $C_1=30°$, $b_1=\sqrt{2}$; $B_2=75°$, $C_2=60°$, $b_2=\sqrt{6}$
4. No solution 5. $A_1=105°$, $C_1=45°$, $a_1=10(\sqrt{6}+\sqrt{2})$; $A_2=15°$, $C_2=135°$, $a_2=10(\sqrt{6}-\sqrt{2})$
6. $B=60°$, $A=90°$, $a=20\sqrt{3}$; or, $B=120°$, $A=30°$, $a=10\sqrt{3}$ 7. $A=30°$, $B=90°$, $b=10$ 8. $63{\cdot}996$ ft.
9. $44°25'39''$ 10. $A=100°34'$, $B=34°26'$
11. $A=75°53'$, $C=67°47'$, or, $A=31°27'$, $C=112°13'$
12. $59°10'35''$ or $120°49'25''$ 13. $B=34°27'$(App.), $C=100°33'$
14. $4{\cdot}56706$ ft. 15. $5°44'21''$
17. $A=31°39'33''$, $C=96°1'27''$, $a=878{\cdot}753$.

Exercise 16

1. (a) $2\cos\frac{1}{2}(\phi_2-\phi_1)$, (b) $\frac{1}{2}(\phi_1+\phi_2)$ 2. $1960{\cdot}95$ yds.
4. $1202{\cdot}08$ yds. 5. 1 or $\frac{1}{3}$ 6. 1 or 4 8. $341\sqrt{2}$ ft.
9. 1 or 2 10. 12 ft. 11. 100 ft. 12. 10000 ft.
13. $73{\cdot}2$ ft, 200 ft. 14. $h\cot\beta\tan\alpha$ 15. $1060{\cdot}5$ ft.
17. At a distance of $141{\cdot}7$ ft. from the hill 18. $200{\cdot}1$ ft.
19. $440\sqrt{6}$ yds. 21. $1185{\cdot}7$ ft. 22. $339{\cdot}4$ ft. 27. $\frac{a}{\sqrt{2}}$
29. $107{\cdot}238$ ft. (App.).

Exercise 17

7. $x=\frac{\pi}{4}$ 8. $x=n\pi+\frac{1}{4}\pi$ 9. $x=\frac{\pi}{4}$ 10. $x=38°10'$ (App.)

11. $x=0$, $x=1{\cdot}17$ radian (nearly) 12. $x=0°$, $60°$, $180°$, $300°$, $360°$

13. $x=0$ 14. $\theta=90°$ and $46°25'$ (App.)

15. The graph approaches nearer and nearer to the line $x=\frac{\pi}{2}$ but will never touch it. So the line $x=\frac{\pi}{2}$ is an asymptote to the curve.

16. [In $y=x$, x should be measured in radians].

The graphs will touch one another at the origin. This suggests that when x is very small, $\sin x=x=\tan x$.

CO-ORDINATE GEOMETRY

Exercise 18

1. (i) $x^2+y^2=9$, (ii) $x^2+y^2+6y-16=0$
(iii) $x^2+y^2-4x-6y-3=0$ (iv) $x^2+y^2+6x-8y+16=0$
(v) $x^2+y^2+6x-4y+7=0$ (vi) $x^2+y^2+2x+4y=0$
(vii) $x^2+y^2-4x-6y-12=0$

2. (i) $(0, 0)$, 2 ; (ii) $(0, 0)$, $\sqrt{5}$; (iii) $(-1, 2)$, $\sqrt{2}$
(iv) $(2, 3)$, 5 ; (v) $(-\frac{3}{4}, \frac{5}{4})$, $\frac{5\sqrt{2}}{4}$; (vi) $(3, -7)$, 5

3. 19 4. 0 ; the centres are collinear 6. $x+3y=0$

7. $2x+5y=6$ 8. Centres : $(1, -1)$, $(3, 1)$, $(4, 2)$, radii : 3, 4, 5 ; $x-y=2$

9. (i) $x^2+y^2-4x-5y=0$ (ii) $x^2+y^2-22x-4y+25=0$
(iii) $x^2+y^2-5x-y+4=0$ (iv) $x^2+y^2-6x-2y+5=0$
(v) $x^2+y^2+5x-5y=0$

10. $x^2+y^2-4x+6y+4=0$ 11. $x^2+y^2+56x+46y-212=0$

12. $x^2+y^2+2x+4y-45=0$ 13. $x^2+y^2-6x-8y=0$
14. $x^2+y^2-4x-3y=0$ 15. $x^2+y^2-3x+4y-12=0$
16. $2x^2+2y^2+5x-6y+7=0$ 17. $x^2+y^2=36$
18. $x^2+y^2+2\sqrt{21}y-4=0$, $x^2+y^2-2\sqrt{21}y-4=0$
19. $x^2+y^2+6y-16=0$ 21. $\frac{3\sqrt{3}}{4}(f^2+g^2-c)$
22. $x^2+y^2-4x-6y-12=0$
23. $x-2y+2=0$, $y=0$, $x+y=1$; $x^2+y^2=x+y$
24. $x^2+y^2-17x-19y+50=0$.

Exercise 19

1. $x^2+y^2-2x-2y+1=0$
2. $x^2+y^2+6x+6y+9=0$, $x^2+y^2-6x-6y+9=0$
3. *Hints* :—Distance between the centres = the sum of the radii
4. *Hints* : Distance between the centres = the difference of the radii.
5. $x^2+y^2-4x-2y+4=0$
6. $x^2+y^2-10x+26y+25=0$, $x^2+y^2-10x-26y+25=0$
7. (3, 4) and (4, 3)
8. The line touches the circle at the point $(-1, -1)$
9. (7, 3) and (2, 8) 10. $(-1, 1)$
11. $(-4, 6)$ 12. $\left(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$
15. $(-3, 1)$ 16. (*a*) 8 (*b*) $\frac{2}{5}\sqrt{1600-c^2}$
17. (*a*) $2x-3y+13=0$ (*b*) $x+2y=5$ 18. $x-y=3$
19. (i) $3x-4y=25$; (ii) $3x+4y=32$; (iii) $12x-5y+96=0$
20. $4x+5y=41$ and $4x-5y=41$
21. (i) $2y=5x$; (ii) $3x-4y=6$ 22. $\pm\frac{5}{12}$
23. $3x+4y+15=0$ and $3x+4y-15=0$
24. $3x+4y+25=0$ and $3x+4y-25=0$
25. (i) $3x+4y\pm25=0$ (ii) $5x\pm12y=65$

26. $4x+3y\pm 25=0$

27. $4x+3y+19=0$ and $4x+3y-31=0$

28. $x+2y+11=0$ and $x+2y-9=0$

29. $5x^2+5y^2-10x+30y+49=0$ 31. (i) 4 ; (ii) 3 ; (iii) $\sqrt{\frac{22}{3}}$

33. $x^2+y^2-x-y=0$ 34. $(2, 2\sqrt{3})$

5. $\left(\dfrac{\sqrt{2}+1}{\sqrt{2}}, \dfrac{\sqrt{2}+1}{\sqrt{2}}\right)$ 36. $x^2+y^2=\dfrac{1}{a^2}$ 37. $\sqrt{3}y=x\pm 10$

38. $x+2y=0$, $2x+y=0$ 39. $c=2(1\pm\sqrt{1+m^2})$.

40. $5x+y+2=0$; $2x-10y+21=0$ 41. $x-2y+7=0$; 4.

Exercise 20

1. (i) 14 ; $(\frac{7}{2}, 0)$; (ii) 5 ; $(\frac{5}{4}, 0)$; (iii) $\frac{7}{2}$; $(\frac{7}{8}, 0)$;
 (iv) 2 ; $(-\frac{1}{2}, 0)$; (v) 4 ; $(0, 1)$; (vi) 8 ; $(0, -2)$
 (vii) $\frac{7}{2}$; $(0, -\frac{7}{8})$; (viii) a, $\left(\dfrac{a^2-4b}{4a}, 0\right)$

2. (a) $(0, 0)$, $y+2=0$; (b) $(2, -5\frac{1}{2})$, $y+6=0$

3. $\frac{1}{2}$; $(\frac{1}{8}, 0)$; 5. $4x+13=0$; $(-\frac{13}{4}, \frac{13}{2})$ and $(-\frac{13}{4}, -\frac{13}{2})$

6. $\left(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a}\right)$; $\left(-\dfrac{b}{2a}, -\dfrac{b^2-4ac-1}{4a}\right)$; $\dfrac{1}{a}$

7. $\frac{7}{5}$, $(\frac{7}{20}, 0)$ 8. $\frac{4}{3}$, $(\frac{1}{3}, 0)$

9. $\frac{4}{3}$, $(\frac{1}{3}, 0)$, the two points of intersection are $(0, 0)$ and $(3, 2)$

10. $(1, 2)$, $(0, 2)$, 4 11. $(-2, -3)$, $(-\frac{3}{2}, -3)$, $2x+5=0$

12. Intersection $(1\frac{1}{5}, 1\frac{1}{5})$, focus $(\frac{3}{10}, 0)$

13. (a) $9x^2+4y^2-12xy-28x+62y+29=0$;
 (b) $16x^2+9y^2+24xy-80x-110y+225=0$
 (c) $x^2=4y$ (d) $(y-2)^2=12(x+2)$;
 (e) $x^2+2xy+y^2-16x+16y-64=0$;
 (f) $x^2-2xy+y^2-20x-20y=0$,
 $x^2-2xy+y^2+20x+20y+200=0$

14. (a) $(y-2)^2=8(x+3)$; (b) $(x-1)^2=8(y+1)$

15. $(x+2y)^2-42x+26y+56=0$

16. $x^2=12y$ 17. $\left(\frac{b^2-c}{2a}, -b\right)$ 18. $y=2x^2+3x+4$; $\frac{1}{2}$

19. $x^2=12y$ 21. $x^2-4xy+4y^2+4x+2y-1=0$; $\frac{2\sqrt{5}}{5}$

22. $x^2+y^2=4$ 23. $\frac{5}{8}$ths of the latus rectum.

Exercise 21

1. (i) (1, 3) ; (4, 6) (ii) (1, 2) ; $(\frac{1}{9}, -\frac{2}{3})$
2. (i) $(\frac{1}{2}, 2)$ (ii) $(\frac{3}{16}, \frac{3}{4})$, (iii) (3, 2) 3. $(\frac{3}{4}, 3)$
4. $al^2=mn$ 5. (4, 2), (9, 3)
6. (i) $x+y=3$, $x-y=9$; (ii) $x\pm y+2=0$; $x\pm y=6$;
(iii) $y=2x+8$; $x+2y=6$; (iv) $4y-x=24$; $4x+y=108$
(v) $y=x$, $x+y=4a$; $y+x=0$, $x-y=4a$
7. (a) $2x-y+12=0$ (b) $x+3y=33$
8. $x+y=3a$, $x-y=3a$ 9. $2x-4y=9$; $(\frac{1}{2}, -2)$
11. $4y=x+28$; (28, 14) 12. $9x+6y+8=0$; $(\frac{8}{9}, -\frac{8}{3})$
13. $x-3y+18=0$, $9x+3y+2=0$, 90° 15. $5\sqrt{5}$ 16. 12
18. $a=3$ 19. 6 20. $(am^2, -2am)$ 21. $(6, -4\sqrt{3})$
22. $2x+y+1=0$, $(\frac{1}{2}, -2)$ and $2y-x-8=0$, (8, 8)
23. $(3a, \pm 2\sqrt{3}a)$ 24. $2x+3y+36=0$ 25. $y=\pm(x+2a)$
26. $(3a, 2\sqrt{3}a)$ 32. $y=1$
33. $y=3x-7$ 34. $4x+3y+1=0$ 37. $8y^2=5x+24y+10$
38. $9x-4y+4=0$; $x-4y+36=0$
41. (i) $y=2x-9$; (3, -3) ; $1\frac{1}{2}$
(ii) $x-\sqrt{2}y+6=0$; 12.

Exercise 22

1. (i) $\frac{\sqrt{7}}{4}$; $(\pm\sqrt{7}, 0)$, $\frac{9}{2}$; (ii) $\frac{1}{2}$, $(\pm 1, 0)$, 3;

(iii) $\frac{3}{5}$, $(0, \pm 3)$, $6\frac{2}{3}$

(iv) $\frac{1}{\sqrt{3}}$, $\left(\pm\frac{2}{\sqrt{3}}, 0\right)$, $\frac{8}{3}$; (v) $\frac{1}{\sqrt{3}}$, $\left(\pm\frac{1}{\sqrt{6}}, 0\right)$, $\frac{2\sqrt{2}}{3}$

2. (a) $\left(\pm\frac{\sqrt{3}}{2}, 0\right)$, $x \pm 2\sqrt{3} = 0$;

(b) $\left(1 \pm \frac{\sqrt{5}}{6}, 2\right)$, $x - 1 \pm \frac{3\sqrt{5}}{10} = 0$

3. Major axis $= 8$, minor axis $= 4\sqrt{3}$, $e = \frac{1}{2}$, foci $= (-2, 0)$, $(2, 0)$

4. $\frac{10}{3}$; $\frac{2}{3}$; $(0, 5)$, $(0, 1)$

5. $(3, 2)$; $\frac{\sqrt{5}}{3}$; $(3 - \sqrt{5}, 2)$, $(3 + \sqrt{5}, 2)$

$x = 3 - \frac{9}{\sqrt{5}}$, $x = 3 + \frac{9}{\sqrt{5}}$ 6. $\frac{1}{\sqrt{2}}$

7. $e = \frac{1}{\sqrt{2}}$, co-ordinates $(\pm 1, 0)$ 8. $\frac{1}{2}$, $(\pm 2, 0)$

9. (i) $9x^2 + 16y^2 = 144$. (ii) $x^2 + 2y^2 = 9$.

(iii) $16x^2 + 24y^2 = 81$ (iv) $2x^2 + 3y^2 = 35$

10. (i) $14x^2 + 11y^2 - 4xy - 58x - 26y + 74 = 0$

(ii) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

11. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ 12. $3x^2 + 5y^2 = 32$

13. $3x^2 + 5y^2 = 32$, $\sqrt{\frac{2}{5}}$ 14. $\frac{3x^2}{32} + \frac{5y^2}{32} = 1$, $\sqrt{\frac{2}{5}}$

15. $5x^2 + 5y^2 - 76x - 88y - 2xy + 506 = 0$ 16. $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

17. Major axis $= 30$ in., minor axis $= 24$ in. 18. $\left(-\frac{16}{5}, \frac{9}{5}\right)$

19. $\left(-\frac{2\sqrt{21}}{21}, \frac{\sqrt{21}}{14}\right)$ 21. $\left(-\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \frac{b^2}{\sqrt{a^2 m^2 + b^2}}\right)$

22. $x+3y=5$; $9x-3y=5$

23. $25x+6y=137$; $6x-25y+20=0$ 24. $\pm\sqrt{7}\,x\pm4y=16$

25. ±1, $(-1, 2)$, $(1, 2)$ 26. $y=3x\pm\frac{1}{2}\sqrt{\frac{155}{3}}$

27. $x-3y+2=0$, $x-3y=2$; $(-1, \frac{1}{3})$; $(1, -\frac{1}{3})$

28. $\frac{x^2}{49}+\frac{y^2}{25}=1$, $\frac{2}{7}\sqrt{6}$, $(\pm2\sqrt{6}, 0)$ 32. (i) $6\frac{1}{4}$ (ii) $x+y=5$

37. $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

Exercise 23

1. (i) On (ii) within (3) outside.
2. (a) $3x-4y=12$ (b $(2, -1)$
3. (a) $(-1, 3)$ and $(\frac{8}{3}, -\frac{2}{3})$ (b) $(\frac{3}{2}, 1)$
4. (a) $\sqrt{2}$ units (b) $7\frac{33}{41}$ units ; $(\frac{75}{41}, \frac{48}{41})$
5. $8x-9y=25$
6. $4x-5y+25=0$, $(-4, \frac{9}{5})$; $x-4y-13=0$, $(\frac{25}{13}, -\frac{36}{13})$
7. $(\pm4, \frac{9}{5})$ 8. $(3, -2)$ 9. $x+10y=0$
10. $9x-10y=19$ 11. $9x-20y=0$ 12. $5x-4y=0$
14. $72x+15y=149$ 11. (i) $4x^2+5y^2-4x-5y=0$
(ii) $4x^2+5y^2-16x+25y=0$
18. $x-3y=0$, $2x+y=0$ and $2x-y=0$, $x+3y=0$
19. There can be no such pair of conjugate diameters.

Exercise 24

1. (i) $9x^2-16y^2=36$ (ii) $25x^2-144y^2=900$

(iii) $65x^2-36y^2=441$ (iv) $x^2-y^2=32$

2. (a) $\dfrac{\sqrt{13}}{3}$; $2\frac{2}{3}$; $(\pm\sqrt{13}, 0)$ (b) $e=\frac{13}{12}$; $(13, 0)$, $(-13, 0)$

3. (i) $(\pm 1, 0)$, $x\pm\frac{1}{4}=0$, 2

(ii) $\left(\pm\dfrac{\sqrt{21}}{2}-3,\ 2\right)$, $x\pm\dfrac{2\sqrt{3}}{\sqrt{7}}+3=0$, $\dfrac{\sqrt{7}}{2}$

4. (i) $x+\sqrt{2}y+2=0$ (ii) $5x+3\sqrt{5}y=27$

5. (i) $24y-30x\pm\sqrt{161}=0$ (ii) $x+2y\pm2\sqrt{2}=0$

6. (a) $c=\pm\sqrt{a^2m^2-b^2}$ (b) $a^2l^2-b^2m^2=n^2$

7. (a) $\left(-\frac{8}{3}, -\frac{7}{3}\right)$ (b) $\left(\dfrac{a^2m}{\sqrt{a^2m^2-b^2}}, \dfrac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

8. $(\pm3\sqrt{2}, 0)$; $1\frac{4}{5}$

10. $4x^2+31y^2+36xy+8x+66y-41=0$

11. $144x^2-25y^2=900$

12. $(5, -\frac{20}{3})$ and $(5, -\frac{20}{3})$ 13. $(-\frac{9}{32}, -\frac{5}{16})$

14. (a, b) and $\left\{\dfrac{a(a+3b)}{b-a}, \dfrac{b(b+3a)}{a-b}\right\}$ 15. $x-2y+1=0$

16. $\dfrac{2ab}{a^2m^2-b^2}\sqrt{c^2-a^2m^2+b^2}.\sqrt{1+m^2}$

17. (a) $\frac{15}{2}\sqrt{2}$ (b) $\dfrac{2ab\sqrt{1+m^2}}{\sqrt{b^2-a^2m^2}}$ 19. $y=\dfrac{b^2}{a^2m}x$

20. $9y=32x$ 23. $(2, 1)$ 24. (a) $x^2+y^2=a^2-b^2$(director circle) (b) $x^2+y^2=a^2$ (auxiliary circle)

25. (i) $3x\pm2y=0$ (ii) $ay\pm bx=0$.

Exercise 25

1. $ay=c^2$ 2. $\frac{\alpha}{x}+\frac{\beta}{y}=1$ 4. $x+y=c$

5. $7x-4y+16=0$, $8x+14y+9=0$ 6. $x+y=4$

7. $3(x^2+y^2)-10ax+3a^2=0$

8. Centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$; radius$=\frac{a}{2}$

9. $2(x^2+y^2)-2cx+c^2-a^2=0$, it is a circle.

10. $3x^2+3y^2-16x-24y+52=0$ 11. $(x-3)^2=12(y-3)$

12. $3x^2+4y^2-16x-16y+32=0$

15. $\frac{x^2}{25}+\frac{y^2}{16}=1$ 18. $y^2-4ax=(a+x)^2$

20. $(x^2+y^2-a^2-b^2)^2=4\cot^2\theta(b^2x^2+a^2y^2-a^2b^2)$

21. $(x^2+y^2)^2=a^2x^2+b^2y^2$

22. $x^2+y^2-2ay\cot\theta-a^2=0$ [Taking base$=2a$ and vertical angle$=\theta$]

24. $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$ 25. $x^2-y^2=a^2-b^2$.

APPENDIX

Parametric Representation

1. In cartesian co-ordinates, any point on a locus is expressed by two separate quantities x_1 and y_1 which are however connected by the following relations,

(i) $ax_1+by_1+c=0$ (straight line)

(ii) $x_1^2+y_1^2=a^2$ (circle)

(iii) $y_1^2=4ax_1$ (parabola)

(iv) $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}=1$ (ellipse)

(v) $\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}=1$ (hyperbola)

But often the point on a locus is conveniently expressed by a single variable which is called a parameter.

2. Parametric Equations.

(*a*) Circle.

Let P (x, y) be any point on the circle $x^2+y^2=a^2$ [Fig. (i)] Join OP and draw PN perpendicular to OX.

If the angle PON be θ, then.

$$x=\text{ON}=\text{OP}\cos\theta=a\cos\theta,$$

$$y=\text{PN}=\text{OP}\sin\theta=a\sin\theta.$$

Thus the co-ordinates of any point on the circle are expressed by the single variable θ, which is known as a parameter.

$\therefore$ $\boldsymbol{x=a\cos\theta}$ and $\boldsymbol{y=a\sin\theta}$ are called the parametric equations of the circle.

From the above two equations when θ is eliminated, the equation to the circle is obtained. The point $(a \cos \theta, a \sin \theta)$ which satisfies the equation $x^2+y^2=a^2$ is often called 'the point θ' on the circle.

When the equation to the circle is $(x-\alpha)^2+(y-\beta)^2=a^2$

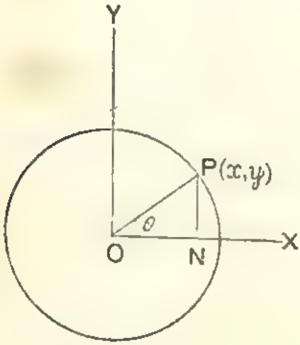


Fig. (i)

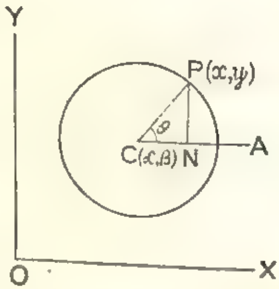


Fig. (ii)

[Fig (ii)], then the point $(\alpha+a \cos \theta, \beta+a \sin \theta)$ evidently satisfies the equation to the above circle and is called 'the point θ' on the circle.

$\therefore$ The parametric equations to the circle in this case are

$$\boldsymbol{x=\alpha+a\cos\theta} \text{ and } \boldsymbol{y=\beta+a\sin\theta}$$

(*b*) **Parabola.**

The equation $y^2=4ax$ is satisfied by $x=at^2$ and $y=2at$ for all values of t. Thus the co-ordinates of any point on the parabola may be expressed by the single variable t and the point $(at^2, 2at)$ is called 'the point t' on the curve.

$\therefore$ The parametric equations to the parabola are $\boldsymbol{x=at^2}$ and $\boldsymbol{y=2at}$.

(c) **Ellipse.**

Auxiliary circle.

Def. The circle which is described on the major axis AA′ of an ellipse as diameter, is called the auxiliary circle of the ellipse.

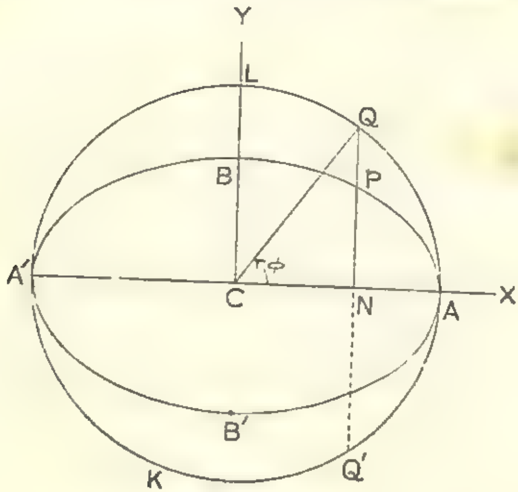


Fig. (iii)

In fig (iii) AKA′L is the auxiliary circle of the ellipse AB′A′B.

Let an ordinate PN from any point P on the ellipse cut the auxiliary circle at Q and Q′ (Fig. (iii). Join CQ.

Now the chord QQ′ of the circle is bisected at N by the diameter AA′, $\therefore \quad QN^2 = AN.NA' \ldots\ldots(1)$

$\because$ CN and PN are the co-ordinates of the point P on the ellipse, we get from the equation of the ellipse,

$$\frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1 \text{ or } \frac{PN^2}{b^2}=1-\frac{CN^2}{a^2}=\frac{a^2-CN^2}{a^2}=\frac{(a+CN)(a-CN)}{a^2}$$

$$\therefore \quad \frac{PN^2}{b^2}=\frac{AN.NA'}{a^2}=\frac{QN^2}{a^2} \quad [\text{ from (1) }]$$

$$\therefore \quad \frac{PN^2}{QN^2}=\frac{b^2}{a^2}, \quad \text{or,} \quad \frac{PN}{QN}=\frac{b}{a}\ldots\ldots(2)$$

Eccentric Angle.

Def. The eccentric angle of any point P on the ellipse is the angle NCQ (Fig. iii) made with the major axis by the straight line CQ joining the centre C to the point Q on the auxiliary circle which corresponds to the point P.

This angle is generally denoted by ϕ.

From Fig. (iii) we have

$$CN = CQ \cos\phi = a\cos\phi \text{ and } QN = CQ \sin\phi = a \sin\phi.$$

From (2) we get $PN = \frac{b}{a} \times QN = \frac{b}{a} \times a \sin\phi = b \sin\phi$.

$\therefore$ The co-ordinates of any point P on the ellipse are

$$(a\cos\phi,\ b\sin\phi)\cdots\cdots(3)$$

This point on the ellipse is often called 'the point ϕ'.

The co-ordinates of the point Q on the auxiliary circle, corresponding to the point P on the ellipse, are

$$(a\cos\phi,\ a\sin\phi)\cdots\cdots(A)$$

Thus we see from the result (3) that any point on the ellipse may be expressed by the single variable ϕ.

$\therefore$ The parametric equations to the ellipse are

$$\mathbf{x = a\cos\phi \text{ and } y = b\sin\phi.}$$

(*d*) **Hyperbola.**

In the case of the hyperbola any ordinate of the curve does not meet the auxiliary circle, *i.e.*, the circle on AA′ as diameter, in real points. Therefore, there is no real eccentric angle as in the case of the ellipse.

The angle 'ϕ' can be easily defined in the following manner.

In Fig. (iv), a circle has been drawn on the transverse axis AA′ of the hyperbola as diameter.

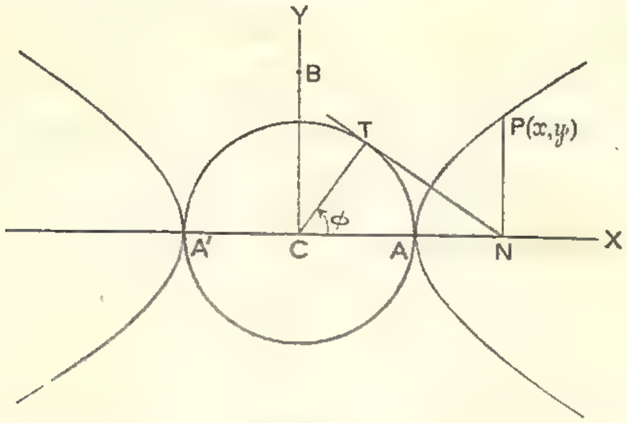


Fig. (iv)

From N, the foot of any ordinate NP of the hyperbola a tangent NT is drawn to the circle and CT is joined.

The angle NCT is defined as ϕ.

Now, CT=CN $\cos\phi$. Taking co-ordinates of P as (x, y), CN$=x$, PN$=y$.

$\therefore$ CN$=x=$CT $\sec\phi=a\sec\phi$, and NT$=$CT $\tan\phi=a\tan\phi$.

Again, $\dfrac{\text{NP}^2}{b^2}=\dfrac{y^2}{b^2}=\dfrac{x^2}{a^2}=1$ $\left[\because \text{ equation of the hyperbola is } \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\right]$

$$=\frac{\text{CN}^2}{a^2}-1=\frac{a^2\sec^2\phi}{a^2}-1=\sec^2\phi-1=\tan^2\phi$$

$$\therefore \quad \frac{\text{NP}^2}{b^2}=\frac{\text{NT}^2}{a^2}, \quad i.e., \quad \text{NP}:\text{NT}=b:a.$$

Thus we see that the ordinate of the hyperbola is in a constant ratio to the length of the tangent drawn from its foot to the auxiliary circle.

Hence, the angle ϕ is a constant angle here. But this angle ϕ is not so important an angle for the hyperbola as the eccentric angle 'ϕ' is for the ellipse.

From above we find that the co-ordinates of any point on the hyperbola may be denoted by ($a \sec \phi$, $b \tan \phi$)$\cdots$(4) and this point on the hyperbola is known as 'the point ϕ'.

The result (4) shows that any point on the hyperbola may be expressed by the single variable ϕ.

$\therefore$ The parametric equations to the hyperbola are

$$\mathbf{x = a \sec \phi \quad \text{and} \quad y = b \tan \phi.}$$

Illustrations

1. Find the equation of the chord of the circle $x^2+y^2=a^2$, joining the points θ_1 and θ_2.

Hence, find the equation of the tangent to the above circle at any point.

$\because$ the co-ordinates of the points θ_1 and θ_2 are $(a \cos \theta_1, a \sin \theta_1)$ and $(a \cos \theta_2, a \sin \theta_2)$ respectively, therefore the gradient of the chord joining θ_1 and θ_2

$$=\frac{a \sin \theta_1 - a \sin \theta_2}{a \cos \theta_1 - a \cos \theta_2}$$

$$=\frac{2 \cos \frac{\theta_1+\theta_2}{2} \sin \frac{\theta_1-\theta_2}{2}}{-2 \sin \frac{\theta_1+\theta_2}{2} \sin \frac{\theta_1-\theta_2}{2}} = -\cot \frac{\theta_1+\theta_2}{2}.$$

$\therefore$ The equation to the chord is

$$y - a \sin \theta_1 = -\cot \frac{\theta_1+\theta_2}{2}(x - a \cos \theta_1)$$

or, $$x \cos \frac{\theta_1+\theta_2}{2} + y \sin \frac{\theta_1+\theta_2}{2}$$

$$= a\left(\cos \theta_1 \cos \frac{\theta_1+\theta_2}{2} + \sin \theta_1 \sin \frac{\theta_1+\theta_2}{2}\right)$$

or, $x \cos \dfrac{\theta_1+\theta_2}{2}+y \sin \dfrac{\theta_1+\theta_2}{2}=a \cos \dfrac{\theta_1-\theta_2}{2}$ which is the required equation of the chord.

Let the pts. θ_1 and θ_2 approach gradually to each other and ultimately coincide at the point θ (say). Then the chord at this stage becomes a tangent to the circle at the point θ.

Thus putting $\theta_1=\theta_2=\theta$ in the above equation, we get the equation of the tangent to the circle at the point θ, as

$$x \cos \theta+y \sin \theta=a.$$

2. Show that the locus of the middle points of the chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the point (α, β) is $(x-\alpha)^2+y(-\beta)^2+(x^2+y^2-a^2)=0$.

Let the co-ordinates of the extremities of the chord AB of the circle $x^2+y^2=a^2$, be respectively $(a \cos \theta_1, a \sin \theta_1)$ and $(a \cos \theta_2, a \sin \theta_2)$.

The chord AB subtends a right angle at the point P(α, β).

To find the locus of the middle point of AB.

Let the middle point of AB be C (h, k).

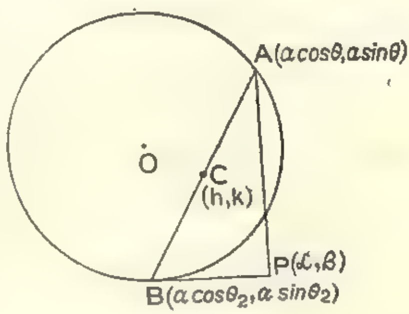


Fig. (v)

$\therefore \quad h=\frac{1}{2}a(\cos \theta_1+\cos \theta_2)$

and $k=\frac{1}{2}a(\sin \theta_1+\sin \theta_2)$

$\therefore$ Squaring and adding the results we get

$$\begin{aligned}4h^2+4k^2&=a^2\{(\cos \theta_1+\cos \theta_2)^2+(\sin \theta_1+\sin \theta_2)^2\}\\&=a^2\{2+2 \cos \theta_1 \cos \theta_2+2 \sin \theta_1 \sin \theta_2\}\\&=2a^2+2a^2 \cos(\theta_1-\theta_2)\cdots\cdots(\text{i})\end{aligned}$$

$\because \quad \angle \text{APB}=90^\circ, \quad \therefore$ Gradient of AP $\times$ Gradient of BP $=-1$

$$i.e., \quad \frac{\beta-a \sin \theta_1}{\alpha-a \cos \theta_1}\times\frac{\beta-a \sin \theta_2}{\alpha-a \cos \theta_2}=-1$$

or, $\alpha^2+\beta^2-a\beta(\sin\theta_1+\sin\theta_2)-a\alpha(\cos\theta_1+\cos\theta_2)$
$+a^2\cos(\theta_1-\theta_2)=0,$

or, $a^2\cos(\theta_1-\theta_2)=-\alpha^2-\beta^2-2\beta k+2\alpha h.$

Putting this value in (i) we get.

$$4h^2+4k^2=2a^2-2(\alpha^2+\beta^2-2\alpha h-2\beta k)$$

or, $h^2-2\alpha h+\alpha^2+k^2-2\beta k+\beta^2+h^2+k^2-a^2=0,$

or, $(h-\alpha)^2+(k-\beta)^2+h^2+k^2-a^2=0.$

$\therefore$ the locus of the middle point (h, k) is

$$(x-\alpha)^2+(y-\beta)^2+(x^2+y^2-a^2)=0.$$

3. Find, in the standard form, the equation of the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2=4ax$. [H. S. '66]

Gradient of the join of the given points $=\dfrac{2at_1-2at_2}{at_1^2-at_2^2}=\dfrac{2}{t_1+t_2}.$

$\therefore$ The equation of the chord is $y-2at_1=\dfrac{2}{t_1+t_2}(x-at_1^2)$

i.e., $y(t_1+t_2)-2x-2at_1t_2=0.$

4. Find the equation of the normal to the parabola $y^2=4ax$ at $(at^2, 2at)$, where t is a parameter.

Prove that this normal meets the centre at $(at_1^2, 2at_1)$ where $t_1=-t-\dfrac{2}{t}.$ [H.S. '66 Compl.]

The equation of the tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is

$$y.2at=2a(x+at^2),\ i.\ e.,\ y=\frac{1}{t}(x+at^2)\cdots\cdots(A)$$

$\therefore$ the equation to the normal at the same point is

$(y-2at)=-t(x-at^2)$ or, $y=-tx+2at+at^3\cdots\cdots(B)$

Since this normai passes through $(at_1^2, 2at_1)$,

$\therefore$ $2at_1=-t.at_1^2+2at+at^3,$

or, $tt_1^2 - t^3 = 2t - 2t_1$, or, $t(t_1 - t)(t_1 + t) = -2(t_1 - t)$

$\therefore \quad t(t_1 + t) = -2 \quad [\because \quad t_1 \neq t]$, i.e., $t_1 = -t - \dfrac{2}{t}$.

5. Prove that the tangents at the ends of a focal chord of a parabola meet at right angles on the directrix.

[U. P. '61 ; Roorki '41, '54, '58]

Let the equation of the parabola be $y^2 = 4ax$, and the co-ordinates of the ends of a focal chord of it be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Equation of the chord joining the above two points is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2) \quad \text{[See illustr. 3(i)]}$$

$\because$ it passes through the focus $(a, 0)$,

$\therefore \quad 0 - 2at_1 = \dfrac{2}{t_1 + t_2}(a - at_1^2)$, or, $\quad -t_1(t_1 + t_2) = 1 - t_1^2$,

$\therefore \quad t_1 t_2 = -1 \cdots\cdots(C)$

Again, equations of the tangents at the two ends are

$$\left.\begin{aligned} y &= \frac{1}{t_1}(x + at_1^2) \cdots\cdots(\text{i}) \\ \text{and } y &= \frac{1}{t_2}(x + at_2^2) \cdots\cdots(\text{ii}) \end{aligned}\right\} \quad \text{[See illus. 4(A)]}$$

$\therefore$ product of the gradients $= \dfrac{1}{t_1} \times \dfrac{1}{t_2} = \dfrac{1}{t_1 t_2} = \dfrac{1}{-1} = -1$

[See result (C)]

$\therefore$ the two tangents are perpendicular to each other.

Now subtracting (ii) from (i) we get

$$0 = \frac{x}{t_1} + at_1 - \frac{x}{t_2} - at_2$$

or, $x\left(\dfrac{1}{t_1} - \dfrac{1}{t_2}\right) + a(t_1 - t_2) = 0$, or, $x\left(\dfrac{t_2 - t_1}{t_1 t_2}\right) + a(t_1 - t_2) = 0$

or, $\quad x(t_1 - t_2) + a(t_1 - t_2) = 0 \quad [\because \quad t_1 t_2 = -1]$

or, $x+a=0$ [$\because$ $t_1 \neq t_2$]. But this is the equation of the directrix of the parabola.

Hence, the tangents at the ends of a focal chord of the parabola intersect at right angles on the directrix.

[N. B. This may be otherwise stated as—

'Prove that the locus of the point of intersection of two perpendicular tangents at the ends of a focal chord of a parabola is the directrix of the parabola'.]

6. If the point $(at_1^2, 2at_1)$ is one extremity of a focal chord of the parabola $y^2=4ax$, find the co-ordinates of the other extremity and show that the length of the chord is $a\left(t_1+\frac{1}{t_1}\right)^2$.

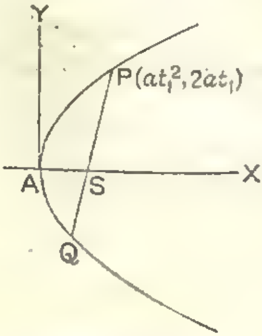


Fig (vi)

Let PQ be the focal chord of the parabola $y^2=4ax$, and the co-ordinates of P be $(at^2_1, 2at_1)$.

If the co-ordinates of Q be $(at_2^2, 2at_2)$,

then $t_1t_2=-1$ [$\because$ PQ is a focal chord, illstr. 5(C)]

$\therefore$ $t_2=-\frac{1}{t_1}$. Hence, the co-ordinates of Q

are $\left\{a\left(-\frac{1}{t_1}\right)^2, 2a\left(-\frac{1}{t_1}\right)\right\}$, *i.e.*, $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.

Again, the length PQ=PS+QS

$=(a+at_1^2)+\left(a+\frac{a}{t_1^2}\right)$ [$\because$ the focal distance of any point on a parabola $=a+x$]

$=a\left(t_1^2+2+\frac{1}{t_1^2}\right)=a\left(t_1+\frac{1}{t_1}\right)^2$.

7. Prove that in a parabola the semi-latusrectum is the harmonic mean of the segments of a focal chord.

From fig. (vi) we get, $SP = a + at_1^2 = a(1+t_1^2)$,

and $SQ = a + \dfrac{a}{t_1^2} = \dfrac{a(1+t_1^2)}{t_1^2}$.

$$\therefore \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(1+t_1^2)} + \frac{t_1^2}{a(1+t_1^2)} = \frac{1+t_1^2}{a(1+t_1^2)} = \frac{1}{a} = \frac{2}{2a}.$$

Hence, $\dfrac{1}{SP}$, $\dfrac{1}{2a}$ and $\dfrac{1}{SQ}$ are in A. P.

$\therefore$ SP, $2a$ and SQ are in H.P. But semi-latus-rectum $= 2a$,

$\therefore$ the semi-latus-rectum is the Harmonic mean between the segments of a focal chord.

8. Show that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points. [Hooghly Collegiate school test (67-68)]

Let the three points on the parabola $y^2 = 4ax$ be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.

Area of the Δ formed by joining these three points

$$= \tfrac{1}{2}\{at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_3^2(2at_1 - 2at_2)\}$$

$$= \tfrac{1}{2} \times 2a^2\{t_1^2(t_2 - t_3) + t_2^2(t_3 - t_1) + t_3^2(t_1 - t_2)\}$$

$$= -a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1).$$

The equations of the tangents at the above three points on the parabola are

$$y = \frac{1}{t_1}(x + at_1^2) \cdots \text{(i)}, \quad y = \frac{1}{t_2}(x + at_2^2) \cdots \text{(ii)}, \quad y = \frac{1}{t_3}(x + at_3^2) \cdots \text{(iii)}$$

Solving these three equations we get three points of intersection of the three tangents as

$$\{at_1t_2,\ a(t_1+t_2)\}\ ;\ \{at_2t_3,\ a(t_2+t_3)\} \text{ and } \{at_3t_1,\ a(t_3+t_1)\}.$$

$\therefore$ the area of the Δ formed by these three points

$$=\tfrac{1}{2}[at_1t_2(at_2-at_1)+at_2t_3(at_3-at_2)+at_3t_1(at_1-at_3)]$$
$$=\tfrac{1}{2}a^2[-\{t_1t_2(t_1-t_2)+t_2t_3(t_2-t_3)+t_3t_1(t_3-t_1)\}]$$
$$=\tfrac{1}{2}a^2[-\{-(t_1-t_2)(t_2-t_3)(t_3-t_1)\}]$$
$$=\tfrac{1}{2}a^2(t_1-t_2)(t_2-t_3)(t_3-t_1).$$

Hence, the area of the former triangle is double that of the latter.

9. Find the equation of the chord joining the pts. $(a\sec\phi_1, b\tan\phi_1)$ and $(a\sec\phi_2, b\tan\phi_2)$ on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$

Hence, deduce the equations of the tangent and the normal at the point 'ϕ' on the hyperbola.

Gradient of the line joining the two given points on the hyperbola

$$=\frac{b(\tan\phi_1-\tan\phi_2)}{a(\sec\phi_1-\sec\phi_2)}=\frac{b}{a}\left[\frac{\dfrac{\sin\phi_1}{\cos\phi_1}-\dfrac{\sin\phi_2}{\cos\phi_2}}{\dfrac{1}{\cos\phi_1}-\dfrac{1}{\cos\phi_2}}\right]=\frac{b}{a}\left[\frac{\sin(\phi_1-\phi_2)}{\cos\phi_2-\cos\phi_1}\right].$$

$$=\frac{b}{a}\left[\frac{2\sin\frac{1}{2}(\phi_1-\phi_2)\cos\frac{1}{2}(\phi_1-\phi_2)}{2\sin\frac{1}{2}(\phi_1-\phi_2)\sin\frac{1}{2}(\phi_1-\phi_2)}\right]=\frac{b}{a}\left(\frac{\cos\dfrac{\phi_1-\phi_2}{2}}{\sin\dfrac{\phi_1+\phi_2}{2}}\right).$$

$\therefore$ The equation to chord joining the points is

$$y-b\tan\phi_1=\frac{b}{a}\left(\frac{\cos\dfrac{\phi_1-\phi_2}{2}}{\sin\dfrac{\phi_1+\phi_2}{2}}\right)(x-a\sec\phi_1),$$

or, $$\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}-\tan\phi_1\sin\frac{\phi_1+\phi_2}{2}$$
$$=\frac{x}{a}\cos\frac{\phi_1-\phi_2}{2}-\sec\phi_1\cos\frac{\phi_1-\phi_2}{2}$$

or, $\dfrac{x}{a}\cos\dfrac{\phi_1-\phi_2}{2}-\dfrac{y}{b}\sin\dfrac{\phi_1+\phi_2}{2}$

$$=\frac{1}{\cos\phi_1}\cos\frac{\phi_1-\phi_2}{2}-\frac{\sin\phi_1}{\cos\phi_1}\sin\frac{\phi_1+\phi_2}{2}$$

$$=\frac{1}{\cos\phi_1}\left[\cos\frac{\phi_1-\phi_2}{2}-\sin\phi_1\sin\frac{\phi_1+\phi_2}{2}\right]$$

$$=\frac{1}{\cos\phi_1}\left[\cos\frac{\phi_1-\phi_2}{2}-\frac{1}{2}\left\{2\sin\phi_1\sin\frac{\phi_1+\phi_2}{2}\right\}\right]$$

$$=\frac{1}{\cos\phi_1}\left[\cos\frac{\phi_1-\phi_2}{2}-\frac{1}{2}\left\{\cos\left(\phi_1-\frac{\phi_1+\phi_2}{2}\right)\right.\right.$$

$$\left.\left.-\cos\left(\phi_1+\frac{\phi_1+\phi_2}{2}\right)\right\}\right].$$

$$=\frac{1}{\cos\phi_1}\left[\cos\frac{\phi_1-\phi_2}{2}-\frac{1}{2}\cos\frac{\phi_1-\phi_2}{2}+\frac{1}{2}\cos\frac{3\phi_1+\phi_2}{2}\right]$$

$$=\frac{1}{2\cos\phi_1}\left[\cos\frac{\phi_1-\phi_2}{2}+\cos\frac{3\phi_1+\phi_2}{2}\right]$$

$$=\frac{1}{2\cos\phi_1}\left[2\cos\phi_1\cos\frac{\phi_1+\phi_2}{2}\right]=\cos\frac{1}{2}(\phi_1+\phi_2)$$

Hence, the required equation is

$$\frac{x}{a}\cos\frac{1}{2}(\phi_1-\phi_2)-\frac{y}{b}\sin\frac{1}{2}(\phi_1+\phi_2)=\cos\frac{1}{2}(\phi_1+\phi_2).$$

Let the pts. ϕ_1 and ϕ_2 approach gradually to each other and ultimately coincide at the point ϕ say. At this stage the chord becomes a tangent to the hyperbola at the point ϕ.

Thus putting $\phi_1=\phi_2=\phi$ in the above equation, we get the required equation of the tangent,

$$\frac{x}{a}-\frac{y}{b}\sin\phi=\cos\phi\cdots\cdots(\text{A})$$

Again, gradient of this tangent $=\dfrac{b}{a\sin\phi}$.

$\therefore$ the equation of the normal at the pt. ϕ is

$$y-b\tan\phi=-\frac{a\sin\phi}{b}(x-a\sec\phi)$$

or, $\dfrac{ax}{\sec\phi}+\dfrac{by}{\tan\phi}=a^2+b^2\cdots\cdots(\text{B})$

10. Any ordinate NP of an ellipse meets the auxiliary circle in Q; prove that the locus of the intersection of the normals at P and Q is the circle $x^2+y^2=(a+b)^2$.

[Hooghly Collegiate School Test 1965-66].

Let the co-ordinates of P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be $(a\cos\phi, b\sin\phi)$. Then the co-ordinates of Q, the corresponding point on its auxiliary circle will be $(a\cos\phi, a\sin\phi)$

[See Art. 2(C) result (A).].

Now, the equation of the normal at P is

$$\frac{x-a\cos\phi}{\frac{a\cos\phi}{a^2}}=\frac{y-b\sin\phi}{\frac{b\sin\phi}{b^2}}, \text{ or, } \frac{ax}{\cos\phi}-a^2=\frac{by}{\sin\phi}-b^2$$

or, $ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2\cdots\cdots$(i)

Since, the pt. Q lies on a circle, therefore the normal at the pt. Q $(a\cos\phi, a\sin\phi)$ will pass through the centre of the circle which in this case is (0, 0).

$\therefore$ the equation of the normal at Q is

$$y=x\tan\phi\cdots\cdots\text{(ii)}$$

Let the co-ordinates of the pt. of intersection of (i) and (ii) be (h, k). $\therefore$ we get

$k=h\tan\phi\cdots$(iii) and $ah\sec\phi-bk\operatorname{cosec}\phi=a^2-b^2\cdots$(iv).

From (iii) we have

$$1+\tan^2\phi=1+\frac{k^2}{h^2}=\frac{h^2+k^2}{h^2}, \quad \therefore \quad \sec\phi=\frac{\sqrt{h^2+k^2}}{h}.$$

Similarly, $\operatorname{cosec}\phi=\frac{\sqrt{h^2+k^2}}{k}$.

Now putting these two values in (iv) we get

$a\sqrt{h^2+k^2}-b\sqrt{h^2+k^2}=a^2-b^2$, or, $\sqrt{h^2+k^2}=(a+b)$

Hence, the locus of the pt. of intersection (h, k) of the normals is $x^2+y^2=(a+b)^2$.

11. If the line $\frac{lx}{a}+\frac{my}{b}=n$ cuts the ellipse $b^2x^2+a^2y^2=a^2b^2$ at the ends of conjugate diameters, prove that $l^2+m^2=2n^2$. (C. U. '49)

Equation of the ellipse is $b^2x^2+a^2y^2=a^2b^2$, or, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P$(a\cos\theta, b\sin\theta)$ be one end of a diameter of the ellipse. Then the co-ordinates of Q, one end of the conjugate diameter, will be $(-a\sin\theta, b\cos\theta)$.

$\therefore$ the equation of PQ is $\frac{y-b\sin\theta}{b\sin\theta-b\cos\theta}=\frac{x-a\cos\theta}{a\cos\theta+a\sin\theta}$,

or, $\frac{x}{a(\cos\theta+\sin\theta)}+\frac{y}{b(\cos\theta-\sin\theta)}$

$$=\frac{\cos\theta}{\cos\theta+\sin\theta}+\frac{\sin\theta}{\cos\theta-\sin\theta}=\frac{1}{\cos^2\theta-\sin^2\theta}\cdots\text{(i)}$$

But the equation $\frac{l}{a}x+\frac{m}{b}y=n$ (ii) passes through P and Q.

$\therefore$ (i) and (ii) are identical.

$$\therefore\quad \frac{\frac{l}{a}}{\frac{1}{a(\cos\theta+\sin\theta)}}=\frac{\frac{m}{b}}{\frac{1}{b(\cos\theta-\sin\theta)}}=\frac{n}{\frac{1}{(\cos^2\theta-\sin^2\theta)}}$$

or, $l(\cos\theta+\sin\theta)=m(\cos\theta-\sin\theta)=n(\cos^2\theta-\sin^2\theta)$.

$\therefore$ $l=n(\cos\theta-\sin\theta)$ and $m=n(\cos\theta+\sin\theta)$.

Hence, $l^2+m^2=n^2\{(\cos\theta-\sin\theta)^2+(\cos\theta+\sin\theta)^2\}=2n^2$.

Exercise 26

[Use parametric forms in the following cases.]

1. Find the equation of the chord of the circle $(x-\alpha)^2+(y-\beta)^2=a^2$, joining the points θ_1 and θ_2.

Hence, deduce the equation of the tangent to the above circle at any point.

2. (*a*) A chord of the circle $x^2+y^2=a^2$ always subtends a right angle at the centre, find the locus of its middle point.

(*b*) Find the locus of the middle points of the chords of the circle $x^2+y^2=a^2$, which subtend a right angle at the point $(k,0)$.

3. (*a*) Prove that the tangent at any point 't' on a parabola bisects the angle between the focal distance of the point and the perpendicular from that point on the directrix.

(*b*) Prove that the portion of a tangent at 't' to a parabola cut off between the directrix and the point of contact subtends a right angle at the focus. [U. P. '47]

4. Prove that the locus of the feet of perpendiculars drawn from the focus to tangents to the parabola $y^2=4ax$ is a tangent at the vertex. [U. P. '46]

5. From any point 't' on the parabola $y^2=4ax$ perpendicular PN is drawn on the axis meeting it at N. Normal at P meets the axis in G. Prove that NG=semi-latusrectum. [U. P. '53]

6. (*a*) Find the point of intersection of the normals at points t_1 and t_2 on the parabola $y^2=4ax$.

(*b*) Find the co-ordinates of the point of intersection of two mutually perpendicular normals at the points t_1 and t_2 on the parabola $y^2=4ax$ and show that the abscissa of this point can never be smaller than $3a$. What is the ordinate when the abscissa is the smallest ? [Raj. '41]

7. The normal to the parabola $y^2=4ax$ at any point $P(at_1^2, 2at_1)$ meets the curve again at $Q\,(at_2^2, 2at_2)$.

If the lines joining the origin to P and Q be mutually perpendicular, then prove that $t_1^2=2$. [Raj. '56]

[*Hints* : Establish first, $t_2=-t_1-\dfrac{2}{t_1}$ then proceed].

8. Prove that any two perpendicular tangents to a parabola intersect on the directrix. [Raj. '52, 55]

9. The normal to the parabola $y^2=4ax$ at $(at_1^2, 2at_1)$ meets it again at $(at_2^2, 2at_2)$, then prove that $t_1^2+t_1t_2+2=0$.

10. XY is a focal chord of a parabola and the line through Y parallel to the axis of the parabola meets the directrix at K. Prove the XK goes through the vertex. [*H. S. Tech.* '66]

11. Prove that the locus of the middle point of the portion of the normal to the parabola $y^2=4ax$ intercepted between the curve and the axis is another parabola. Find its vertex and the length of the latus rectum.

12. Find the area of the triangle formed by the normals to the parabola $y^2=4ax$ at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.

13. Find the equation to the chord joining the points ϕ_1 and ϕ_2 on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Hence, deduce the equations of the tangent and the normal at the point ϕ.

14. Find the points on the ellipse such that the tangents at each of them make equal angles with the axes.

Prove also that the length of the perpendicular from the centre on either of these tangents is $\sqrt{\dfrac{a^2+b^2}{2}}$.

15. If p be the perpendicular from the centre of an ellipse upon the tangent at any point P on it, and d be the distance of P from the centre, show that $a^2+b^2-d^2=\dfrac{a^2b^2}{p^2}$.

16. The perpendicular from the centre upon the tangent at any point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ meets it at the point Q. Show that the locus of Q is given by the equation $(x^2+y^2)^2=a^2x^2-b^2y^2$.

ANSWERS

Exercise 26

1. $(x-\alpha)\cos\frac{\theta_1+\theta_2}{2}+(y-\beta)\sin\frac{\theta_1+\theta_2}{2}=a\cos\frac{\theta_1-\theta_2}{2}$;

$(x-\alpha)\cos\theta+(y-\beta)\sin\theta=a$ [Taking the pt. of coincidence to be θ].

2. (a) $x^2+y^2=\frac{1}{2}a^2$ (b) $2(x^2+y^2)-2Kx+K^2-a^2=0$.

6. (a) $\{2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2)\}$.

(b) $\{2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2)\}$; ordinate is zero.

11. locus is $y^2=a(x-a)$; vertex $(a, 0)$ and latusrectum$=a$

12. $\frac{a^2}{2}(t_1-t_2)(t_2-t_3)(t_3-t_1)(t_1+t_2+t_3)^2$.

13. $\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$.

Tangent : $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Normal : $ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2$.

14. $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

Question Papers

HIGHER SECONDARY EXAMINATION, 1960

ELECTIVE MATHEMATICS

FIRST PAPER

Group A—Algebra [60 Marks]

Answer **any four** *questions. All questions carry equal marks.*

1. (*a*) If $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$, $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$,

find the value of $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$. [Ans. $\frac{15}{13}$]

(*b*) Simplify : $\left[\sqrt[3]{4}\times\dfrac{1}{\sqrt[9]{8}}\times\sqrt[12]{16^{-1}}\right]^{\frac{1}{2}}$ [Ans. 1]

(*c*) Find the square root of $28-6\sqrt{3}$.

[Ans. $\pm(3\sqrt{3}-1)$]

2. Solve the equations :

(*a*) $\left.\begin{aligned}2x^2+3xy+y^2&=15\\ 5x+2y&=12\end{aligned}\right\}$ [Ans. $x=2$, $y=1$ or, $x=14$, $y=-29$]

(*b*) $\left.\begin{aligned}3x+4y&=5xy\\ 2y+3z&=2yz\\ 5z+2x&=6zx\end{aligned}\right\}$ [Ans. $x=y=z=0$; or, $x=1, y=3, z=2$]

3. (*a*) A class consists of a number of boys whose ages are in Arithmetical Progression, the common difference being 3 months. If the youngest boy is just seven years old and the sum of the ages of the boys is 153 years, find the number of boys in the class. [Ans. 17]

(*b*) If S_1, S_2, S_3 denote respectively the sum of the first n terms, first $2n$ terms and first $3n$ terms of a series in Geometrical Progression, prove that $S_1(S_3-S_2)=(S_2-S_1)^2$.

4. (*a*) Find the cube roots of unity. If ω be an imaginary cube root of unity, prove that $1+\omega+\omega^2=0$.

$\left[\text{Ans. } 1,\ \dfrac{-1\pm\sqrt{-3}}{2}\right]$

(*b*) The area of a circle varies as the square of its radius. If the area is $38\frac{1}{2}$ sq. ft. when the radius is 3 ft. 6 in., find the area when the radius is 4 ft. 8 in. [Ans. $68\frac{4}{9}$ sq. ft.]

5. (*a*) If x be real, prove that the value of the expression $\frac{x^2+x+2}{x^2+2x+4}$ must lie between $\frac{1}{2}$ and $\frac{7}{6}$.

(*b*) If α and β are the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

[Ans. $qx^2-p(q+1)x+(q+1)^2=0$.]

6. (*a*) Find the value of the term independent of x, in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$. [Ans. 495]

(*b*) Apply the Binomial Theorem to find the value of $(\cdot 999)^3$ to 5 places of decimals. [Ans. $\cdot 997002$]

7. (*a*) Simplify :

$$\log_{10}\frac{384}{5}+\log_{10}\frac{81}{32}+3\log_{10}\frac{5}{3}+\log_{10}\frac{1}{9}.$$ [Ans. 2]

(*b*) If x, y, z are in geometrical progression, prove that $\log_{10} x$, $\log_{10} y$ and $\log_{10} z$ are in Arithmetical Progression.

8. (*a*) Find the number of permutations of n different things taken r at a time, where r is less than or equal to n.

(*b*) How many numbers lying between 3,000 and 4,000 can be formed with the digits 1, 2, 3, 4, 5 and 6 ? [Ans. 60 or 216]

Group B—Trigonometry [40 Marks]

Answer **any three** *questions. All questions carry equal marks.*

9. (*a*) Prove that the radian is a constant angle. Find its value in degrees, minutes, etc. $[\pi=\frac{22}{7}]$ [Ans. $57^\circ 17' 44'' 8$]

(*b*) The angles of a triangle are in Arithmetical Progression and the number of degrees in the least is to the number of radians in the greatest as 60 to π. Find the angles in degrees.

[Ans. 30°, 60°, 90°]

10. (*a*) If A, B, $A+B$ are all acute angles, prove (geometrically) that $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

(*b*) Find the value of—

$\sin^2 60^\circ+\cos^2 150^\circ+\tan^2 120^\circ+\cos 180^\circ-\tan 135^\circ$.

[Ans. $4\frac{1}{2}$]

11. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $2\sin^2\theta+3\cos\theta=0$. [Ans. 120°, 240°]

(*b*) If $A+B=90^\circ$, prove that $\dfrac{\cos 2B-\cos 2A}{\sin 2A}=\tan A-\tan B$.

12. (*a*) In a triangle ABC prove that $a=b\cos C+c\cos B$.

(*b*) In a triangle, the angles are to one another as $1:2:3$; prove that the corresponding sides are as $1:\sqrt{3}:2$.

13. Two vertical pillars, the height of one of which is double that of the other, are at a distance of 150 ft. from each other. At a point between the pillars and on the line joining their feet the angular elevations of the tops of the taller and the shorter pillar are found to be 60° and 30° respectively. Find the heights of the pillars and the position of the point.

[Ans. Height $=60\sqrt{3}$ ft., $30\sqrt{3}$ ft., distance $=60$ ft. from the taller tower.]

14. Draw the graph of $\sin x$ between the values $x=-\pi$ and $x=\pi$ and find, from the graph, the value of $\sin 120^\circ$. [Ans. ·87]

SECOND PAPER

1. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Prove that in any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle, diminished by twice the rectangle contained by one of these sides and the projection on it of the other side.

(*b*) Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

(c) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(b) A straight line AB is divided in a given ratio internally at C and externally at D. If P be a point where CD subtends a right angle, prove that PC bisects the angle APB.

2. (a) Show that the angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(b) ABC is a triangle inscribed in a circle ; AD, AE are lines drawn to the base BC parallel to the tangents at B, C respectively ; prove that $BD : CE = AB^2 : AC^2$.

Or, (b) Tangents AB, AC are drawn to a circle ; CE is perpendicular to the diameter BD through B ; prove that AD bisects CE.

3. Draw an equilateral triangle, each side of which is 2 inches. Now proceed to construct a square equal in area to this triangle.

Or, Draw circles of radii 4 cms. and 2·5 cms. respectively, with their centres at a distance 10 cms. apart. Proceed to construct a transverse common tangent to the two circles.

[*Statement of construction, and full, neat and distinct traces are to be given in either case, but no proof.*]

4. Answer *either* (a) and (b), *or* (c) and (d) :

(a) Obtain the co-ordinates of the point which divides the straight line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$.

$$\left[\text{Ans. } \frac{m_1x_2+m_2x_1}{m_1+m_2},\ \frac{m_1y_2+m_2y_1}{m_1+m_2}\right]$$

(b) If A, B, C, D are points whose co-ordinates are $(-2, 3)$, $(8, 9)$, $(0, 4)$, and $(3, 0)$ respectively, and AB and CD are joined, find the ratio of the segments into which AB is divided by CD.

[Ans. 11 : 47]

(*c*) Obtain the equation of the straight line whose intercepts on the axes OX, OY are a and b respectively.

$$\left[\text{Ans. } \frac{x}{a}+\frac{y}{b}=1\right]$$

(*d*) Determine the equation of the straight line which passes through the intersection of the lines given by $3x-4y+1=0$ and $5x+y=1$, and has equal intercepts of the same sign on the axes.

[Ans. $x+y=\frac{11}{23}$]

5. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Find the length of the chord of the circle $x^2+y^2=64$, intercepted on the straight line $3x+4y-c=9$.

[Ans. $\frac{2}{5}\sqrt{1600-c^2}$]

(*b*) Obtain the co-ordinates of the point of contact of any one of the two tangents to the above circle $x^2+y^2=64$, parallel to the line $3x+4y-c=0$. [Ans. $(\frac{24}{5}, \frac{32}{5})$, $(-\frac{24}{5}, -\frac{32}{5}$]

(*c*) Find out the eccentricity and the co-ordinates of the foci of the ellipse $9x^2+25y^2=225$. [Ans. $\frac{4}{5}$; $(4, 0)$, $(-4, 0)$]

(*d*) Find the distance from the origin of the point where the tangent at the extremity of a latus rectum of the above ellipse $9x^2+25y^2=225$, intersects the major axis. [Ans. $\frac{25}{4}$]

6. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Find out the equation of the tangent of the parabola $y^2=4ax$ at the extremity of the latus rectum.

[Ans. $y=x+a$, $y=-x-a$]

(*b*) A double ordinate of the parabola $y^2=4ax$ is of length $8a$. Prove that the lines joining the vertex to its two ends are at right angles.

(*c*) Obtain the equation to the hyperbola whose focus is $(a, 0)$, directrix is the straight line $x=\frac{1}{2}a$ and eccentricity is $\sqrt{2}$.

$$\left[\text{Ans. } x^2-y^2=\frac{a^2}{2}\right]$$

(*d*) A rod of length 6 units slides with its extremities always on the co-ordinate axes. Prove that its middle point traces out a circle, whose equation you are to determine. [Ans. $x^2+y^2=9$]

7. Answer *any two* of the following questions (*a*), (*b*), (*c*), (*d*) :

(*a*) A thick hollow cylindrical pipe is 6 inches in length, and its whole surface (outer and inner curved surfaces and the plane edges) is 308 sq. inches. If the external diameter of the pipe is 8 inches, and if its material weighs 4 ozs. per cubic inch, find its weight. [Take $\pi = \frac{22}{7}$] [Ans. 528 ozs.]

(*b*) When is (i) a straight line, (ii) a plane said to be perpendicular to a given plane ?

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane containing them.

(*c*) Prove that in any triangle, the middle points of the sides and the middle points of the lines joining the orthocentre to the vertices lie on a circle.

Prove also that the distance of the orthocentre from any angular point of the triangle is double the distance of the circum-centre from the opposite side.

(*d*) Obtain the co-ordinates of the centre of the circle passing through the points (1, 2), (3, $-$4), (5, $-$6), and determine the length of its diameter. [Ans. Centre (11, 2) ; diameter = 20]

Is the origin inside, or outside the circle ? [Ans. Outside]

H. S. Exam. (Compartmental)—1960

ELECTIVE MATHEMATICS (FIRST PAPER)

Group A—Algebra—[60 Marks]

Answer **any four** *questions. All questions carry equal marks.*

1. (*a*) Given $\sqrt{2} = 1{\cdot}414$, find to three decimal places, the value of $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$. [Ans. 14·070]

(*b*) Simplify : $\left[81^{-\frac{3}{4}} \times \frac{16^{\frac{1}{4}}}{6^{-2}} \times \left(\frac{1}{27}\right)^{-\frac{4}{3}}\right]^{\frac{1}{3}}$ [Ans. 6]

(*c*) Find the square root of $17+12\sqrt{2}$. [Ans. $\pm(3+2\sqrt{2})$]

2. (a) Solve the equations : $x+2y=4$, $2xy-y^2=3$.

[Ans. $x=2, y=1$; $x=\frac{14}{5}, y=\frac{3}{5}$]

(b) The distance through which a heavy body falls from rest varies as the square of the time of its fall. If a body falls 64 feet in two seconds, how far does it fall in 8 seconds ?

[Ans. 1024 ft.]

3. (a) One hundred stones being placed in a straight line on the ground at a distance of one yard from one another, how far will a person travel, who shall bring them, one by one, to a basket, placed in the same straight line at the distance of a yard from the first stone ? [Ans. 5m. 1300 yds.]

(b) If a, b, c be respectively the p^{th}, q^{th} and r^{th} terms of a geometric series, prove that $a^{q-r}.b^{r-p}.c^{p-q}=1$.

4. (a) If α and β are the roots of the equation $x^2+8x+15=0$, find the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.

[Ans. $x^2-68x+256=0$]

(b) Show that, if x is real, the value of the expression $\frac{x^2-2x+4}{x^2+3x+4}$ lies between 7 and $\frac{1}{7}$.

5. (a) Find the middle term in the expansion of $(x-2y)^8$.

[Ans. $1120x^4y^4$]

(b) Expand $(2a-3x)^6$.

[Ans. $64a^6-576a^5x+2160a^4x^2-4360a^3x^3+4860a^2x^4$
$-2916ax^5+729x^6$]

6. (a) Find the logarithms of (i) 324 to the base $3\sqrt{2}$. (ii) $\frac{1}{9}$ to the base $9\sqrt{3}$. [Ans. (i) 4 (ii) $-\frac{4}{5}$]

(b) Prove that $\log\frac{81}{8}-2\log\frac{3}{2}+3\log\frac{2}{3}+\log\frac{3}{4}=0$.

7. (a) Find the number of permutations of n dissimilar things taken r at a time, where r is less than or equal to n.

(b) There are 12 ferrysteamers plying between Chandpalghat and Botanical Gardens. In how many ways, can a man go from Chandpalghat to Botanical Gardens and return by a different steamer ? [Ans. 132]

Group B—Trigonometry—[40 marks]

Answer ANY THREE *questions. All questions carry equal marks.*

8. (*a*) The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians; express these angles in degrees. [Ans. 81°, 9°]

(*b*) If s is the length of the arc of a circle whose radius is r and θ is the radian measure of the angle at the centre, standing on the arc, prove that $\theta = \frac{s}{r}$.

9. (*a*) If A and B are both acute angles and A is greater than B, prove (geometrically) that

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

(*b*) If $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ where A and B are acute angles, find the value of $\frac{\tan A - \tan B}{1 + \tan A \tan B}$. [Ans. $\frac{16}{63}$]

10. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$. [Ans. 60° and 300°]

(*b*) If $A+B+C=180°$, prove that

$$\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

11. In a triangle ABC, prove that

(*i*) $\cos A = \frac{b^2+c^2-a^2}{2bc}$. (*ii*) $a \cos\frac{B-C}{2} = (b+c)\sin\frac{A}{2}$.

12. The upper part of a straight tree broken over by the wind but not completely separated, makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet. What was the height of the tree? [Ans. $50\sqrt{3}$ ft.]

13. Draw the graph of $\cos x$ between the values of $x = -\pi$ and $x = \pi$ and read off from the graph, the value of cos 150°.

SECOND PAPER

1. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :—

(*a*) If two triangles are equiangular, prove that their corresponding sides are proportional.

(*b*) Prove that the line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at the point.

(*c*) Prove that in a triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

(*d*) Show that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

2. (*a*) If two chords of a circle intersect outside the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

(*b*) Prove that if the common chord of two intersecting circles be produced, it will bisect their common tangent.

Or, ABC is a triangle right-angled at A; AD is perpendicular to BC. Show that $AB^2 = BD.BC$.

3. Draw a circle of radius 2 cms. Construct an equilateral triangle circumscribing this circle.

Or, Draw a triangle with sides 3, 4 and 5 cms. Now construct a square equal in area to this triangle.

[*Statement of construction, and full, neat and distinct traces are to be given in either case, but no proof*].

4. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Find the distance between the points whose co-ordinates are (x_1, y_1) and (x_2, y_2).

(*b*) Prove that the points whose co-ordinates are $(-2, -2)$, $(2, 2)$ and $(4, -4)$ are the vertices of an isosceles triangle.

(*c*) Find the angle between the straight lines whose equations are $y = m_1x + c_1$ and $y = m_2x + c_2$.

(*d*) Obtain the equation to the straight line passing through the point $(-1, 2)$ and perpendicular to the line $3x + 4y = 5$.

[Ans. $4x - 3y + 10 = 0$]

5. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Obtain the equation to a circle having its centre at (3, 7) and radius 5. [Ans. $x^2+y^2-6x-14y+33=0$]

(*b*) Find the equation of the tangent of the circle $x^2+y^2=a^2$ at any point (x_1, y_1) on it.

(*c*) Find the equation to the tangent of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ at the point } (x_1, y_1) \text{ on it.}$$

(*d*) Show that $x-3y=13$ touches the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

6. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Find the equation to the normal at (x_1, y_1) of the parabola $y^2=4ax$. [Ans. $y_1(x-x_1)+2a(y-y_1)=0$]

(*b*) Prove that the length intercepted on the x-axis of the parabola $y^2=4ax$, between the foot of the ordinate of any point of it and the point of intersection of the normal at the point with the x-axis is constant.

(*c*) Obtain the length of the chord of the hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$ passing through the origin and making equal angles with the axes. [Ans. $\frac{15}{2}\sqrt{2}$]

(*d*) Find the co-ordinates of the foci of the hyperbola $x^2-y^2=9$. [Ans. $(\pm 3\sqrt{2}, 0)$]

7. Answer *any two* of the following questions :—

(*a*) Prove that all straight lines drawn perpendicular to a given straight line at a given point of it are coplanar.

(*b*) The volume of a right circular cone whose height is 24 inches is 1232 cu. inches. Find the area of its slant surface. [Ans. 550 sq. in.]

(*c*) AB is a diameter of a circle ; AC and AD are any two chords cutting the tangent at B in P and Q ; prove that $\angle PCQ = \angle PDQ$.

(*d*) A straight line is drawn through the point (3, 5) such that the point bisects the portion of the line intercepted between the axes. Find the equation of the line, and calculate its perpendicular distance from the origin. [Ans. $5x+3y=30$; $\frac{15}{17}\sqrt{34}$]

HIGHER SECONDARY EXAMINATION, 1961

ELECTIVE MATHEMATICS

FIRST PAPER—Group A (Algebra)

Answer *any four* questions.

1. (*a*) Simplify : $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

[Ans. $\sqrt{3}$ or, 1·732]

(*b*) Simplify : $\sqrt[ln]{\dfrac{x^l}{x^n}}\times\sqrt[mn]{\dfrac{x^n}{x^m}}\times\sqrt[lm]{\dfrac{x^m}{x^l}}$. [Ans. 1]

(*c*) Find the sq. root of $33-4\sqrt{35}$. [Ans. $\pm(2\sqrt{7}-\sqrt{5})$]

2. (*a*) Solve the equations :

$$\left.\begin{array}{r}x+y=3\\ 2x^2-5xy+2y^2=0\end{array}\right\}$$

$\left[\begin{array}{l}\text{Ans. } x=1,\ y=2\ ;\\ \quad\text{or, } x=2,\ y=1.\end{array}\right]$

(*b*) The length of a pendulum varies inversely as the square of the number of beats it makes per minute. If a pendulum 16 ft. long makes 27 beats per minute, find the length of the pendulum that makes 24 beats per minute. [Ans. $20\frac{1}{4}$ ft.]

3. (*a*) A person lends Rs. 1000 to a friend agreeing to charge no interest and also recover the amount by monthly instalments decreasing successively by Rs. 2. In how many months will the loan be paid up, if the first instalment be Rs. 64 and its payment be made one month after the sum is lent ? [Ans. 25]

(*b*) If $1, \omega, \omega^2$ are the three cube roots of unity, prove that

$$(1+\omega-\omega^2)^3=(1-\omega+\omega^2)^3=-8.$$

4. (*a*) If p, q are the roots of the equation $2x^2-5x+2=0$, find the equation whose roots are $p+mq$ and $q+mp$.

[Ans. $4x^2-10(1+m)x+4m^2+17m+4=0$]

(*b*) Find the maximum and minimum values of $\dfrac{x^2-x+1}{x^2+x+1}$ for real values of x. [Ans. 3 and $\frac{1}{3}$]

5. (*a*) Expand $\left(\dfrac{2x}{3}-\dfrac{3}{2x}\right)^6$.

$$\left[\text{Ans. } \frac{64}{729}x^6-\frac{32}{27}x^4+\frac{20}{3}x^2-20+\frac{135}{4x^2}-\frac{243}{8x^4}+\frac{729}{64x^6}\right]$$

(*b*) Write down the coefficient of x^{10} in $(x-2y)^{13}$.

[Ans. $-2288y^3$]

6. (*a*) Given log 2 = ·30103 and log 3 = ·4771213, find the logarithm of ·015. [Ans. $\bar{2}$·1760913]

(*b*) Prove that $7 \log \frac{10}{9} - 2 \log \frac{25}{24} + 3 \log \frac{81}{80} - \log 2 = 0$.

7. (*a*) Find the number of combinations of n dissimilar things taken r at a time. $\left[\text{Ans.}\ \dfrac{\lfloor n}{\lfloor r \lfloor n-r}\right]$

(*b*) How many numbers lying between 10 and 100 can be formed with the digits 3, 4, 0, 5, 8 ? [Ans. 16]

GROUP B—Trigonometry

Answer *any three* questions.

8. (*a*) The radius of a circle is 10 cm., find the angle in degrees and minutes, subtended at its centre by an arc 6 cm. in length. $[\pi = \frac{22}{7}]$ [Ans. $34^\circ 21' 8'$]

(*b*) The angles of a triangle are in A. P. If the number of degrees in the greatest angle is the same as the number of grades in the least, find the angles in degrees. [Ans. $63^\circ\frac{3}{19}$, 60°, $55^\circ\frac{16}{19}$]

9. (*a*) If A, B and $A-B$ are positive acute angles, prove geometrically that $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

(*b*) Find the value of $\sin 330^\circ + \tan 45^\circ - 4 \sin^2 120^\circ + 2 \cos^2 135^\circ + \sec^2 180^\circ$. [Ans. $-\frac{1}{2}$]

10. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\sqrt{3} \sin \theta + \cos \theta = 1$. [Ans. 0°, 120°, 360°]

(*b*) If $A+B+C = 180^\circ$, prove that

$$\tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$$

11. In a triangle ABC, prove that

(*a*) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

(*b*) $a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0$.

12. On a straight coast there are three objects A, B and C such that $AB = BC = 4$ miles. A steamer approaches B in a line perpendicular to the coast and at a certain point AC is found to subtend an angle 60° ; after sailing in the same direction for ten

minutes, AC is found to subtend an angle of $120°$; find the rate at which the steamer is going. [Ans. $16\sqrt{3}$ miles per hr.]

13. Draw the graph of sin x between the values of $x=0°$ and $x=360°$ and read off from the graph the value of sin $240°$.

[Ans. $-\cdot 87$]

SECOND PAPER

1. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, prove that the triangles are similar.

(*b*) If two triangles are similar, prove that their areas are proportional to the squares on their corresponding medians.

(*c*) Prove that the ratio of the areas of similar triangles is equal to the ratio of the squares on their corresponding sides.

(*d*) If ABC be a triangle inscribed in a circle, and the tangent at A meets BC produced in D, prove that $BD : CD = AB^2 : AC^2$.

2. (*a*) If from a point outside a circle, a secant and a tangent be drawn to the circle, prove that the rectangle contained by the segments of the secant is equal to the square on the tangent.

(*b*) If the diagonals of a cyclic quadrilateral are at right angles, show that the perpendicular from the point of intersection to any side when produced backwards bisects the opposite side.

Or, (*b*) From the extremities of any chord AB of a circle, perpendiculars AQ, BR are drawn to the tangent at any point P. If PM is perpendicular to AB, prove that $PM^2 = AQ.BR$.

3. Draw a circle of radius 1 inch, and then construct a regular hexagon circumscribing the circle.

Or, Take a straight line of length 2 inches and divide it into two parts such that the square on one part may be double the square on the other part.

[*Statement of construction and distinct traces are to be given in either case, but no proof.*]

4. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Obtain the area of the triangle whose vertices are points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

[Ans. $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$]

(*b*) Find the area of the triangle whose vertices A, B, C are respectively $(3, 4)$, $(-4, 3)$ and $(8, -6)$; hence or otherwise find the length of the perpendicular from A on BC.

[Ans. 37·5 units of area ; 5 units of length]

(*c*) Obtain the equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2). $\left[\text{Ans. } \dfrac{y-y_1}{y_1-y_2} = \dfrac{x-x_1}{x_1-x_2}.\right]$

(*d*) Obtain the equation to the perpendicular bisector of the line joining the points $(-2, 7)$ and $(8, -1)$. At what distance is this perpendicular-bisector from the origin?

[Ans. $5x-4y-3=0$, distance $=\dfrac{3}{\sqrt{41}}$ units of length]

5. (*a*) Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Obtain the equation to the circle passing through the points $(3, 4)$, $(3, -6)$, $(-1, 2)$ and determine its centre and radius.

[Ans. $(x-3)^2+(y+1)^2=5^2$; centre $(3, -1)$, radius $=5$ units of length]

(*b*) Prove that the straight line $y=x+a\sqrt{2}$ touches the circle $x^2+y^2=a^2$, and find its point of contact.

$\left[\text{Ans. } \left(\dfrac{-a\sqrt{2}}{2}, \dfrac{a\sqrt{2}}{2}\right)\right]$

(*c*) Obtain the equation to the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point (x_1, y_1) on the ellipse.

$$\left[\text{Ans. } \frac{x-x_1}{\dfrac{x_1}{a^2}} = \frac{y-y_1}{\dfrac{y_1}{b^2}}\right]$$

(*d*) Find the equation to the tangent of the ellipse $9x^2+16y^2=144$ having equal positive intercepts on the axes.

[Ans. $x+y=5$]

6. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Find out the equation to the parabola whose focus is $(-3, 4)$ and directrix is $6x-7y+5=0$.

[Ans. $(7x+6y)^2+450x-610y+2100=0$]

(*b*) The two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles. Find the locus of P. [Ans. $x+1=0$]

(*c*) In the hyperbola $4x^2-9y^2=36$, find the lengths of the axes, the co-ordinates of the foci, the eccentricity and the' length of the latus rectum.

[Ans. lengths$=6$ and 4 units of length ; foci $(\pm\sqrt{13}, 0)$.
$e=\frac{\sqrt{13}}{3}$, latus rectum$=2\frac{2}{3}$ units of length]

(*d*) Find the condition that $y=mx+c$ may touch the hyperbola $x^2-y^2=a^2$. [Ans. $c=\pm a\sqrt{m^2-1}$]

7. Answer *any two* of the following questions (*a*), (*b*), (*c*) and (*d*) :

(*a*) A and B are two fixed points whose co-ordinates are (2, 4) and (2, 6) respectively ; ABP is an equilateral triangle on the side of AB opposite to the origin. Find the co-ordinates of P.

[Ans. $(2+\sqrt{3}, 5)$]

(*b*) B and C are fixed points having co-ordinates (3, 0) and $(-3, 0)$ respectively. If the vertical angle BAC be 90°, show that the locus of the centroid of the triangle ABC is a circle whose equation you are to determine. [Ans. $x^2+y^2=1$]

(*c*) With the material of a hollow sphere of outer diameter 10 cms. and thickness 2 cms. is made a solid right circular cone of height 8 cms. Find the surface area of its curved surface to the nearest square centimetre. [$\pi=\frac{22}{7}$]. [Ans. 234 sq. cm.]

(*d*) How is the angle between two intersecting planes defined? When is a plane perpendicular to another plane ?

If two straight lines are parallel, and if one of them is perpendicular to a plane, prove that the other is also perpendicular to the same plane.

Group B—Trigonometry

8. (a) Define a radian. Taking $\pi = 3{\cdot}1416$, show that a radian contains 206265 seconds approximately.

(b) One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{2}{3}x$ degrees, whilst the third is $\frac{x\pi}{75}$ radians; express them all in degrees. [Ans. 24°, 60°, 96°]

9. (a) If A, B and $A-B$ are all positive acute angles, prove geometrically that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

(b) Find the value of

$$\frac{2\tan^2 30^\circ}{1-\tan^2 30^\circ} + (\sec^2 45^\circ - \cot^2 45^\circ) - (\cos^2 60^\circ + \sin^2 120^\circ).$$

[Ans. 1]

10. (a) Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

(b) If $A+B+C=180^\circ$, prove that
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

11. In a triangle ABC, prove that

(a) $c = a\cos B + b\cos A$.

(b) $(b-c)\cos\frac{A}{2} = a\sin\frac{B-C}{2}$.

12. Two vertical poles are 120 feet apart and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary. Find their heights.
[Ans. $30\sqrt{2}$ ft., $60\sqrt{2}$ ft.]

13. Draw the graph of $\cos x$ between the values of $x=0^\circ$ and $x=360^\circ$ and read off from the graph the value of $\cos 300^\circ$.

SECOND PAPER

1. Answer *either* (a) and (b), *or* (c) and (d) :—

(a) Prove that the bisector of the exterior angle of a triangle divides the opposite side externally in the ratio of the other two sides.

(b) In a quadrilateral, if the bisectors of one pair of opposite angles meet on one diagonal, prove that the bisectors of the other pair of opposite angles will meet on the other diagonal.

H. S. Exam. (Compartmental)—1961

FIRST PAPER

Group A—Algebra—60

1. (*a*) Simplify :—

$$\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$$ [Ans. 0]

(*b*) Find the square root of $37-20\sqrt{3}$. [Ans. $\pm(5-2\sqrt{3})$]

2. (*a*) Solve the equations : $\frac{x}{y}+\frac{y}{x}=2\frac{1}{2}$; $x+y=6$.

[Ans. $x=4$, $y=2$, or, $x=2$, $y=4$]

(*b*) If x varies as y^2 and $y=4$ when $x=8$, find y when $x=32$.

[Ans. $y=\pm 8$]

3. (*a*) If a, b, c be in Arithmetical Progression and x, y, z in Geometrical Progression, prove that $x^{b-c}y^{c-a}z^{a-b}=1$.

(*b*) If $x=3+4i$ and $y=3-4i$, where $i=\sqrt{-1}$, find the value of x^3+y^3.

[Ans. -234]

4. (*a*) If α and β be the roots of the equation $x^2+4x+3=0$, show that the equation whose roots are $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$ is $3x^2-16x+16=0$.

(*b*) If x is real, prove that

$$\frac{x}{x^2-5x+9} \text{ must lie between 1 and } -\tfrac{1}{11}.$$

5. (*a*) Write down the coefficient of y in the expansion of $\left(y^2+\frac{c^3}{y}\right)^5$.

[Ans. $10c^9$]

(*b*) Apply the Binomial Theorem to find the value of $(1{\cdot}03)^4$ to 4 places of decimals.

[Ans. 1·1255...]

6. (*a*) If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, prove that $xyz=1$.

(*b*) Show that

$$\log_{10} 2+16\log_{10}\frac{16}{15}+12\log_{10}\frac{25}{24}+7\log_{10}\frac{81}{80}=1.$$

7. (*a*) Find the number of permutations of n dissimilar things taken r at a time, where r is less than or equal to n.

(*b*) There are 26 stations on a certain railway line. How many kinds of different single third class tickets have to be printed, in order that it may be possible to travel from any station to any other ?

[Ans. 650]

(c) If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, prove that the triangles on each side of the perpendicular are similar to one another. Hence deduce that the perpendicular is a mean proportional between the segments of the hypotenuse.

(d) In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, show that the segments of the hypotenuse have the same ratio as the squares on the sides containing the right angle.

2. (a) Prove that the obtuse angle between the tangent at a point of a circle and a chord through the point of contact is equal to the angle in the alternate segment.

Or, If from any point on the circumcircle of a triangle perpendiculars are drawn to the sides of the triangle, prove that the feet of the perpendiculars are collinear.

(b) If two circles intersect, show that their common tangent subtends supplementary angles at the points of intersection.

Or, Two radii of a circle are perpendicular to each other, and a tangent cuts them when produced ; prove that the other tangents drawn to the circle from these points of intersection are parallel.

3. Take a straight line of length 6 cms. ; divide it into two segments such that the rectangle contained by the segments may be equal to a square on a side of length 2 cms.

Or, Draw a circle of radius 1 inch. Find out a point outside this circle such that the two tangents from it to the circle and the line joining the points of contact may form an equilateral triangle.

[*Statement of construction, and full, neat and distinct traces are to be given in either case, but no proof.*]

4. Answer *either* (a) and (b) *or* (c) and (d) :—

(a) Obtain the distance between the points whose rectangular cartesian co-ordinates are (x_1, y_1) and (x_2, y_2).

(b) Show that the triangle whose vertices are the points $(-2, -5)$, $(4, -1)$ and $(-1, 0)$ is isosceles.

(c) Obtain the equation to a straight line which is inclined to the x-axis at an angle θ, and whose intercept on the y-axis is c.

(d) Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear.

5. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) :—

(*a*) The extremities of a diameter of a circle have co-ordinates $(-4, 3)$ and $(12, -1)$; find the equation to the circle.

[Ans. $x^2+y^2-8x-2y-51=0$]

(*b*) Find the condition that the straight line $y=mx+c$ may touch the circle $x^2+y^2=a^2$.

(*c*) An ellipse has its major axis along the x-axis and the minor axis along the y-axis. Its eccentricity is $\frac{1}{2}$ and the distance between the foci is 4. Find its equation and show that the ellipse passes through the point (2, 3).

(*d*) Find the equation to the tangent at the point (x_1, y_1) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) :—

(*a*) Show that the straight line $y=mx+\frac{a}{m}$ is a tangent to the parabola $y^2=4ax$, whatever m may be.

(*b*) Show that the foot of the perpendicular from the focus of the parabola $y^2=4ax$ on any tangent lies on the y-axis.

(*c*) Prove that in the hyperbola $x^2-y^2=a^2$, the difference between the focal distances of any point on it is constant.

(*d*) Find the length of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ along the line $y=mx$.

$\left[\text{Ans. } 2ab\sqrt{\frac{1+m^2}{b^2-a^2m^2}}\right]$

7. Answer *any two* of the following questions :—

(*a*) A and B are two fixed points on a plane, and a point P moves on the plane in such a way that $PA=2PB$ always. Prove either geometrically or analytically that the locus of P is a circle.

(*b*) OA, OB, OC are three straight lines on a plane. If OP be perpendicular to OA and OB, prove that it is perpendicular to OC also.

(*c*) A solid right circular cylinder, whose height is 9 inches and diameter of the base 4 inches, is deformed into a sphere. Find the surface area of this sphere. [Ans. $113\frac{1}{7}$ sq. inches]

(*d*) Find the equation of the straight line which passes through the intersection of the lines $3x-7y+5=0$, $x-2y-7=0$, and has equal intercepts of the same sign along the axes.

[Ans. $x+y=85$]

HIGHER SECONDARY EXAMINATION, 1962

ELECTIVE MATHEMATICS

FIRST PAPER—Group A—Algebra

Answer *any four* questions

1. (a) Given $x=\frac{\sqrt{3}}{2}$, find the value of

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \qquad \left[\text{Ans. } \frac{1}{\sqrt{3}}\right]$$

(b) Find the square root of $17-12\sqrt{2}$. [Ans. $\pm(3-2\sqrt{2})$]

(c) Simplify : $(8x^3 \div 27a^{-3})^{\frac{2}{3}} \times (64x^6 \div 27a^{-3})^{-\frac{2}{3}}$ [Ans. $\frac{1}{4}$]

2. (a) Solve the equations : $\left.\begin{array}{r} 3x-5y=2 \\ xy=8 \end{array}\right\}$

$$\left[\text{Ans. } x=4,\ y=2\ ;\ \text{or, } x=-\frac{10}{3},\quad y=-\frac{12}{5}\right]$$

(b) Given that the area of a circle varies as the square of its radius and that the area of a circle is 154 sq. feet, when the radius is 7 ft., find the area of a circle whose radius is 10 ft. 6 inches.

[Ans. 346·5 sq. ft.]

3. (a) If S_1, S_2, S_3 be the sums of n terms of three Arithmetic series, the first term being 1 and the respective common differences 1, 2, 3, prove that $S_1+S_3=2S_2$.

(b) If 1, ω, ω^2 are the three cube roots of unity, prove that $(x+y)^2+(x\omega+y\omega^2)^2+(x\omega^2+y\omega)^2=6xy$.

4. (a) If α, β be the roots of the equation $ax^2+x+b=0$, show that $\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\beta}\right)=\frac{1}{ab}$.

(b) If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

5. (a) Find the value of the term independent of x in $\left(2x^2-\frac{1}{x}\right)^{12}$

Ans. 7920]

(b) Expand $(1+2x)^{-3}$ to five terms.

[Ans. $1-6x+24x^2-80x^3+240x^4$]

6. (a) Given $\log 2=\cdot30103$ and $\log 3=\cdot4771213$, find the logarithms of (i) $5^{\frac{1}{16}}$ and (ii) ·1875.

[Ans. (i) ·7043652, (ii) $\bar{1}$·2730013]

(b) Find the value of $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25}$.

[Ans. $\log 3$]

7. (a) Same as Q. 8(a) of H. S., 1960.

(b) How many permutations can be made out of the letters of the word TRIANGLE ? How many of these will begin with T but end with E ? [Ans. 40320, 720]

Group B—Trigonometry—40

8. (a) The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. Find the number of degrees and minutes in the third angle. [$\frac{22}{7}$ radians = 2 right angles]. [Ans. $132^\circ 16' \cdot 36$]

(b) The diameter of a graduated circle is 6 ft. and the graduation on its rim are $15'$ apart ; find the distance (in inches correct to two places of decimals) from one graduation to another next to it. [$\pi = \frac{22}{7}$] [Ans. $\cdot 16$ in.]

9. (a) Same as Q. 10(a) of H. S., 1960.

(b) Show that

$$\sin 420^\circ \cos 390^\circ + \cos(-300^\circ) \sin(-330^\circ) = 1.$$

10. (a) Find the values of θ between 0° and 360° which satisfy the equation $\cos^2\theta - \sin\theta = \frac{1}{4}$. [Ans. 30°, 150°]

(b) $A+B+C=180^\circ$, prove that

$$\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

11. In a triangle ABC, prove that

(a) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. (b) $a \sin\left(\dfrac{A}{2}+C\right) = (b+c) \sin \dfrac{A}{2}$.

12. The angle of elevation of the top of a tower is observed to be 60° from a point in the horizontal plane through the foot of the tower ; at the point 40 ft. vertically above the first point of observation, the elevation is found to be 45°. Find the height of the tower and its horizontal distance from the points of observation. [Ans. $h = 94{\cdot}6$ ft., $d = 54{\cdot}6$ ft.]

13. Draw the graph of $\cos x$, between the values of $x = -\pi$ and $x = \pi$ and read off from the graph the value of $\cos 120^\circ$.

[Ans. $-{\cdot}5$]

SECOND PAPER

Group A—Plane Geometry—40

Answer *any three* questions.

1. (*a*) Prove that in an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle, together with twice the rectangle contained by one of these sides and the projection of the other side on it.

(*b*) Same as Q. 1 (*d*) of H. S. 1960 (Compl.)

2. (*a*) If two chords of a circle intersect inside the circle, prove that the rectangle contained by the parts of one, is equal to the rectangle contained by the parts of the other.

(*b*) Through any point X on the common chord of two intersecting circles, chords AB and CD are drawn one in each circle. Prove that $AX.XB=CX.XD$.

3. (*a*) Same as Q. 1(*a*) of H. S. 1960 (Compl.).

(*b*) In the trapezium $ABCD$, AB is parallel to DC and the diagonals intersect at O. Show that $OA : OC=OB : OD$.

4. (*a*) Same as Q. 1 (*c*) of H. S. 1960.

(*b*) AD is a median of the triangle ABC, and the angles ADB, ADC are bisected by lines which meet AB, AC at E and F respectively. Show that EF is parallel to BC.

5. Construct a regular hexagon circumscribing a circle of radius 1·5 inches. Measure a side of the hexagon. [Ans. 1·73″]

[*Statement of construction, traces of construction as well as justification are to be given.*]

Group B—Co-ordinate Geometry—40

Answer *any three* questions.

6. (*a*) Same as Q. (4) (*a*) of H. S. 1960.

(*b*) The co-ordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). Find the co-ordinates of the point where the medians of the triangle intersect.

[Ans. $x=\frac{1}{3}(x_1+x_2+x_3)$, $y=\frac{1}{3}(y_1+y_2+y_3)$]

7. (*a*) Find the angle between the straight lines whose equations are $y=m_1x+c_1$ and $y=m_2x+c_2$.

[Ans. $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$.]

(*b*) Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the line $5x-2y+7=0$.

[Ans. $2x+5y+1=0$]

8. (*a*) Find the equation of the circle passing through the origin which makes intercepts 6 and 8 on the positive sides of the axes of x and y respectively. [Ans. $x^2+y^2-6x-8y=0$]

(*b*) Prove that the centres of the three circles $x^2+y^2-2x+6y=-1$, $x^2+y^2+4x-12y=9$, and $x^2+y^2-16=0$ lie on a straight line.

9. (*a*) Find the equation of the parabola, whose focus is at the point (5, 0) and whose directrix is the line $3x-4y+2=0$.

[Ans. $16x^2+24xy+9y^2-262x+16y+621=0$]

(*b*) Same as Q. 6 (*a*) of H. S., 1961 (Compl.).

10. (*a*) Find the equation of the ellipse whose major and minor axes lie along the axes of co-ordinates *ox*, *oy* respectively and whose eccentricity is $\dfrac{1}{\sqrt{2}}$ and latus rectum 3.

[Ans. $x^2+2y^2=9$]

(*b*) Show that the line $x-y=5$ touches the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$

Group C—Solid Geometry and Mensuration—20

Answer *any two* questions.

11. Same as Q. 7. (*a*) of H. S. 1960 (Compl.)

12. If a right angle rotates about one of its arms, then the other arm describes a plane.

13. Find the volume and the lateral surface of a right prism 8 inches long, standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side 6 inches.

[Ans. vol. = 96 cu. in., surface = 128 sq. inches.]

14. A right pyramid stands on a rectangular base whose sides are 12 inches and 9 inches ; and the length of each of the slant edges is 8·5 inches. Find the height and the volume of the pyramid. [Ans. $h=4''$, $v=144$ cubic inches.]

H. S. Exam. (Compartmental)—1962

ELECTIVE MATHEMATICS

FIRST PAPER

Group A—Algebra—60

1. (a) Simplify :—

$$\frac{3\sqrt{8}-2\sqrt{12}+\sqrt{20}}{3\sqrt{18}-2\sqrt{27}+\sqrt{45}}.$$ $\left[\text{Ans. } \frac{2}{3}\right]$

(b) Find the square root of $8+\sqrt{60}$. [Ans. $\pm(\sqrt{5}+\sqrt{3})$]

(c) Simplify :—

$$\left(\frac{x^b}{x^c}\right)^{\frac{1}{bc}}\times\left(\frac{x^c}{x^a}\right)^{\frac{1}{ca}}\times\left(\frac{x^a}{x^b}\right)^{\frac{1}{ab}}.$$ [Ans. 1]

2. (a) Solve the equations :

$$\left.\begin{aligned}x-y&=2\\x^2+y^2&=34\end{aligned}\right\}$$ $\left[\text{Ans. } \left.\begin{aligned}x&=5\\y&=3\end{aligned}\right\} \text{ or, } \begin{aligned}x&=-3\\y&=-5\end{aligned}\right]$

(b) When a body falls from rest, its distance from the starting point varies as the square of the time elapsed. If a body falls from rest through $402\frac{1}{2}$ ft. in 5 seconds, how far does it fall in 10 seconds? [Ans. 1610 ft.]

3. (a) The fifth term of a G. P is 81 and the second term is 24 ; find the series. [Ans. 16, 24, 36, 54, 81,......]

(b) If $i=\sqrt{-1}$, show that $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}=-\frac{8}{29}$.

4. (a) Find the equation, whose roots are the squares of the roots of the equation $3x^2-7x-5=0$. [Ans. $9x^2-79x+25=0$]

(b) If x is real, prove that the value of

$\frac{(x-1)(x+3)}{(x-2)(x+4)}$ does not lie between $\frac{4}{9}$ and 1.

5. (a) Find the value of the middle term in the expansion of $(3x-2y)^{18}$. [Ans. $-\frac{\underline{|18}}{\underline{|9}\,\underline{|9}}.3^9.2^9.x^9.y^9$]

(b) Apply the Binomial Theorem to find the value of $(99)^4$ [Ans. 96059601]

6. (a) Find the logarithms of

(i) 5832 to the base $3\sqrt{2}$ [Ans. 6]

(ii) 81 to the base $\sqrt[3]{9}$. [Ans. 6]

(*b*) Show that $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$.

7. (*a*) Same as Q. 8(*a*) of H. S. 1960.

(*b*) Four travellers arrive at a town where there are five hotels. In how many ways, can they take up their quarters, each at a different hotel ? [Ans. 120]

Group B—Trigonometry—40

8. (*a*) Same as Q. 9(*a*) of H. S. 1960.

(*b*) Find the ratio of the radii of two circles, at the centres of which, two arcs of the same length subtend angles of 60° and 75°. [Ans. 5 : 4]

9. (*a*) Same as Q. 9 (*a*) of H. S. 1961.

(*b*) Show that

$\cos A + \sin (270° + A) - \sin (270° - A) + \cos (180° + A) = 0$.

10. (*a*) Find the values of θ between 0° and 360°, which satisfy the equation $\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta$. [Ans. 60° or 300°]

(*b*) Prove that $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta \cos 4\theta$.

11. (*a*) Same as Q. 11(*a*) of H. S. 1961.

(*b*) The sides of a triangle are 56, 65 and 33 feet ; find its greatest angle. [Ans. 90°]

12. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 50 feet. What was the height of the tree ? [Ans. $50\sqrt{3}$ ft.]

13. Draw the graph of sin x, between the values of $x = -\pi$ and $x = \pi$ and read off from the graph the value of sin 150°.

SECOND PAPER

Group A—Plane Geometry—40

1. (*a*) Same as Q. 1(*a*) of H. S. 1960.

(*b*) ABC is an isosceles triangle in which $AB = AC$; and BE is drawn perpendicular to AC. Shew that $BC^2 = 2AC.CE$.

2. (*a*) Prove that the angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

(*b*) AB is the common chord of two circles, one of which passes through O, the centre of the other ; prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

3. (*a*) Same as Q. 1(*c*) of H. S. 1961.

(*b*) A trapezium $ABCD$ has its sides AB, CD parallel and its diagonals intersect at O. If AB is double of CD, find the ratio of the areas of the triangles AOB and COD. [Ans. 4 : 1]

4. (*a*) Same as Q. 1(*c*) of H. S. 1961 (Compl.).

(*b*) At the extremities of a diameter of a circle, whose centre is C, tangents are drawn ; these are cut in Q and R by any third tangent whose point of contact is P. Shew that QR subtends a right angle at C.

5. Construct a square equal in area to a given rectangle whose adjacent sides are 1·5 in. and 2·5 in. Measure the side of the square. [*Statement of the construction and the traces are to be given.*]

Group B—Co-ordinate Geometry—40

6. (*a*) Same as Q. 4. (*a*) of H. S. 1960 (Compl.).

(*b*) Shew that the straight line joining the two points $(-7, 3)$ and $(14, -6)$ passes through the origin.

7. (*a*) Same as Q. 4. (*c*) of H. S. 1961.

(*b*) Show that the three lines $3x+y=5$, $x+5y+3=0$ and $5x-2y=12$ meet in a point.

8. (*a*) Find the equation of the circle which passes through the points $(0, -3)$, $(1, -2)$ and $(5, -8)$.

[Ans. $x^2+y^2-6x+10y+21=0$]

(*b*) Shew that the straight line $x-y+2=0$ touches the circle $x^2+y^2=2$ and find the co-ordinates of the point of contact.

[Ans. $(-1, 1)$]

9. (*a*) Find the co-ordinates of the points at which the straight line $x+y=7$ cuts the circle $x^2+y^2=25$. [Ans. (3, 4), (4, 3)]

(*b*) Find the equation of the circle which passes through the origin and the points at which the straight line $3x+4y=12$ meets the axes. [Ans. $x^2+y^2-4x-3y=0$]

10. (*a*) Find the equation of the parabola whose vertex is the origin and whose focus is the point (0, 5). Find the length of its Latus Rectum. [Ans. $x^2=20y$; 20]

(*b*) The straight line $y=3x-1$ cuts the parabola $y^2=4x$ at A and B. Find the length AB. [Ans. $\frac{8}{9}\sqrt{10}$]

Group C—Solid Geometry and Mensuration—20

11. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane in which they lie.

12. From O, the centre of a circle, a perpendicular OA is erected to the plane of a circle. Prove that all points on the circumference are equidistant from any point on the perpendicular OA.

13. The length, breadth and height of a rectangular block are in the ratio 4 : 3 : 2, and the whole surface of the block is 1872 sq. in. Find the dimensions of the block and its volume. [Ans. 24″, 18″, 12″ ; 5184 cubic inches]

14. Find the curved surface and the volume of a right circular cylinder whose height is 8 in. and the radius of whose base is 5 in. [$\pi=\frac{22}{7}$]. [Ans. $251\frac{3}{7}$ sq. in. ; $628\frac{4}{7}$ cubic inches.]

H. S. Examination—1963

FIRST PAPER

Group A—Algebra

Answer *any four* questions.

1. (*a*) Simplify :—

$$\sqrt{\frac{(\sqrt{12}-\sqrt{8})(\sqrt{3}+\sqrt{2})}{5+\sqrt{24}}}.$$ [Ans. $\sqrt{6}-2$]

(b) If $a^x = m$, $a^y = n$ and $a^2 = (m^y n^x)^z$, prove that $xyz = 1$.

(c) Find the square root of $\frac{2+\sqrt{3}}{2}$. [Ans. $\pm\frac{1}{2}(\sqrt{3}+1)$]

2. (a) Solve the equations :

$$x+\frac{4}{y}=1, \quad y+\frac{4}{x}=25. \qquad \left[\text{Ans. } \begin{rcases} x=\frac{1}{5} \\ y=5 \end{rcases} \text{ or, } \begin{rcases} x=\frac{4}{5} \\ y=20 \end{rcases}\right]$$

(b) The volume of a pyramid varies jointly as its height and the area of its base ; and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet ? [Ans. 45 sq. ft.]

3. (a) If a, b, c, d be in $G. P.$, show that

$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2.$$

(b) If $1, \omega, \omega^2$ be the three cube roots of unity, prove that

(i) $1+\omega+\omega^2=0$;

(ii) $(3+3\omega+5\omega^2)^6=(3+5\omega+3\omega^2)^6=64$.

4. (a) If the roots of the equation $lx^2+nx+n=0$ be in the ratio $p : q$, prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

(b) Find the maximum and minimum values of

$\frac{x^2-x+1}{x^2+x+1}$ for real values of x. [Ans. 3, $\frac{1}{3}$]

5. (a) Find the coefficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$. [Ans. 1365]

(b) Find the two middle terms in the expansion of $(a+x)^{2n+1}$.

6. (a) Write the series for e^x ; hence expand $e^x+\frac{1}{e^x}$ in a series of ascending powers of x.

$$\left[\text{Ans. } \frac{\lfloor 2n+1}{n\,!\,(n+1)\,!}a^{n+1}x^n \text{ and } \frac{\lfloor 2n+1}{(n+1)\,!\,n\,!}a^n x^{n+1}\right]$$

6. (b) Given log 2 = ·30103 and log 3 = ·4771213, find (i) $\log_5 75$ and (ii) log 4500. [Ans. (i) 1·8750613 (ii) 3·6532126]

7. (*a*) Find the number of combinations of n dissimilar things taken r at a time.

(*b*) From a company of 15 men, how many selections of 9 men can be made, so as to (*i*) exclude three particular men, (*ii*) to include three particular men ? [Ans. (*i*) 220, (*ii*) 924]

GROUP B—Trigonometry

Answer *any three* questions.

8. Same as Q. 9(*a*) of H.S. 1961 (Compl.)

(*b*) If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, prove that

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$$

9. (*a*) Find the values of θ between $0°$ and $360°$ which satisfy the equation $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}$. [Ans. 15° and 105°]

(*b*) If $A+B+C=180°$, prove that

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

10. (*a*) Prove that in a triangle ABC, with the usual notations

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

(*b*) The sides of a triangle are 20″, 21″ and 29″. Find its greatest angle. [Ans. 90°]

11. A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is found to be 60° ; assuming that the boat moves uniformly, how soon will it reach the shore ? [Ans. $1\frac{1}{2}$ mins.]

12. Draw the graph of $\sin 2x$ between the values of $x = -\pi$ and $x = \pi$ and read off from the graph the value of $\sin 150°$.

SECOND PAPER

GROUP A—Plane Geometry

Answer *question 4* and *any two* others from this group.

1. (*a*) If two triangles have their sides proportional, when taken in order, prove that they are equiangular.

(*b*) Prove that the areas of two similar triangles are proportional to the squares on their circum-radii.

2. (*a*) If the base of a triangle be divided externally in the ratio of the other two sides, prove that the line joining the vertex to this point of division bisects the vertical angle externally.

(*b*) Prove that the external bisector of two angles and the internal bisector of the third of a triangle are concurrent.

3. (*a*) Show that the acute angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(*b*) Two circles intersect at A and B, and through P, any point on one of them, straight lines PAC and PBD are drawn to cut the other at C and D. Show that CD is parallel to the tangent at P.

4. Construct, to the scale, an isosceles triangle with each of the equal sides equal to 2 inches, and each base angle double the vertical angle.

Or, Divide a straight line of length 2 inches into two parts such that the square on one part may be three times the square on the other.

[*Statement of construction and full, neat traces are to be given in any one of the above cases, but no proof.*]

Group B—Co-ordinate Geometry

Answer *question 5* and *any two* others from this group.

5. (*a*) Same as Q. 4(*a*) of H. S., 1960 (Compl.).

(*b*) Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the successive angular points of a rectangle.

6. (*a*) Obtain the perpendicular distance from the point (x_1, y_1) to the straight line $ax+by+c=0$.

(*b*) Find the ortho-centre of the triangle whose angular points are $(2, 7)$, $(-6, 1)$ and $(4, -5)$. [Ans. $(-\frac{10}{9}, \frac{40}{27})$]

7. (*a*) Same as Q. 5(*b*) of H. S. 1960 (Compl.).

(*b*) Obtain the equation to the circle which passes through the point $(0, 4)$ and touches the x-axis at the point $(2, 0)$.

[Ans. $x^2+y^2-4x-5y+4=0$]

8. (*a*) A tangent to the parabola $y^2=12x$ makes an angle 45° to the axis. Find the co-ordinates of its point of contact.

[Ans. (3, 6)]

(*b*) The co-ordinates of the foci of a hyperbola are (5, 0) and (−5, 0), and its eccentricity is $\frac{5}{3}$. Find its equation.

$$\left[\text{Ans.}\quad \frac{x^2}{9}-\frac{y^2}{16}=1.\right]$$

9. (*a*) Show that the locus of the middle points of a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a straight line passing through its centre.

(*b*) Find the equation to the normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ at an extremity of a latus rectum.

[Ans. $25x-15y=27$]

GROUP C—Solid Geometry and Mensuration

10. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :—

(*a*) Same as Q. 7(*b*) Second part of H. S., 1960.

(*b*) If $PA=PB=PC$, where P is a point outside the plane of the triangle ABC, and if PO be drawn perpendicular to the plane, prove that O is the circum-centre of the triangle ABC.

(*c*) If two straight lines are both perpendicular to a plane, show that they are parallel.

(*d*) If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

11. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

Or, A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised ? [Ans. 1 cm.]

H. S. EXAMINATION (Compartmental)—1963

First Paper

Group A—Algebra—60

Answer *any four* questions,

1. (*a*) Simplify : $\dfrac{\sqrt{2}(\sqrt{3}+1)(2-\sqrt{3})}{(\sqrt{2}-1)(3\sqrt{3}-5)(2+\sqrt{2})}$.

[Ans. $2+\sqrt{3}$]

(*b*) Simplify : $\left\{\sqrt[3]{4}\times\dfrac{1}{\sqrt[6]{8}}\times\sqrt[12]{2^{-1}}\right\}^{\frac{1}{4}}$ [Ans. $2^{\frac{1}{48}}$]

(*c*) Find the square root of $\sqrt{18}-\sqrt{16}$.

[Ans. $\pm\sqrt[4]{2}(\sqrt{2}-1)$]

2. (*a*) Solve the equations :

$$\left.\begin{array}{l} x+3y=2 \\ x^2+2y^2+3xy=0 \end{array}\right\}$$

[Ans. $x=-1, y=1$,
or, $x=-4, y=2$]

(*b*) If the volume of a cone whose height is 12 inches and base 30 sq. inches be 120 cubic inches, find the volume of another cone whose height is 20 inches and base one square foot, the volume of a cone varying as the height and the base jointly.

[Ans. 960 cu. in.]

3. (*a*) If S be the sum, P the product and R the sum of the reciprocals of n terms in $G. P.$,

prove that $P^2=\left(\dfrac{S}{R}\right)^n$.

(*b*) Find the three cube roots of unity. If $1, \omega, \omega^2$ are the cube roots, prove that

$$(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^3+b^3+c^3-3abc.$$

4. (*a*) The roots of the equation $2x^2-5x+4=0$ are m and n ; form the equation whose roots are

$m+\dfrac{1}{n}$ and $n+\dfrac{1}{m}$. [Ans. $4x^2-15x+18=0$]

(*b*) Prove that the expression $3x^2-4x+10$ is positive for all real values of x. Find the minimum value of the expression.

[Ans. $\frac{26}{3}$]

5. (*a*) Prove that the term independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$ is $\dfrac{\lfloor 2n}{(\lfloor n)^2}$.

(*b*) Expand $(1-3x)^{\frac{2}{3}}$ to 4 terms. [Ans. $1-2x-x^2-\frac{4}{3}x^3$]

6. (*a*) Given log $2 = \cdot30103$ and log $3 = \cdot4771213$, find the logarithms of (i) $\cdot00015$ and (ii) $5\frac{1}{16}$. [Ans. (i) $\bar{4}\cdot1760913$ (ii) $\cdot7043652$]

(*b*) The logarithm of a certain number to a certain base is 6 and the logarithm of 8 times the number to the base formed by the product of the first base and 25 is 3. Find the first base.

[Ans. 12·5]

7. (*a*) Same as Q. 7 (*a*) of H. S. 1960 (Compl.).

(*b*) In how many ways can the letters of the word VALEDICTORY be arranged, so that the vowels may never be separated? [Ans. 967680]

GROUP B—Trigonometry—40

Answer *any three* questions

8. (*a*) Same as Q. 10. (*a*) of H. S., 1960.

(*b*) Simplify :

$$\frac{\sin(B-C)}{\cos B \cos C}+\frac{\sin(C-A)}{\cos C \cos A}+\frac{\sin(A-B)}{\cos A \cos B}. \qquad \text{[Ans. 0]}$$

9. (*a*) Find the value of θ between 0° and 360° which satisfy the equation $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$. [Ans. 150°, 210°]

(*b*) If $A+B+C=180^\circ$, prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

10. (*a*) Same as Q. 11 (i) of H. S. 1960 (Compl.).

(*b*) The sides of a triangle are 7, 5 and 3 cms. respectively; prove that its greatest angle is 120°.

11. At the foot of a mountain, the elevation of its top is found to be 45°; after proceeding one mile towards the top up a slope of 30° inclination to the horizontal, the elevation of the top is found to be 60°. Find the height of the mountain.

[Ans. $\frac{1}{2}(\sqrt{3}+1)mi.$ or 1·37 miles (App.)]

12. Draw the graph of $\cos x$ between the values of $x=0^\circ$ and $x=360^\circ$ and read off from the graph the value of $\cos 240^\circ$.

Second Paper

GROUP A—Plane Geometry—40

Answer **Question 4** *and any* **two** *others from this group*

1. (*a*) Prove that in any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

(*b*) $ABCD$ is a rectangle, and P is any point within it. Prove that $PA^2+PC^2=PB^2+PD^2$.

2. (*a*) If a straight line is drawn parallel to one side of a triangle, prove that the other two sides are divided proportionally.

(*b*) $ABCD$ is a trapezium in which AB is parallel to DC. If the diagonals AC, BD intersect at O, and through O, the straight line POQ be drawn parallel to AB or DC, meeting AD, BC at P and Q respectively, prove that PQ is bisected at O.

3. (*a*) If from any point outside a circle, two secants are drawn to the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

(*b*) $ABCD$ is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC : shew that $AB.AD=BC.CD$.

4. (*a*) Draw two circles of radii 1 cm. and 2 cms. with their centres 5 cms. apart, and construct a direct common tangent to these circles.

Or, Draw an equilateral triangle each of whose sides is 4 cms. in length, and then construct a square equal in area to this triangle.

[*Statement of construction and full neat traces are to be given in any one of the above cases, and the construction must be accurate to the scale. No proof is necessary.*]

GROUP B—Co-ordinate Geometry—40

Answer **Question 5** *and any* **two** *others from this group*

5. (*a*) Same as Q. 4 (*a*) of H. S., 1961.

*(*b*) Show that the line joining $(-4, -5)$ and $(9, 8)$ bisects the line joining $(2, 1)$ and $(6, 5)$. [*This Question is defective.]

6. (*a*) Same as Q. 4 (*c*) of H. S., 1961.

(*b*) Find the equation to the st. line which passes through the point $(-5, -8)$ and has equal intercepts of opposite signs on the axes. [Ans. $x-y=3$]

7. (*a*) Prove that the equation $x^2+y^2+2gx+2fy+c=0$ always represents a circle, whose centre and radius you are to find.

(*b*) Find the equation to the tangent to the above circle at (x_1, y_1).

8. (*a*) Prove that in a parabola, the length of the axis intercepted between the tangent at a point and the foot of the ordinate of that point, is bisected at the vertex.

(*b*) Find the equation to the hyperbola, referred to its axes as axes of co-ordinates, whose eccentricity is $\sqrt{2}$, and distance between its foci 16. [Ans. $x^2-y^2=32$]

9. (*a*) Find the condition that the straight line $y=mx+c$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(*b*) Prove that the point of intersection of any two perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lies on the circle $x^2+y^2=a^2+b^2$.

GROUP C—Solid Geometry and Mensuration

10. How is the angle between two intersecting planes defined ? When is a plane said to be perpendicular to another plane ?

Show that if a straight line is perpendicular to a plane, then any plane passing through the st. line is perpendicular to that plane.

Or, If PN be drawn perpendicular to a plane XY from an outside point P, and from the foot N of the perpendicular, a line NM is drawn perpendicular to the st. line AB in the plane XY, prove that PM is perpendicular to AB.

11. Two solid copper spheres of radii 1 cm. and 3 cms. are melted, and a solid right circular cone of height 7 cms. is formed of the material. Find the radius of its base. [Ans. 4 cm.]

Or, The external length, breadth and height of a closed box are 10 cms., 9 cms., 7 cms. respectively, and the total inner surface is 262 sq. cms. If the walls of the box be uniformly thick, find the thickness. [Ans. 1 cm.]

HIGHER SECONDARY EXAMINATION—1964

FIRST PAPER

GROUP A—Algebra (*Answer any* **four** *questions*)

1. (*a*) Simplify : $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$. [Ans. 0]

(*b*) If $x^{\frac{1}{a}}=y^{\frac{1}{b}}=z^{\frac{1}{c}}$ and $xyz=1$, prove that $a+b+c=0$.

(*c*) Find the square root of

$8+2\sqrt{2}-2\sqrt{5}-2\sqrt{10}$. [Ans. $\pm(1+\sqrt{2}-\sqrt{5})$]

2. (*a*) Solve the equations :

$$\left.\begin{aligned}2x-3y&=4\\ \frac{1}{x}+\frac{1}{y}&=\frac{7}{10}\end{aligned}\right\}$$

[Ans. $x=5,\ y=2$; or, $x=\frac{4}{7}$, $y=-\frac{20}{21}$]

(*b*) Given that the illumination from a source of light varies as the square of the distance, how much farther from a candle must a book, which is now 6 inches off, be removed so as to receive just half as much light ? [Ans. $8(\sqrt{2}-1)$ in.]

3. (*a*) A man arranges to pay off a debt of £3600 by 40 annual instalments which form an arithmetical series. When 30 of these instalments have been paid, he dies leaving a third of his debt unpaid ; find the value of the first instalment. [Ans. £51]

(*b*) If ω is an imaginary cube root of unity, prove that

$(1-\omega^2)(1-\omega^4)(1-\omega^8)(1-\omega^{10})=9$.

4. (*a*) If α and β are the roots of the equation $ax^2-bx+c=0$, form the equation whose roots are

$\alpha+\dfrac{\alpha^2}{\beta}$ and $\beta+\dfrac{\beta^2}{\alpha}$ [Ans. $a^2cx^2-(b^3-2abc)x+b^3c=0$]

(*b*) If the roots of the equation $ax^2+2bx+c=0$ be α, β and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+m$ and $\beta+m$, show that $\dfrac{b^2-ac}{B^2-AC}=\left(\dfrac{a}{A}\right)^2$.

5. (*a*) Find the $(r+1)$th term in the expansion of $(1-x)^{-4}$.

[Ans. $\dfrac{(r+1)(r+2)(r+3)}{6}x^r$]

(*b*) Write down the coefficient of x^{18} in the expansion of $(2x-3x^2)^{10}$. [Ans. $-2^7.3^8.120$]

6. (*a*) Find the total number of permutations of n dissimilar things taken r at a time $(r<n)$ in which a particular thing always occurs.

(*b*) How many numbers of four digits greater than 5000 can be formed out of the digits 3, 4, 5, 6 and 7, if no digit is repeated ? [Ans. 72]

7. (*a*) Given $\log_{10} 165=2{\cdot}2175$ and $\log_{10} 6974=3{\cdot}8435$, find the value of $\sqrt[5]{{\cdot}00000165}$. [Ans. ·06974]

(*b*) Write down the exponential series for e^x ; hence obtain a series for $e+\dfrac{1}{e}$.

GROUP B—Trigonometry

Answer **Question 8** *and any* **two** *from the rest*

8. (*a*) Same as Q. 10(*a*) of H.S., 1960.

(*b*) If $\tan\theta+\sec\theta=x$, prove that $\sin\theta=\dfrac{x^2-1}{x^2+1}$.

9. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $3(\sec^2\theta+\tan^2\theta)=5$.

[Ans. 30°, 150°, 210°, 330°]

(*b*) If $A+B+C=180°$, prove that

$$\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$$
$$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$

10. (*a*) Prove that, in a triangle ABC, with the usual notations $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

(*b*) The sides of a triangle are 9, 10 and 11 ; find the angle opposite to the side 10, given L tan $29°30'=9{\cdot}7526420$, L tan $29°29'=9{\cdot}7523472$, log $2={\cdot}30103$. [Ans. 58°59′33″ or 34″]

11. Two chimneys are of equal height. A person standing between them in the line joining their bases, which is horizontal, observes the elevation of the nearer one to be 60°. After walking 80 feet in a horizontal direction perpendicular to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height of the chimneys.

[Ans. $40\sqrt{6}$ ft. or $20\sqrt{6}$ ft.]

12. Draw the graph of $y=\cos x-\sin x$ between the values of $x=-\pi$ and $x=\pi$ and find from the graph the value of x for which $\tan x=1$.

$\left[\text{Ans. } x=\frac{\pi}{4}\right]$

SECOND PAPER

Group A—Plane Geometry

Answer **question 1** *and* **two** *others*

1. Construct a square equal in area to a given rectangle.

Or, Construct a regular hexagon about a given circle.

[*Traces of construction only are required in either of the two constructions.*]

2. Same as Q. 1 (*a*) of H. S., 1960.

ABC is an isosceles triangle and AY is drawn to cut the base internally at Y. Show that $AY^2=AB^2-BY.YC$.

3. Same as Q. 1(*a*) of H.S., 1960 (Compl.).

Prove that the altitudes of two similar triangles are proportional to the corresponding sides.

4. Same as Q. 2(*a*) of H.S., 1962.

In a $\triangle ABC$, perpendiculars AP and BQ are drawn from A and B to opposite sides and intersect at O. Prove that $AO.OP=BO.OQ$.

Group B—Co-ordinate Geometry

Answer **Question 5** *and* **two** *others from this* **group**

5. Find the co-ordinates of the point which divides the st. line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m:n$.

Write down the co-ordinates of the middle point of the st. line joining the points $(7, -4)$ and $(-5, 6)$. [Ans. $(1, 1)$]

6. Find the equation of the straight line passing through the intersection of the st. lines $2x-7y+11=0$ and $x+3y-8=0$, if it

(a) passes through the origin. [Ans. $27x-23y=0$]

(b) is perpendicular to the st. line $2x-5y+6=0$. [Ans. $5x+2y-13=0$]

(c) makes equal intercepts on the two axes. [Ans. $13x+13y-50=0$]

7. Prove that the straight line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y-12=0$ and find the co-ordinates of the point of contact. [Ans. $(-1, -1)$]

8. Find the focus, vertex and directrix of the parabola $(y+3)^2=2(x+2)$. [Ans. $(-\frac{3}{2}, -3), (-2, -3)$; $2x+5=0$]

9. What do you understand by the term eccentricity as applied to a hyperbola ?

Find the equation of the hyperbola whose focus is $(2, 3)$, and directrix, the line $x+2y=1$ and eccentricity $\sqrt{3}$. [Ans. $2x^2-7y^2-12xy-14x-18y+62=0$]

GROUP C—Solid Geometry and Mensuration

10. Give instances from the sides and edges of a cube of :

(a) parallel planes, (b) planes perpendicular to one another, (c) lines parallel to a plane, (d) lines perpendicular to a plane, (e) pairs of skew lines.

Or, Same as Q. 11 of H. S., 1962 (Compl.).

11. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively. Find the height and the area of the total surface of the prism. [Ans. height$=13\frac{1}{3}$ ft., area$=172$ sq. ft.]

Or, A conical tent is required to accommodate 4 people ; each person must have 20 sq. ft. of space on the ground and 100 cu. ft. of air to breathe. Find the height and radius of the tent. [$\pi=\frac{22}{7}$] [Ans. height$=15$ ft., radius$=5{\cdot}05$ ft.]

H. S. Exam. (Compartmental)—1964

First Paper—Group A—Algebra

1. (*a*) Simplify : $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)}-\dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$. $\left[\text{Ans. } \dfrac{\sqrt{6}}{3}\right]$

(*b*) Simplify : $\left\{\dfrac{4^{m+\frac{1}{4}}\times\sqrt{2.2^m}}{2.\sqrt{2^{-m}}}\right\}^{\frac{1}{m}}$. [Ans. 8]

(*c*) If $x=7+4\sqrt{3}$, find the value of $\sqrt{x}-\dfrac{1}{\sqrt{x}}$. [Ans. $2\sqrt{}$[illegible]]

2. (*a*) Solve the equations : $\left.\begin{array}{l}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2}\\ x+y=10\end{array}\right\}$

[Ans. $x=8, y=2$; or, $x=2, y=8$]

(*b*) Assuming that the area of a triangle varies as the altitude and base jointly, and that when the altitude is 18 ft. and base 33 ft. 2 in., the area is $33\frac{1}{6}$ sq. yds. ; find the area of a triangle whose base is 10 ft. 4 in. and altitude 2 ft. 9 in.

[Ans. 1 sq. yd. 5 sq. ft. 30 sq. in.]

3. (*b*) If s_1, s_2, s_3 are the sums of n terms, $2n$ terms and to infinity of a series in G. P., prove that $s_1(s_1-s_3)=s_3(s_1-s_2)$.

(*b*) Prove that $\left\{\dfrac{1+\sqrt{-3}}{2}\right\}^6=1$.

4. (*a*) Find the greatest value of the expression $5+8x-8x^2$, for real values of x. [Ans. 7]

(*b*) If r be the ratio of the roots of the equation $ax^2+bx+c=0$, prove that $\dfrac{(r+1)^2}{r}=\dfrac{b^2}{ac}$.

5. (*a*) Expand $(1+x)^{\frac{1}{5}}$ as far as x^4.

[Ans. $1+\frac{1}{5}x-\frac{2}{25}x^2+\frac{6}{125}x^3-\frac{21}{625}x^4+\cdots$]

(*b*) Find the two middle terms of $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^7$. $\left[\text{Ans. } -35\dfrac{x}{y}, 35\dfrac{y}{x}\right]$

6. (*a*) Find the number of permutations of n dissimilar things taken r at a time ($r<n$).

(*b*) Find the number of triangles which can be formed by joining the angular points of a decagon. [Ans. 120]

7. (*a*) Given log 2 = ·3010300 and log 3 = ·4771213, obtain the logarithms of (i) ·0054 and (ii) $(\cdot 405)^{\frac{1}{5}}$.

[Ans. (*i*) $\bar{3}\cdot7323939$, (*ii*) $\bar{1}\cdot9345759$]

(*b*) Write down the exponential series for e^x; hence expand $\dfrac{e^{5x}+e^x}{e^{3x}}$ in a series of ascending powers of x.

$$\left[\text{Ans. } 2\left\{1+\frac{2^2x^2}{\lfloor 2}+\frac{2^4x^4}{\lfloor 4}+\frac{2^6x^6}{\lfloor 6}+\ldots\ldots\text{to}\infty\right\}\right]$$

Group B—Trigonometry

8. (*a*) Same as Q. 9(*a*) of H. S., 1960 (Compl).

(*b*) If $\tan\theta+\sin\theta=m$, $\tan\theta-\sin\theta=n$, prove that $m^2-n^2=4\sqrt{mn}$.

9. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\cot\theta+\tan\theta=2\sec\theta$. [Ans. $\theta=30°, 90°, 150°, 270°$]

(*b*) If $A+B+C=180°$, prove that

$$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C) = 4\sin A\sin B\sin C.$$

10. (*a*) Same as Q. 11(*a*) or H. S., 1961.

(*b*) The sides of a triangle are 5 cms., 7 cms., and 3 cms. Find its greatest angle. [Ans. 120°]

11. From an observation balloon 7,200 ft. above the sea, the angle of depression of two cruisers are 30° and 45°. Find the distance between the cruisers if one is east and the other south of the balloon. [Ans. 14400 ft.]

12. Draw the graph of cos $2x$ between the values of $x=0°$ and $x=360°$ and read off from the graph the value of cos 120°.

SECOND PAPER, GROUP A—Plane Geometry

1. Draw a mean proportional between two given st. lines. (*Traces and statement only are required but no proof.*)

Or, Draw a direct common tangent to two given circles. (*Traces and statement only are required but no proof*).

2. Same as Q. 3 (*a*) of H. S., 1963.

A triangle ABC is inscribed in a circle and the tangents to the circle at A, B, C form a triangle PQR, QR being the tangent at A and RP the tangent at B. If the angle $P=54°$ and the angle $Q=46°$, calculate the angles of the $\triangle ABC$.

[Ans. $\angle A=63°$, $\angle B=67°$, $\angle C=50°$]

3. Same as Q. 1 (*c*) of H. S., 1961.

XY is drawn parallel to BC, the base of the triangle ABC, cutting the sides AB, AC at X, Y respectively. Given that $AX=3$ in., $XB=1·8$ in. Calculate the ratio of the area of the triangle AXY to that of the quadrilateral $BCYX$. [Ans. 25 : 39]

4. Same as Q. 1(*c*) of H. S., 1960.

Hence prove that in any triangle, the internal bisectors of the three angles are concurrent.

GROUP B—Co-ordinate Geometry

5. What do the following equations represent ?

(*i*) $x^2+y=0$. (*ii*) $x^2+y^2=4$. (*iii*) $x^2+4y^2=4$

Roughly sketch the curves.

[Ans. (*i*) a parabola, (*ii*) a circle, (*iii*) an ellipse]

6. Show that the points $A(3, 3)$, $B(-3, -5)$ and $C(-5, -3)$ are the vertices of an isosceles triangle and find the length of its base.

[Ans. base$=4\sqrt{2}$ units of length]

7. The sides BC, CA, AB of a $\triangle ABC$ have respectively the equations (*i*) $2x+y+1=0$, (*ii*) $2x+3y+1=0$, (*iii*) $3x+4y+3=0$. Find the equation of the altitude through A.

[Ans. $x-2y+11=0$]

8. Show that the tangent at the point (3, 4) to the circle $x^2+y^2=25$ touches the circle $x^2+y^2-8x+6y=0$.

9. Find the equation to the parabola having the point $(1, -3)$ for focus and the straight line $x-2y+3=0$ for directrix.

Show that the straight line $y=mx+\frac{a}{m}$ is a tangent to the parabola $y^2=4ax$. [Ans. $4x^2+y^2+4xy-16x+42y+41=0$]

GROUP C—**Solid Geometry & Mensuration**

10. Same as Q. 7 (*d*) Second part of H. S., 1961.

Or, Same as Q. 7 (*a*) of H. S. (Compl.), 1960.

11. Find (*a*) the slant surface, (*b*) the volume of a right pyramid 15 cm. high, standing on a square base whose side is 16 cm. [Ans. slant surface$=544$ sq. cm., volume$=1280$ c.c.]

Or, A zinc cistern (open at the top) measures externally 3 ft. 3 in. long, 2 ft. 3 in. broad and 2 ft. 1 in. deep and its capacity is 75 gallons. If the bottom of the cistern is 1 in. thick, find the thickness of the sides. [1 cu. ft.$=6\frac{1}{4}$ gallons.] [Ans. $1\frac{1}{2}$ in.]

H. S. EXAMINATION—1965

FIRST PAPER

GROUP A—**Algebra** (Answer *any four* questions)

1. (*a*) If $x=(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}$, find the value of $x^3+3bx-2a$. [Ans. 0]

(*b*) Simplify : $\left\{81^{-\frac{3}{4}}\times\frac{16^{\frac{1}{4}}}{6^{-2}}\times\left(\frac{1}{27}\right)^{-\frac{4}{3}}\right\}^{\frac{1}{3}}$. [Ans. 6]

(*c*) Find the square root of $18+6\sqrt{5}$. [Ans. $\pm(\sqrt{15}+\sqrt{3})$]

2. (*a*) Solve : $\left.\begin{aligned}\frac{x}{y}+\frac{y}{x}&=2\frac{1}{2}\\ x+y&=6\end{aligned}\right\}$ $\left[\text{Ans. } \left.\begin{aligned}x&=4\\y&=2\end{aligned}\right\} \text{ or } \left.\begin{aligned}x&=2\\y&=4\end{aligned}\right\}\right]$

(*b*) If $(x+y)$ varies as $(x-y)$, show that (x^3+y^3) varies as (x^3-y^3).

3. (*a*) *A* sets out from a place and travels at the rate of 5 miles an hour. *B* sets out $4\frac{1}{2}$ hours after *A* and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second hour, 4 miles the third hour and so on. Find in how many hours *B* will overtake *A*. [Ans. 15 hrs.]

(b) If 1, w, w^2 are the three cube roots of unity, prove that $(3+3w+5w^2)^6=(3+5w+5w^2)^6=64$.

4. (a) If α, β be the roots of the equation $ax^2+bx+c=0$, show that the equation whose roots are

$$\frac{1}{\alpha+\beta} \text{ and } \frac{1}{\alpha}+\frac{1}{\beta} \text{ is } bcx^2+(ac+b^2)x+ab=0.$$

(b) Find the algebraically greatest value of $\dfrac{x^2+14x+9}{x^2+2x+3}$ for real values of x.

[Ans. 4]

5. (a) Find the term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ and also its value. [Ans. 9th term, 495]

(b) Find the $(r+1)^{\text{th}}$ term in the expansion of $(1-2x)^{-\frac{3}{2}}$

$\left[\text{Ans. } \dfrac{3.5.7\ldots\ldots(2r+1)}{\lfloor r}x^r\right]$

6. (a) Find the number of permutations of n things taken all together, when the things are not all different.

(b) How many different arrangements of letters can be made by using all the letters of the word CONTACT ? [Ans. 1260]

7. (a) Calculate the numerical value of the expression $\log\left\{\dfrac{(10\cdot8)^{\frac{1}{2}}\times(\cdot24)^{\frac{5}{3}}}{(90)^{-2}}\right\}$, given $\log 2=\cdot3010300$ and $\log 3=\cdot4771213$.

[Ans. : 3·3922160]

(b) Write down the exponential series e^x ; hence obtain a series for $\frac{1}{2}(e^{ix}+e^{-ix})$ in ascending powers of x where $i=\sqrt{-1}$.

Group B—Trigonometry

Answer Question 8 and *any two* from the rest.

8. (a) Same as Q. 9(a) of H. S. 1961.

(b) Show that $\dfrac{1+\sin\theta}{1-\sin\theta}=\tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$.

9. (a) Find the values of θ between 0° ond 180° which satisfy the equation $\sin\theta+\sin 5\theta=\sin 3\theta$. [Ans. : 30°, 60°, 120°, 150°]

(b) If $A+B+C=\frac{\pi}{2}$, prove that

$\tan A \tan B+\tan B \tan C+\tan C \tan A=1$.

10. (a) In any triangle ABC, prove that $\cos A=\frac{b^2+c^2-a^2}{2bc}$.

(b) Prove that $2 \tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}$.

11. Two angles of a triangle are $41°13'22''$ and $71°19'5''$ and the side opposite the first angle is 55 ; find the side opposite the other angle, given $\log 55=1\cdot7403627$, $\log 79063=4\cdot8979775$, $L \sin 41°13'22''=9\cdot8188779$, $L \sin 71°19'5''=9\cdot9764927$.

[Ans. 79·063]

12. Draw the graph of $y=\sin x+\cos x$, as x ranges from 0 to π.

SECOND PAPER

GROUP A—Plane Geometry

1. Draw a transverse common tangent to two given circles.

Or, Construct an equilateral triangle circumscribing a given circle.

[*Traces of construction only are required in either of the two cases.*]

2. (a) Same as Q. 2(a) of H. S., 1963 (Compl.)

(b) Prove that the straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

3. Same as Q. 2(a) of H. S. 1960 (Compl.)

Show that each rectangle is equal to the square on the tangent from the point of intersection.

4. Same as Q. 1(c) of H. S., 1961 (Compl.).

ABC is a triangle right-angled at A and AC' is drawn perpendicular to the hypotenuse ; also $C'A'$ is drawn parallel to CA. If $AC=15$ cm. and $AB=20$ cm. find, geometrically, the lengths of AC' and $C'A'$.

[Ans. $AC'=12$ cm., $A'C'=9\cdot6$ cm.]

Group B—Co-ordinate Geometry

5. Show that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose sides are each of length $2a$. Calculate its area. [Ans. : area $=\sqrt{3}a^2$ sq. units]

6. What do the following equations represent ?

(i) $x^2+y^2-6x+8y=0$, [Ans. a circle]

(ii) $(y+4)^2=4(x-3)$, [Ans. a parabola]

(iii) $(x-3)(y+4)=0$. [Ans. a pair of st. lines]

Roughly sketch the graphs.

7. Express in the form $\frac{x}{a}+\frac{y}{b}=1$, the equation of the st. line passing through the point (3, 2) and the intersection of the line $3x+y-5=0$ and $x+5y+3=0$. $\left[\text{ Ans. : } \frac{x}{\frac{7}{8}}+\frac{y}{-7}=1 \right]$

Find the area of the triangle cut off from the co-ordinate axes by this line. [Ans. : $8\frac{1}{8}$ sq. units]

8. Prove that the centres of the three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=6$ and $x^2+y^2-12x+4y=9$ are collinear and that their radii are in arithmetical progression.

9. Find the centre, the eccentricity, the foci and the lengths of the axes of the ellipse $9x^2+25y^2=225$.

[Ans. centre (0, 0), $e=\frac{4}{5}$, foci $(\pm 4, 0)$, lengths of axes $=10$ and 6 units]

Group C—Solid Geometry & Mensuration

10. (*a*) How do you measure the angle which a straight line makes with a given plane ?

(*b*) When is a st. line said to be perpendicular to a given plane ? (*c*) When are two planes perpendicular to one another ?

(*d*) What are skew lines ?

Illustrate your answer by suitable diagrams.

Or, Same as Q. 7(*a*) of H. S., 1960 (Compl.)

11. The base of a right prism is a triangle whose sides are 1′9″, 1′8″ and 1′1″ long. If the height is 8′, find the whole surface and volume of the prism.

[Ans. : whole surface $=37\frac{3}{4}$ sq. ft., volume $=7$ cu. ft.]

Or, Find, to the nearest tenth of a metre, the height of a conical tent which stands on a circular base of diameter 8·0 metres and which contains 90·478 cubic metres of air [$\pi=3·1416$]

[Ans. : 5·4 metres.]

H. S. EXAMINATION, 1966

First Paper—Group A—Algebra

Answer any four questions

1. (*a*) Prove that the modulus of the product of two complex quantities is equal to the product of their moduli.

(*b*) If $\sqrt[3]{x+iy}=a+ib$, where x, y, a, b, are real and $i=\sqrt{-1}$, show that $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$.

(*c*) Simplify : $\frac{3\sqrt{7}}{\sqrt{5}+\sqrt{2}}-\frac{5\sqrt{5}}{\sqrt{2}+\sqrt{7}}+\frac{2\sqrt{2}}{\sqrt{7}+\sqrt{5}}$. [Ans. 0]

2. (*a*) If α, β are the roots of the equation $x^2-px+q=0$, find the equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.

[Ans. $qx^2-(p^3-3pq)x+q^2=0$]

(*b*) In an integer of two digits, the sum of the digits is 16 and their product 63, the digit in the tens' place being the lesser. Find the integer.

[Ans. 79]

3. (*a*) Find the sum of $1^2+3^2+5^2+\cdots+(2n-1)^2$,

[Ans : $\frac{1}{3}n(4n^2-1)$]

(*b*) The distance through which a heavy body falls from rest varies as the square of the time it falls. A body falls from rest 64 ft. in 2 seconds, how far does it fall in 7 seconds ?

[Ans. ; 784 ft.]

4. (a) Find the number of combinations of n different things taken r at a time, without assuming any formula.

(b) Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

5. (a) If $(r+1)^{th}$ term in the expansion of $\left(9x^2 - \dfrac{1}{3x}\right)^{12}$ be independent of x, find r and also the simplified value of the term.

[Ans. $r=8$, value$=495$]

(b) If $(1+x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$,
prove that $c_0 + 2c_1 + 3c_2 + 4c_3 + \cdots + (n+1)c_n = 2^n + n.2^{n-1}$.

6. (a) Expand $(2-3x)^{-3}$ in ascending powers of x up to the fifth term and state the limit within which x should lie.

[Ans. $\frac{1}{8}\{1 + \frac{9}{2}x + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \frac{1215}{16}x^4\}$, $-\frac{2}{3} < x < \frac{2}{3}$.]

(b) Define e. Write down the exponential series for e^x. Show that $1 + \dfrac{2^3}{\lfloor 2} + \dfrac{3^3}{\lfloor 3} + \dfrac{4^3}{\lfloor 4} + = \cdots = 5e$.

7. (a) Simplify : $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}$, the base of the logarithm being 10.

[Ans. 1]

(b) From the formula $y = Ke^{-0 \cdot 038t}$, find, correct to two decimal places, the value of t when $y = \dfrac{K}{2}$, given

$\log_{10}2 = \cdot 3010$, $\log_{10}e = \cdot 4343$. [Ans. 18·24]

Group B—Trigonometry

Answer question 8 and *any two* from the rest

8: (a) If A, B, $A+B$ are positive acute angles, prove geometrically that $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(b) Prove that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

9. (a) $\sin A = m \sin B$, prove that $\tan\dfrac{A-B}{2} = \dfrac{m-1}{m+1} \tan \dfrac{A+B}{2}$.

(b) Simplify : $\tan^{-1}\dfrac{b-c}{1+bc} + \tan^{-1}\dfrac{c-a}{1+ca} + \tan^{-1}\dfrac{a-b}{1+ab}$.

[Ans. 0]

10. (*a*) Solve the equation $\sin x + \sin 5x = \sin 3x$.

[Ans. $\frac{1}{3}n\pi$, $n\pi \pm \frac{\pi}{6}$, or, $n\pi$, $\frac{n\pi}{2} \pm \frac{\pi}{6}$.]

(*b*) Prove that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

11. (*a*) In a triangle *ABC*, prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circumcircle of the triangle.

(*b*) The angles of a triangle are in the ratio 2 : 3 : 7 and its circumradius is 10 cms. Find the sides of the triangle.

[Ans. 10 cm., $10\sqrt{2}$ cm., $5(\sqrt{6}+\sqrt{2})$ cm.]

12. (*a*) Draw the graph of $\sin 2x$ between the values of $x = 0^\circ$ and $x = 180^\circ$.

(*b*) The shadow of a tower standing vertically on a horizontal plane is 50 ft. longer when the altitude of the sun is 30° than when it is 45°. Find the height of the tower. [Ans. $25(\sqrt{3}+1)$ ft.]

Second Paper

Group A—Plane Geometry

Answer question 1 and *two* others from this group.

1. Draw a square equal in area to a given triangle.

Or, Draw a regular hexagon circumscribing a given circle.

(*Only traces of construction are required in each case.*)

2. First part :—Same as Q. 1(*a*) of H. S., 1962.

The medians of a triangle *ABC* meet at *O*. Prove that

$$AB^2 + BC^2 + CA^2 = 3(AO^2 + BO^2 + CO^2).$$

3. Same as Q. 2(*a*) of H. S., 1960.

If *A*, *B*, *C* are three points on the circumference of a circle such that the chord *AB* is equal to the chord *AC*, prove that the tangent at *A* bisects the exterior angle between *AB* and *AC*.

4. Same as Q. 1(*c*) of H. S., 1961.

The tangent at the vertex *A* to the circumcircle of the triangle *ABC* meets the side *BC* in the point *T*. Prove that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$.

Group B—Co-ordinate Geometry

Answer question 5 ond *two* others from this group.

5. The points $A(4, -1)$, $B(3, 2)$, $C(-1, -2)$ are the vertices of a triangle. Find the length of BC and of the altitude AD. Hence calculate the area of the triangle. Find this area also by co-ordinate geometry and check your result.

[Ans. : $BC=4\sqrt{2}$; $AD=2\sqrt{2}$; Area$=8$.]

6. (*a*) Show that the square of the distance between the two points (x_1, y_1) and (x_2, y_2) on the circle $x^2+y^2=a^2$ is equal to $2(a^2-x_1x_2-y_1y_2)$.

(*b*) Find the equation of the circle which has as diameter the join of the points $(0, -1)$ and $(2, 3)$. Also find the intercept made by the circle on the x-axis.

[Ans. $x^2+y^2-2x-2y-3=0$; Intercept$=4$]

7. (*a*) Find, in the standard form, the equation of the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2=4ax$.

[Ans. $y(t_1+t_2)=2(x+at_1t_2)$]

(*b*) PQ is a chord of the parabola $y^2=4ax$, such that the ordinate of P is twice that of Q. Show that the mid-point of PQ lies on the parabola $5y^2=18ax$.

8. (*a*) Find, in the simplest form, the equation to the ellipse, whose focus is the point $(-1, 1)$, whose directrix is the straight line $x-y+3=0$, and whose eccentricity is $\frac{1}{2}$.

[Ans. $7x^2+2xy+7y^2+10x-10y+7=0$]

(*b*) Show that the line $x+y-3=0$ is a tangent to the ellipse $3x^2+6y^2=18$, and find the point of contact. [Ans. $(2, 1)$]

9. (*a*) Find the equation of the tangents common to the hyperbola $\frac{x^2}{4}-\frac{y^2}{2}=1$ and the circle $x^2+y^2=1$.

[Ans. $y=\pm x\pm\sqrt{2}$]

(*b*) The ends B, C of the base of a triangle are $(a, 0)$, $(-a, 0)$. Find the locus of the vertex A when $AB^2+AC^2=2c^2$, where c is a constant.

[Ans. $x^2+y^2=c^2-a^2$]

GROUP C—Solid Geometry & Mensuration

10. Same as Q. 7(*d*) of H. S., 1961.

Or, Explain with the help of figures the possible relations between—(*a*) a given straight line and a plane ; (*b*) two given straight lines in space.

Show how through any point a plane can be drawn parallel to a given plane. (Proof not required)

11. If a solid homogeneous iron ball of 4 inches diameter weigh 9 lbs., what is the weight of an iron shell, made of same material, whose external and internal diameters are 9 inches and 6 inches respectively ? [Ans. $72\frac{9}{64}$ lbs.]

Or, A sphere has the same volume as a right circular cone with its height equal to twice the radius of its base. Find the ratio of the curved surface of the cone to the surface of the sphere.

[Ans. $5^{\frac{1}{2}} : (16)^{\frac{1}{3}}$.]

H. S. EXAMINATION, 1967

First Paper—GROUP A—Algebra

Answer *any four* questions.

1. (*a*) Simplify :

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}. \qquad [\text{Ans. } \sqrt{2}]$$

(*b*) If $x=1+3^{\frac{2}{3}}+3^{\frac{1}{3}}$, prove that $x^3-3x^2-6x-4=0$.

2. (*a*) Solve the equations :

$$\left.\begin{aligned} 3x+4y&=18 \\ \frac{1}{x}+\frac{1}{y}&=\frac{5}{6} \end{aligned}\right\} \qquad \left[\text{Ans. } \left.\begin{aligned} x&=2 \\ y&=3 \end{aligned}\right\} \text{ or } \left.\begin{aligned} x&=\tfrac{18}{5} \\ y&=\tfrac{9}{5} \end{aligned}\right\}\right]$$

(*b*) Find the square root of $a+b+\sqrt{2ab+b^2}$.

$$\left[\text{Ans. } \pm\frac{1}{\sqrt{2}}\left(\sqrt{2a+b}+\sqrt{b}\right)\right]$$

3. (*a*) The volume of a pyramid varies jointly as its height and the area of its base ; and when the area of the base is 60 sq. feet and the height 14 feet, the volume is 280 cubic feet. Find, by variation, the area of the base of the pyramid whose volume is 390 cu. ft. and whose height is 26 feet. [Ans : 45 sq. ft.]

(*b*) Find the sum to n terms of the series

$$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\cdots$$

[Ans. $n(4n^2+9n+6)$]

4. (*a*) If x is real, prove that the expression $\dfrac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

(*b*) Determine the values of m for which $3x^2+4mx+2=0$ and $2x^2+3x-2=0$ may have a common root.

$\left[\text{Ans. } m=\dfrac{7}{4} \text{ or } -\dfrac{11}{8}\right]$

5. (*a*) Find the $(r+1)^{th}$ term in the expansion of $(1-2x)^{-4}$.

$\left[\text{Ans. } \dfrac{(r+1)(r+2)(r+3)}{\underline{|3}}2^r.x^r.\right]$

(*b*) Show that $1+\dfrac{1}{4}+\dfrac{1\,.\,3}{4\,.\,8}+\dfrac{1\,.\,3\,.\,5}{4\,.\,8\,.\,12}+\cdots=\sqrt{2}$.

6. (*a*) Find the number of permutations of n things taken all together, when the things are not all different.

(*b*) Find how many odd numbers of five significant digits can be formed with the digits 3, 2, 7, 4, 0, each digit not occurring more than once in each number.

[Ans. 36]

7. (*a*) Prove that $\log_a m=\text{tog}_b m\times\log_a b$.

(b) Calculate the value of

$$\left\{\frac{(\cdot 32)^3 \times (625)^4}{(\cdot 00432)^2 \times (\cdot 3125)^3 \times 25}\right\}^{\frac{1}{8}} \text{ given } \log 2 = \cdot 3010300,$$

$\log 3 = \cdot 4771213$ and $\log 259569 = 5\cdot 4142524$, (correct to 7 places of decimals). [Ans. 259·569]

Group B—Trigonometry

Answer *question 8* and *any two* from the group.

8. (a) Same as Q. 10 (a) of H. S., 1960.

(b) Find the value of $\cos^2\theta + \cos^2(120° - \theta) + \cos^2(120° + \theta)$. [Ans. $\frac{3}{2}$]

9. (a) Solve the equation $\cot\theta - \tan\theta = 2$. $\left[\text{Ans. } \theta = (4n+1)\frac{\pi}{8}.\right]$

(b) If $A+B+C=180°$, prove that

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

10. (a) If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, prove that

$\cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$. [See sum No. 26 of Ex. 5 of this Book]

(b) Prove that $2\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}\right) = \cos^{-1}\frac{3}{5}$.

11. (a) Same as Q. 10 (a) of H. S., 1964.

(b) Same as Q. 10 (b) of H. S., 1964.

12. Draw the graph of $y = \sin x - \cos x$ as x ranges from $-\pi$ to $+\pi$. Read off from the graph, the value of x for which $\tan x = 1$. $\left[\text{Ans. } x = \frac{\pi}{4}, -\frac{\pi}{4}.\right]$

SECOND PAPER

GROUP A—Plane Geometry

Answer *question 1* and *two others* from this group.

1. Same as Q. 1 of H. S., 1964.

Or, Same as Q. 1. (*or*) of H. S., 1964 (Compl.).

2. (*a*) Same as Q. 1 (*c*) of H. S., 1961.

(*b*) ABC is a triangle, L and M are points in AB and AC respectively such that LM is parallel to BC. If $AL = 5$ cms. and $LB = 4$ cms., find the value of $\dfrac{\text{area } ALM}{\text{area } LMCB}$. $\left[\text{Ans. } \dfrac{25}{56}\right]$

3. (*a*) Same as Q. 2(*a*) of H. S., 1962.

(*b*) Same as Q. 2 (*b*) of H. S., 1960 (Compl.).

4. (*a*) If G be the centroid of the $\triangle ABC$, show that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

(*b*) Same as Q. 2(*b*) *or*, of H. S., 1960 (Compl.).

GROUP B—Co-ordinate Geometry

Answer *question 5* and *two others* from this group

5. Show that the three points (3, 1), (5, −5) and (−1, 13) are collinear. Find the equation of the straight line on which they lie. [Ans. $3x + y = 10$]

6. (*a*) Find the co-ordinates of the centroid of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

(*b*) The vertices of a triangle are (2, 7), (5, 1) and (x, 3); its area is 18. What is the value of x? [Ans. $x = 10$ or -2]

7. (*a*) Obtain the equation of the circle whose centre is the point (2, 3) and which passes through the centre of the circle

$x^2 + y^2 + 2x + 2y - 5 = 0$. [Ans. $x^2 + y^2 - 4x - 6y - 12 = 0$]

(*b*) Find the equation of the tangents to the circle

$$x^2 + y^2 - 6x - 8y + 9 = 0$$

which are parallel to the line $3x + 4y + 5 = 0$

[Ans. $3x + 4y = 5$ and $3x + 4y = 45$]

8. (*a*) Find the latus rectum and the co-ordinates of the focus of the parabola $3y^2 = 4x$. Determine the points at which it is met by the straight line $2x = 3y$.

[Ans. $LL' = \frac{4}{3}$; $S(\frac{1}{3}, 0)$; Points (0, 0) and (3, 2)]

(*b*) Show that for any value of m, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$; also find the co-ordinates of the point of contact. $\left[\text{Ans. } \left(\frac{a}{m^2}, \frac{2a}{m}\right)\right]$

9. (*a*) If p, p' are the lengths of perpendiculars from the foci upon a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $pp' = b^2$.

(*b*) Show that the straight line $y = x + \sqrt{5}$ is a tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$; find the co-ordinates of the point of contact.

$\left[\text{Ans. } \left(-\frac{9}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right).\right]$

Group C—Solid Geometry and Mensuration

10. Same as Q. 7 (*d*) Second part of H. S., 1961.

Or, (*a*) When is a st. line said to be perpendicular to a given plane ? Illustrate your answer by a suitable diagram.

(*b*) Same as Q. 12 of H. S., 1962 (Compl.).

11. A right circular cone, 20 cm. high, has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis and if the radius of the original cone be 4 cm., find the volume of the truncated cone. ($\pi = \frac{22}{7}$). [Ans. $293\frac{1}{3}$ c.c.]

Or, Determine the volume of a pyramid whose height is $10\sqrt{7}$ ft., and which stands on a triangle of sides 16 ft., 11 ft. and 9 ft. [Ans. 420 cu. ft.]

H. S. EXAMINATION—1968

FIRST PAPER—GROUP A—Algebra

Answer *any four* questions.

1. (*a*) Simplify : $\dfrac{\sqrt{5}}{\sqrt{3}+\sqrt{2}}-\dfrac{3\sqrt{3}}{\sqrt{2}+\sqrt{5}}+\dfrac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}$. [Ans. 0]

(*b*) If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, prove that

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}.$$

2. (*a*) Same as Q. 2(*a*) of H. S. 1963.

(*b*) Find the fraction such that the product of its numerator and denominator is 21 and the denominator when diminished by 1 becomes double the numerator. [Ans. $\frac{3}{7}$]

3. (*a*) The volume of a cone varies jointly as its height and the area of its circular base. The volume of the cone is 50 c. ft. when its height is 15 ft. and the area of its base is 10 sq. ft. Find the radius of its circular base when the volume of the cone is 770 c. ft. and its height is 15 ft. ($\pi=\frac{22}{7}$). [Ans. 7 ft.]

(*b*) Same as Q. 3(*b*) of H. S. 1960 (compl.).

4. (*a*) Prove that the roots of the equation $ax^2+bx+c=0$, where a, b and c are real, are both real or both imaginary.

(*b*) If $x^2+px+q=0$ and $x^2+qx+p=0$ have a common root, show that either $p=q$ or $p+q+1=0$.

5. (*a*) Same as Q. 7.(*a*) of H. S. 1961.

(*b*) In how many ways can 5 boys and 3 girls be arranged so that no two girls may come together ? [Ans. 14400 ways]

6. (*a*) Find the simplified value of the term independent of x in the expansion of $\left(9x^2-\dfrac{1}{3x}\right)^{12}$. [Ans. 495]

(*b*) In the expansion of $(1+x)^{m+n}$, where m and n are positive integers, prove that the co-efficients of x^m and x^n are equal.

7. (*a*) If $a>0$, $b>0$, prove that $\log_a b^r = r \log_a b$, r being any number.

(*b*) Same as Q. 6(*b*) of H. S., 1962 (compl.).

(*c*) Find the number of digits in 2^{64}, given $\log 2 = \cdot 30103$.

[Ans. 20]

Group B—Trigonometry

Answer *question 8* and *any two* from the rest.

8. (*a*) Same as Q. 8(*a*) of H. S., 1966.

(*b*) Show that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.

9. (*a*) Solve : $\tan \theta + \cot 2\theta = 2$.

$\left[\text{either } \frac{n\pi}{2} + (-1)^n \frac{\pi}{12};\ \text{or, } n\pi + \frac{\pi}{12},\ n\pi + \frac{5\pi}{12}\right]$

(*b*) If $\tan^2\theta = 1 + 2\tan^2\phi$, prove that $\cos 2\phi = 1 + 2\cos 2\theta$.

10. (*a*) Solve : $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$.

[Ans. $x = \frac{1}{2}$, $y = 1$]

(*b*) Same as Q. 10(*b*) of H. S. 1966.

11. (*a*) If a, b, c be the sides of a triangle, find an expression for the area of the triangle in terms of a, b and c.

(*b*) Same as Q. 10(*b*) of H. S. '64.

12. (*a*) Draw the graph of $\cos x$ between $x = -\pi$ to $x = \pi$.

(*b*) The angular elevation of a tower at a place A due South of it is 30° and at a place B due West of A and at a distance a from A, the elevation is 18°. Show that the height of the tower is $\frac{a}{\sqrt{2+2\sqrt{5}}}$.

SECOND PAPER

Group A—Plane Geometry

Answer question *1* and *two* others from this Group.

1. Draw a direct common tangent to two given circles.

Or, Draw a square equal in area to a given triangle.

(Only traces of construction are to be given in each case).

2. Same as Q. 1(*a*) of H. S., 1960.

ABC is an isosceles acute-angled triangle with $AB = AC$. CD is drawn perpendicular from C on AB. Prove that $BC^2 = 2AB.BD$.

3. Same as Q. 2(*a*) of H. S. 1960 (Compl.).

If the chord AB and tangent PT of a circle intersect at an external point P, then show that $PT^2 = PA.PB$.

4. Same as Q. 1(*a*) of H. S. 1960 (compl.).

Prove that the line joining the middle points of any two sides of a triangle is parallel to the third side and is equal to half that side in length.

GROUP B—Co-ordinate Geometry

Answer question *5* and *two* others from this Group.

5. The co-ordinates of two vertices of a triangle are (3, 2) and (5, 6). Find the equation to the locus of the third vertex if the area of the triangle is 12 square units.

[Ans. $y = 2x + 8$ and $y = 2x - 16$]

6. Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the straight line $5x - 2y + 7 = 0$ and express it in the form $\frac{x}{a} + \frac{y}{b} = 1$.

$$\left[\text{Ans. } 2x + 5y + 1 = 0, \text{ and } \frac{x}{-\frac{1}{2}} + \frac{y}{-\frac{1}{5}} = 1.\right]$$

Show that the centres of the circles,

$$x^2 + y^2 = 1,\ x^2 + y^2 + 6x - 2y - 1 = 0,\ x^2 + y^2 - 12x + 4y - 1 = 0$$

are collinear.

7. Find the equation of the circle which touches the x-axis in the first quadrant at a distance 5 units from the origin and cuts off a chord of length 24 units from the y-axis.

[Ans. $x^2 + y^2 - 10x \pm 26y + 25 = 0$]

Show that the straight line $y = mx + \frac{a}{m}$, $(m \neq 0)$ touches the parabola $y^2 = 4ax$. Find the co-ordinates of the point of contact.

[Ans. $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$]

8. A tangent to the parabola $y^2 = 12x$ makes an angle $45°$ with the axis. Find its equation and its point of contact with the parabola.

[Ans. $y = x + 3$ and $(3, 6)$]

Prove that the straight line $x + y - 3 = 0$ touches the ellipse $3x^2 + 6y^2 = 18$ and find the co-ordinates of the point of contact.

[Ans. $(2, 1)$]

9. Find the condition that the straight line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[Ans. $a^2l^2 + b^2m^2 = n^2$]

Find the equation of the tangents to the hyperbola $3x^2 - y^2 = 3$ which are parallel to the straight line $2x - y = 1$

[Ans. $2x - y \pm 1 = 0$]

GROUP C—Solid Geometry and Mensuration

10. Same as Second part of Q. 7(*b*) of H. S. 1960.

Or, (*a*) How is the angle between two planes measured ? When is one plane said to be perpendicular to another ? (Give suitable diagrams with your answers).

(*b*) If a right angle revolves round one of its arms, prove that the locus of the other arm is a plane.

11. The dimensions of a rectangular parallelopiped are in the ratio 5 : 3 : 2. Determine the dimensions if the whole surface of the parallelopiped is 558 sq. cm.

[Ans. 15 cm., 9 cm., 6 cm.]

Or, The base of a right pyramid is a rectangle of dimensions 24″ and 18″. If the slant edge of the pyramid is 17″, find its height and volume.

[Ans. $h = 8$ in. ; vol. $= 1152$ cu. in.]

C. U. PRE-UNIVERSITY EXAMINATION—1961

Group A—Algebra

1. (*a*) Find the square root of $5-\sqrt{10}-\sqrt{15}+\sqrt{6}$.

[Ans. $\pm\left(1+\sqrt{\frac{3}{2}}-\sqrt{\frac{5}{2}}\right)$]

(*b*) In a "Geometrical Progression", the first term is 5, the last term is 320 and the sum of the series is 635 ; find the 4th term.

[Ans. 40]

2. (*a*) If α, β are roots of the equation $2x^2+3x+3=0$, find the value of $\alpha^3\beta^5+\alpha^5\beta^3$.

[Ans. $-\frac{81}{32}$]

(*b*) Use logarithmic tables to calculate the value of $2{\cdot}41\times(1{\cdot}24)^{\frac{1}{3}}\div(0{\cdot}78)^{\frac{1}{2}}$, correct to two places of decimals.

[Ans. 2·93]

3. (*a*) Find the number of permutations of n dissimilar things taken p at a time ($n>p$).

(*b*) Calculate the number of ways in which the letters of the word DRAUGHT can be arranged so that the vowels are always together.

[Ans. 1440]

4. (*a*) State and prove the "Binomial Theorem" for a positive integral index.

(*b*) Calculate the coefficient of x^{-11} in the expansion of

$$\left(x^2-\frac{1}{x^3}\right)^{12}.$$

[Ans. -792]

Group B—Trigonometry

5. (*a*) Show geometrically, that if A and B are positive acute angles and $A>B$, then $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

(*b*) If A, B, C are the angles of a triangle, shew that

$$\sin\frac{A}{2}+\cos\frac{B-C}{2}=2\cos\frac{B}{2}\cos\frac{C}{2}.$$

6. (*a*) Shew that $\dfrac{1+\sin\theta}{1-\sin\theta}=\tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$.

(*b*) Shew that

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

7. (a) Same as Q. 10(a) of H. S., 1963.

(b) The sides of a triangle are 8 cms., 15 cms., 17 cms. Find its greatest angle. [Ans. $90°$]

8. (a) Find the value of $\sin 18°$. [Ans. $\frac{1}{4}(\sqrt{5}-1)$]

(b) Draw a neat graph of $\sin x$ in the region $-\pi \leqslant x \leqslant \pi$.

Group C—Co-ordinate Geometry

9. (a) The co-ordinates of A, B, C are $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the middle points of BC, CA, AB respectively. Calculate the area of the triangle DEF.

[Ans. 4 sq. units]

(b) Obtain the equation of the straight line through the point $(2, 1)$ and perpendicular to the line joining the points $(2, 3)$ and $(3, -1)$. [Ans. $x-4y+2=0$]

10. (a) Obtain the equation of the locus of a point which moves in the plane of (xy) in such a way that its distance from the point $(2, 3)$ is always two-thirds of its distance from y-axis.

[Ans. $5x^2+9y^2-36x-54y+117=0$]

(b) Find the equation of the tangent to the parabola $y^2=4ax$ at the point (x', y').

11. (a) Show that the centres of the following three circles are in a straight line :

$$x^2+y^2-2x-6y-5=0,\ x^2+y^2-4x-10y-7=0,$$
$$x^2+y^2-6x-14y-9=0.$$

(b) Find the eccentricity and the co-ordinates of the foci of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$. [Ans. $e=\frac{4}{5}$; co-ordinates $=(\pm 4, 0)$]

12. (a) Obtain the equations of the lines which bisect the angles between the lines (i) $a_1x+b_1y+c_1=3$ and
(ii) $a_2x+b_2y+c_2=0$.

(b) Obtain the equation of the circle which has its centre at the point $(3, 4)$ and touches the straight line $5x+12y=1$.

[Ans. $x^2+y^2-6x-8y+\frac{381}{169}=0$]

Group D—Solid Geometry & Mensuration

13. Same as Q. 7(*b*) Second Part of H. S., 1960.

14. (*a*) Find the locus of a point in space equidistant from two given points.

(*b*) Three solid spheres of gold whose radii are 1 cm., 6 cms., and 8 cms. respectively are melted into a single gold sphere. Find the radius of the sphere so formed. [Ans. 9 cm.]

C. U. PRE-UNIVERSITY EXAM.—1962

Group A—Algebra

1. (*a*) If $x+iy=\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$, where $i=\sqrt{-1}$, find x and y.

$$\left[\text{Ans. } x=\frac{4}{7}, y=-\frac{\sqrt{6}}{14}\right]$$

(*b*) In an *A. P.* the first term is 2, the last term 29, the sum 155 ; find the common difference. [Ans. 3]

2. (*a*) If P varies as the sum of two quantities of which one varies directly as x and the other inversely as x, and if $P=6$ when $x=4$ and $P=3\frac{1}{3}$ when $x=3$, find P when $x=2$. [Ans. 0]

(*b*) Use logarithmic tables to calculate the value of

$$\frac{(48{\cdot}7)^{\frac{1}{3}}\times({\cdot}00321)^{\frac{1}{2}}}{0{\cdot}372}. \qquad [\text{ Ans. } {\cdot}5563]$$

3. (*a*) If x be real, show that $\dfrac{x^2+2x-3}{x^2+2x-8}$ cannot lie between $\frac{4}{9}$ and 1.

(*b*) If $C_0, C_1, C_2, \ldots, C_n$ denote the coefficients in the expansion of $(1+x)^n$ where n is a positive integer, shew that

$$C_1-2C_2+3C_3-\cdots\cdots+n(-1)^{n-1}C_n=0.$$

4. (*a*) Find the number of ways in which m things may be arranged among themselves, taking them all at a time, when b of the things are alike of one kind, c of the things alike of a second kind and the rest all different.

(*b*) Find the number of ways in which n books can be arranged on a shelf so that two particular books are not together.

[Ans. $(n-2)\underline{|n-1}$]

Group B—Trigonometry

5. (*a*) Shew, geometrically, that if A and B are positive acute angles and $(A+B)<90^\circ$, then

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

(*b*) Show that $\cos A+\cos B+\cos C+\cos(A+B+C)$

$$=4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$$

6. (*a*) Solve and find a general value for x, where

$\cos x+\sqrt{3}\sin x=1.$ $\left[\text{Ans. } x=2n\pi+\frac{2\pi}{3}, \text{ or, } 2n\pi.\right]$

(*b*) Shew that $\tan^{-1}x+\cot^{-1}y=\tan^{-1}\dfrac{xy+1}{y-x}.$

7. (*a*) If a, b, c are the sides and A, B, C the angles opposite to the sides a, b, c respectively, in a triangle ABC, shew that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

(*b*) A triangular courtyard has two of the sides of length 32 and 48 metres respectively and the angle included between those sides is $64^\circ 36'$. Calculate the angles at the remaining vertices of the courtyard, having given

$\log_{10} 2=\cdot30103$; $L\cot 32^\circ 18'=10\cdot19916$;

$L\tan 17^\circ 33'=9\cdot50004$; $L\tan 17^\circ 34'=9\cdot50048$.

[Ans. $B=75^\circ 15' 20\cdot5''$; $C=40^\circ 8' 39\cdot5''$]

8. (*a*) The shadow cast by a telegraph post is 6 metres longer when the sun's altitude is 30° than when it is 45° ; shew that the height of the post is $3(1+\sqrt{3})$ metres.

(*b*) Draw a neat graph of $\cos x$, in the region $-\pi\leqslant x\leqslant\pi$.

Group C—Co-ordinate Geometry

9. (*a*) Find the angle between the two st. lines

$$y=mx+c,\ y=m'x+c'.$$

(*b*) Obtain the equations to the st. lines each of which passes through the point $(2, -1)$ and intersects the axes of co-ordinates

at points equidistant from the origin and calculate the angle between them. [Ans. $x+y=1$, $x-y=3$; a right angle.]

10. (*a*) Obtain the equation to the circle which passes through the points $(2, -1)$ and $(3, -2)$ and has its centre on the straight line $2x+4y-3=0$. [Ans. $3x^2+3y^2-19x+5y+28=0$]

(*b*) A conic is represented by the equation $4x^2-9y^2=36$; calculate its eccentricity, length of latus rectum and the co-ordinates of the foci.

$$\left[\text{Ans. } e=\frac{\sqrt{13}}{3} \text{ ; latus rectum}=\frac{8}{3} \text{ ; foci are } (\pm\sqrt{13}, 0) \right]$$

11. (*a*) Shew that the chord of a parabola $y^2=4ax$, whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$, is a normal to the parabola, and find the co-ordinates of the point of the parabola at which it is the normal. [Ans. Co-ordinates are $(2a, -2a\sqrt{2})$]

(*b*) Find possible values of k for which the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10x$.

[Ans. 40 and -10]

12. (*a*) Same as Q. 5(*d*) of H. S., 1961 (Compl.).

(*b*) A point P moves in the plane of (xy) in such a way that its distances from the lines $12x+5y-4=0$ and $3x+4y+7=0$ are equal ; obtain the equation to the locus traced out by P.

[Ans. $7x-9y-37=0$, or, $99x+77y+71=0$]

Group D—Solid Geometry & Mensuration

13. (*a*) Same as Q. 7 (*a*) of H. S., 1960 (Compl.).

(*b*) The diagonal of a rectangular block is 10 cms., and the sum of the lengths of its edges is 80 cms., calculate the total area of the outer surface of the block. [Ans. 6300 sq. cm.]

14. (*a*) Shew that if a st. line is perpendicular to a plane, then every plane passing through it is perpendicular to that plane.

(*b*) A right circular cone is 10 cms. high and its slant height is 15 cms. Calculate the volume of the cone. $[\pi=\frac{22}{7}]$.

[Ans. $1309\frac{11}{21}$ cu. cms.]

C. U. PRE-UNIVERSITY EXAM.—1963

Group A—Algebra

1. (a) Express $(3+i)(4+3i)(5+7i)$ in the form of $A+iB$, where $i=\sqrt{-1}$. [Ans. $-46+128i$ where $-46=A$, $128=B$]

(b) If x, y, z be the p^{th}, q^{th} and r^{th} terms respectively of a G. P., prove that $x^{q-r}\, y^{r-p}\, z^{p-q}=1$.

2. (a) Solve $x+y=5$, $x^2+2y^2=17$.

[Ans. $x=3$, $y=2$, or, $x=\frac{11}{3}$, $y=\frac{4}{3}$]

(b) If α and β be the roots of the equation $3x^2+6x+2=0$, find the equation whose roots are $-\dfrac{\alpha^2}{\beta}$ and $-\dfrac{\beta^2}{\alpha}$.

[Ans. $3x^2-18x+2=0$]

(3). (a) Prove the Binomial Theorem for a positive integral index.

(b) If the $(r+1)th$ term in the expansion of $\left(x+\dfrac{1}{x}\right)^{10}$ be independent of x ; find r.

[Ans. $r=5$]

4. (a) Find the number of ways in which n different books can be arranged on a shelf so that two particular books are always together.

[Ans. $2\,\underline{|n-1}$]

(b) Prove that $7 \log \frac{16}{15}+5 \log \frac{25}{24}+3 \log \frac{81}{80}=\log 2$.

Group B—Trigonometry

5. (a) Prove geometrically that if A and B are positive acute angles and $A+B<90^\circ$, $\cos (A+B)=\cos A \cos B-\sin A \sin B$.

(b) Express $\cos 4\theta$ in terms of $\cos \theta$.

[Ans. $8 \cos^4\theta-8 \cos^2\theta+1$]

6. (a) Solve the equation $2(\cos^2 x-\sin^2 x)=1$.

$\left[\text{Ans.}\quad x=n\pi\pm\dfrac{\pi}{6}\right]$

(b) Show that $\tan 3\theta=\dfrac{3 \tan \theta-\tan^3\theta}{1-3 \tan^2\theta}$.

7. (*a*) Same as Q. 10(*a*) of H. S., 1964.

(*b*) Same as Q. 10(*b*) of H. S., 1964.

8. (*a*) The angle of elevation of the top of a tower is 45° ; on walking 100 feet nearer the elevation is found to be 60°. Show that the height of the tower is $50(3+\sqrt{3})$ feet.

(*b*) Draw the graph of $\sin x$ from $-\pi \leqslant x \leqslant \pi$.

Group C—Co-ordinate Geometry

9. (*a*) Same as Q. 4 (*a*) of H. S., 1961.

(*b*) If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the co-ordinates of the vertices A, B, C respectively of the triangle ABC and if D is the middle point of BC, find the co-ordinates of G which divides AD such that $AG=2GD$.

[Ans. $\{\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)\}$]

10. (*a*) Find the equation of the st. line passing through the point (3, 5) and parallel to the line $4x-3y+1=0$.

[Ans. $4x-3y+3=0$]

(*b*) Find the equations of the tangents to the circle $x^2+y^2=9$ which are parallel to the line $3x+4y=0$.

[Ans. $3x+4y+15=0$, $3x+4y-15=0$]

11. (*a*) Find the locus of the point whose distance from the point $(-1, 1)$ is equal to its perpendicular distance from the st. line $x+y+1=0$. [Ans. $x^2+y^2-2xy+2x-6y+3=0$]

(*b*) Find the co-ordinates of the focus of the parabola $y^2=4ax$ which passes through the point of intersection of the lines $\frac{x}{3}+\frac{y}{2}=1$ and $\frac{x}{2}+\frac{y}{3}=1$.

[Ans. $(\frac{3}{10}, 0)$]

12. (*a*) If the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be equal to the distance between the foci, find the eccentricity of the ellipse.

$\left[\text{Ans. } \frac{1}{\sqrt{2}}\right]$

(*b*) Find the equation of the ellipse referred to its centre as origin and major axis as x-axis, whose latus rectum is 5 and eccentricity is $\frac{2}{3}$.

[Ans. $20x^2+36y^2=405$]

Group D—Solid Geometry & Mensuration

13. (*a*) Prove that two intersecting planes cut one another in a straight line and in no point outside it.

(*b*) Same as Q. 7 (*b*) Second Part of H. S., 1960.

14. (*a*) Three solid gold spherical beads of radii 3, 4, 5 cms. respectively are melted into one solid spherical bead. Find its radius. [Ans. 6 cms.]

(*b*) The volume of a right circular cylinder and a right circular cone standing on the same base are as 3 : 2. Show that the height of the cone is double the height of the cylinder.

C. U. PRE-UNIVERSITY—1964

Group A—Algebra

1. (*a*) If $x=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $y=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of $3x^2-5xy+3y^2$. [Ans. 289]

(*b*) Express $(\sqrt{3}-\sqrt{2}i)(2\sqrt{3}-\sqrt{2}i)$ in the form of $A+iB$, where $i=\sqrt{-1}$ and A, B are real.

[Ans. $4-3\sqrt{6}i$ where $A=4$, $B=-3\sqrt{6}$]

2. (*a*) Same as Q. 4 (*b*) of H. S. 1964.

(*b*) Solve $\left.\begin{array}{l} x^2+y^2=25 \\ x+y=7 \end{array}\right\}$ $\left[\text{Ans. } \left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\}\right]$

3. (*a*) Find the number of permutations of n things taken all together when the things are not all different.

(*b*) If $(1+x)^n=c_0+c_1x+c_2x^2+\cdots+c_nx^n$, prove that $C_0^2+C_1^2+C_2^2+\cdots+C_n^2=\lfloor 2n/(\lfloor n)^2$.

4. (*a*) Simplify : $16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}+\log 2$, the base of the logarithm being 10. [Ans. 1]

(*b*) Evaluate (using logarithmic tables) :

$\dfrac{5\cdot631\times42\cdot13\times\cdot2783}{2\cdot451\times\cdot8392\times12\cdot61}$. [Ans. 2·545]

Group B—Trigonometry

5. (a) Same as Q. 5 (a) of C. Pre-U. 1961.

(b) If A, B, C be the angles of a triangle, prove that $\sin 2A+\sin 2B+\sin 2C=4 \sin A \sin B \sin C$.

6. (a) Same as Q. 6 (a) of H. S. 1966.

(b) Solve :

$$\tan^{-1}x+\tan^{-1}2x+\tan^{-1}3x=\pi.$$ [Ans. $x=0$ or ± 1]

7. (a) Same as Q. 7(a) of C. Pre-U. 1962.

(b) Two sides of a triangle are 11 cm. and 9. cm. and the included angle is 60°. Find the remaining angles, given

$\log 3=\cdot 4771213$; $L \tan 9°49'=9\cdot 2381203$; diff. for $1'=7514$.

[Ans. 69°49′35·2″, 50°10′24·8″]

8. (a) Draw the graph of $\cos x$ between $x=-\pi$ and $x=\pi$.

(b) From an aeroplane vertically above a st. horizontal road, the angles of depression of two consecutive mile-stones are observed to be 30° and 45°. If the mile-stones be on the opposite sides of the aeroplane, show that its height, in miles, above the road is $\frac{1}{1+\sqrt{3}}$.

Group C—Co-ordinate Geometry

9. (a) The three points, whose co-ordinates are (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie in a st. line, show that

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

(b) Find the angle between the st. lines,

$x-\sqrt{3}y=1$ and $\sqrt{3}x-y=4$. [Ans. 30° or 150°.]

10. (a) Find the equation of the circle, having its centre at $(1, -2)$ and passing through the point of intersection of the straight lines $3x+y=14$ and $2x+5y=18$.

[Ans. $x^2+y^2-2x+4y-20=0$]

(b) The angle between the two tangents drawn from a point P to the circle $x^2+y^2=a^2$ is 120°. Prove that the locus of P is $x^2+y^2=4a^2/3$.

11. (a) Same as Q. 9 (a) of H. S. 1962.

(b) Find the equation to the normal to the parabola $y^2=4ax$ at the point $(am^2, 2am)$. [Ans. $y+mx-2am-am^3=0$]

12. (a) Find the latus rectum, the eccentricity and the co-ordinates of the foci of the ellipse

$\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$. [Ans. latus rectum $=3\frac{3}{5}$, $e=\frac{4}{5}$, foci $(\pm 4, 0)$]

(b) Obtain the equations of the tangents to the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$, which are parallel to the straight line $\sqrt{3}x-2y=0$.

[Ans. $2y=\sqrt{3}x\pm\sqrt{39}$]

Group D—Solid Geometry & Mensuration

13. (a) Same as Q. 7(a) of H. S., 1960 (Compl.)

(b) If a triangle revolves about its base, prove that the locus of its vertex is a circle.

14. (a) A right prism stands on a triangular base whose sides are 18", 20", 34". If the height of the prism is 10", find the area of its total surface. [Ans. 1008 sq. in.]

(b) How many square feet of canvas are required for a conical tent 24 feet high, the diameter of the base being 14 feet ? [Ans. 550 sq. ft.]

[$\pi \doteq \frac{22}{7}$]

B. U. ENTRANCE EXAMINATION,—1961

Group A—Algebra

1. (a) If the roots of the equation $ax^2+bx+c=0$ bears to one another the ratio 3 : 4, prove that $12b^2=49ac$.

(b) Solve $x+y+3\sqrt{x+y}=x^2+y^2=10$.

[Ans : $x=3$, $y=1$; or, $x=1$, $y=3$]

(ii) $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2}$, $\frac{1}{x+y}=\frac{1}{3}$ [Ans : $x=2$, $y=1$; or, $x=1$, $y=2$]

2. (*a*) Prove ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$

and ${}^nC_1+{}^nC_2+{}^nC_3+\cdots\cdots+{}^nC_n=2^n-1$.

(*b*) How many triangles can be formed by joining the angular points of a decagon ? How many diagonals has it ? [Ans. 120, 35]

3. Expand $(1+x)^n$ in ascending powers of x when n is a positive integer. Is the expansion valid when n is not a positive integer ? If not, state the condition under which it is valid.

Show that the middle term of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.7\ldots\ldots(2n-1)}{\underline{|n}}2^n$.

4. (*a*) Prove that $\log_b a\times\log_c b\times\log_a c=\log_a a$.

(*b*) Find S_n, when $S_n=1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

$\left[\text{Ans : } 2-\frac{1}{2^{n-1}}\right]$

Calculate the lowest value of n so that $2-S_n<\frac{1}{100}$. [Ans : 8]

Group B—Trigonometry

5. (*a*) Prove geometrically

$\cos(A+B)=\cos A\cos B-\sin A\sin B$,

when $0<A<\frac{\pi}{2}$, $0<B<\frac{\pi}{2}$ and $A+B<\frac{\pi}{2}$.

(*b*) If $A+B+C=\pi$, prove that

$\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.

6. (*a*) Prove $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15$.

(*b*) Solve $\tan x+\tan 2x+\tan 3x=\tan x\tan 2x\tan 3x$.

$\left[\text{Ans. } x=\frac{n\pi}{6}\right]$

7. (*a*) Same as Q. 7(*a*) of C. Pre-U., 1962.

(*b*) Prove that in any triangle $\cos A+\cos B+\cos C=1+\frac{r}{R}$

(r=in-radius and R=circum-radius)

Group C—Co-ordinate Geometry

8. (a) Find the equation to the straight line passing through the intersection of lines $x+2y-3=0$ and $3x+4y+7=0$ and perpendicular to the straight line $y-x=8$. [Ans : $x+y+5=0$]

(b) Show that three points (4, 2), (7, 5) and (9, 7) are in one straight line and find the equation of the line of collinearity.

[Ans : $x-y=2$]

9. (a) Find the equation of the circle described on the diameter whose end points are (x_1, y_1) and (x_2, y_2).

(b) Find the equation of the tangent to the circle $x^2+y^2=25$ making an angle of 30° with the straight line $3x+4y=0$.

$$\left[\text{Ans. } y=mx\pm5\sqrt{1+m^2} \text{ where } m=\frac{25\sqrt{3}-48}{39}\right]$$

10. (a) Find the condition that $y=mx+c$ will be a normal to the parabola $y^2=4ax$.

(b) Prove that the tangent to the parabola $y^2=8x$ at a given point on it makes equal angles with the focal distance of the point and its perpendicular distance from the directrix.

11. (a) Find the eccentricity, length of the latus rectum and the co-ordinates of the foci of the ellipse $16x^2+9y^2=144$.
[Ans. $e=\frac{1}{4}\sqrt{7}$, latus rectum $=\frac{9}{2}$ units, foci are the pts. $(0, \pm\sqrt{7})$]

(b) Prove that the tangent to an ellipse at a point on it makes equal angles with the focal distances of the point.

Group D—Solid Geometry & Mensuration

12. (a) Same as Q. 7(a) of H. S., 1960 (Compl.)

(b) Same as Q. 13 of H. S., 1962.

13. (a) Same as Q. 14(a) of C. Pre-U., 1962.

(b) If a straight line is parallel to a given plane, prove that the line of intersection of any plane through the given line with the given plane is parallel to the given line.

B. U. ENTRANCE EXAM.—1962

Group A—Algebra (Answer *any two* questions)

1. (*a*) Represent geometrically the numbers $3+2i$, $6+4i$ and $9+6i$. Find their moduli and amplitudes and show that these are collinear points. [Ans. Moduli $= +\sqrt{13}, +2\sqrt{13}, +3\sqrt{13}$, Amplitudes $= \tan^{-1}\frac{2}{3}, \tan^{-1}\frac{2}{3}, \tan^{-1}\frac{2}{3}$]

(*b*) Solve : (i) $x^2+y^2+xy=84$, $x+y=10$.

[Ans : $x=8, y=2$; or $x=2, y=8$]

(ii) $y^x=4$, $y^2=2^x$.

[Ans : $x=2, y=\pm 2$; or, $x=-2, y=\pm\frac{1}{2}$]

2. (*a*) Find the conditions for the roots of the quadratic equations $ax^2+bx+c=0$ to be real and positive.

[Ans. (*i*) $b^2 \geqslant 4ac$, (*ii*) a and c have like signs opposite to that of b]

(*b*) If x is real, prove that—

$\dfrac{2x^2+4x+1}{x^2+4x+2}$ is capable of having all real values.

3. (*a*) Find the number of permutations of n different things taken r at a time. $(n \geqslant r)$.

(*b*) How many numbers of not more than four digits can be formed with the digits 2, 3, 4 ? [Ans. 120]

4. (*a*) If the second, third and fourth terms in the expansion of $(a+x)^n$ be 240, 720 and 1080 respectively, find a, x, n.

[Ans. $a=2, x=3, n=5$]

4. (*b*) If $\dfrac{\log x}{b-c}=\dfrac{\log y}{c-a}=\dfrac{\log z}{a-b}$, prove that $x^a y^b z^c=1$.

Group B—Trigonometry

(Answer *any two* questions)

5. (*a*) Same as Q. 9(*a*) of H. S., 1960 (Compl.)

(*b*) If $A+B+C=\pi$, prove that

$\cos^2 A+\cos^2 B+2\cos A\cos B\cos C=\sin^2 C$.

6. (*a*) Solve : $\sin 4\theta=\cos 3\theta+\sin 2\theta$ $(0<\theta<\pi)$.

[Ans. 30°, 90°, 150°]

(*b*) Show that : $(\tan^{-1} 1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3})=\dfrac{\pi}{2}$.

7. (a) Same as Q. 10(a) of H. S., 1963.

(b) Same as Q. 10(b) of H. S., 1964.

8. (a) Find out the value of sin 18°.

(b) A tower leans towards the north. If α and β are the angles of elevation of the top from two observation points due south of the tower at distances a and b from its foot, prove that its inclination to the horizontal is.

$$\cot^{-1}\left(\frac{b\cot\alpha - a\cot\beta}{b-a}\right).$$

Group C—Co-ordinate Geometry

(Answer *any two* questions)

9. (a) Prove analytically that the line joining the middle points of any two sides of a triangle is half the third side.

(b) Find the equations of the medians of the triangle whose vertices are the points (3, 2), (1, −1), (−19, −9) and show that they are concurrent.

[Ans. $7x-12y+3=0$, $5x-18y-23=0$, $19x-42y-17=0$]

10. (a) Find the equations of the tangents to the circle $(x-1)^2+(y+1)^2=16$ parallel to the line $y=3x+10$.

[Ans. $y=3x-4\pm4\sqrt{10}$]

(b) Prove that the points (1, 1), (2, 0), (3, −3), (−5, −7) are concyclic and find the equation of the circle passing through them.

[Ans. $x^2+y^2+4x+6y-12=0$]

11. (a) Find the equation to the parabola whose focus is at the origin and whose directrix is the straight line $2x+y-1=0$. Find out its vertex.

[Ans. $x^2+4y^2-4xy+4x+2y-1=0$. vertex $(\frac{1}{5}, \frac{1}{10})$]

(b) Show that the tangents to the parabola $y^2=8x$ at the points (2, 4) and (2, −4) intersect on the x-axis. Also find the angle between the tangents.

$\left[\text{Ans. angle} = \frac{\pi}{2}\right]$

12. (*a*) Starting from the focus-directrix properties of an ellipse, show that the equation of the ellipse is of the form :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(*b*) Find the length of the latus rectum, the foci and the equations of the directrices of the hyperbola $3x^2-4y^2=48$.

[Ans. latus rectum $=6$, foci $=(\pm 2\sqrt{7}, 0)$, directrices are $x\pm\frac{8}{\sqrt{7}}=0$]

Group D—Solid Geometry

13. (*a*) Same as Q. 7 (*d*) second part of H. S., 1961.

(*b*) Show that all st. lines drawn perpendicular from a given point to a system of parallel st. lines in space are coplanar.

14. (*a*) How many solid circular cylinders of length 8 in. and diameter 6 in. can be made out of the material of a solid sphere of radius 6 inches ? [Ans. 4]

(*b*) A right circular cone 15 cm. high, the radius of the base being 8 cm., has its upper part cut off by a plane through the middle point of its axis and parallel to the base. Find the volume of the truncated cone. [Ans. 880 cu. cm.]

B. U. Entrance Examination, 1963

Group A—Algebra

1. (*a*) Solve (*i*) $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}, x+y=5.$ $\left[\text{Ans. } \begin{matrix} x=4,\ y=1\ ; \\ \text{or } x=1,\ y=4 \end{matrix}\right]$

(ii) $x^{\frac{1}{3}}+y^{\frac{1}{3}}=3, x+y=9$ [Ans : $x=8$, $y=1$; or, $x=1$, $y=8$]

(iii) $x^y=y^2, y^{2y}=x^4$. [Ans. $x=\pm 2$, $y=2$, or, $x=\pm\frac{1}{2}$, $y=-2$.]

(*b*) Three globes of gold whose radii are 3, 4 and 5 inches are melted and formed into a new globe. Find the radius of the new globe, given that the volume of a globe varies as the cube of its radius.

[Ans. 6 in.]

2. (a) Find the conditions that the roots of the Quadratic equation $ax^2+bx+c=0$ should be (i) reciprocals, (ii) both positive, (iii) both negative.

[Ans. (i) If $c=a$; (ii) if a, c have like signs opposite to that of b; (iii) if a, b, c have like signs.]

(b) If the roots of $lx^2+mx+m=0$ be in the ratio of $p:q$, prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{m}{l}}=0$.

3. (a) Without assuming the formula for ^nP_r find the number of combinations of n different things taken r things at a time $(r\leqslant n)$.

(b) Show that in $^{2n}C_n$ the number of combinations in which a particular thing occurs, is equal to the number in which it does not occur.

4. (a) Expand $(a+x)^n$ in ascending powers of x, n being a positive integer.

(b) Same as Q. 5(b) of H. S., 1966.

5. (a) Write down (without proof) the series for $\log_e(1+x)$, stating the condition for its validity. Hence prove that

$$\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)$$ [Ans. Valid if $-1<x\leqslant1$]

(b) Prove that

$$\left(\frac{1}{5}+\frac{1}{7}\right)+\frac{1}{3}\left(\frac{1}{5^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{1}{5^5}+\frac{1}{7^5}\right)+\cdots\text{to }\infty=\log_e\sqrt{2}.$$

Group B—Trigonometry

6. (a) Same as Q. 5 (a) of C. Pre-U., 1963.

(b) If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, prove that $\tan\frac{\theta-\phi}{2}=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$.

7. (a) If $\alpha+\beta=\gamma$, show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$.

(b) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz$.

8. (a) Solve $\tan x+\tan 2x+\tan 3x=0$.

$\left[\text{Ans. } x=\frac{n\pi}{3} \text{ or } n\pi\pm\alpha \text{ where } \tan\alpha=\frac{1}{\sqrt{2}}\right]$

(b) In any triangle prove that $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

9. (a) Two sides of a triangle are 3 ft. and 5 ft. respectively and the included angle is 120°, find the other angles, having given $\log 4\cdot8=\cdot6812412$, $L\tan 8°12'=9\cdot1586706$, diff. for $60''=8940$.

[Ans. 38°12′47·5″, 21°47′12·5″]

(b) Solve graphically the equation

$\tan x=2x$ between $x=0$ and $x=\frac{\pi}{2}$.

Group C—Co-ordinate Geometry

10. (a) Show that the area of the triangle, the co-ordinates of whose angular points are

$\left(a, \frac{1}{a}\right)$, $\left(b, \frac{1}{b}\right)$ and $\left(c, \frac{1}{c}\right)$ is $\frac{(b-c)(c-a)(a-b)}{2abc}$.

(b) If the centroid of a triangle is (1, 4) and two of its vertices are (4, −3) and (−9, 7), find the other vertex. [Ans. (8, 8)]

11. (a) Show that the equation

$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$, where λ is an arbitrary constant, represents any straight line passing through the points of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

(b) Find the equation of the straight line which passes through (x_1, y_1) and is perpendicular to the join of (x_2, y_2) and (x_3, y_3).

[Ans. $(y-y_1)(y_3-y_2)+(x-x_1)(x_3-x_2)=0$]

12. (a) Find the equation of the circle which is concentric with the circle $x^2+y^2-4x+6y-3=0$ and which passes through the point (5, −2).

[Ans. $x^2+y^2-4x+6y+3=0$]

(b) If $y=x\sin\alpha+a\sec\alpha$ be a tangent to the circle $x^2+y^2=a^2$, then prove that $\cos^2\alpha=1$.

13. (*a*) Find the condition that the straight line $y=mx+c$ should touch the parabola $y^2=4ax$.

(*b*) Find the equation of the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point $(-3, 1)$ and has the eccentricity $\sqrt{\frac{2}{5}}$. [Ans. $3x^2+5y^2=32$]

Group D—Solid Geometry

14. (*a*) If two intersecting planes are each perpendicular to a third plane, prove that their line of intersection is also perpendicular to the plane.

(*b*) If of three lines of intersection of three planes, two be parallel, show that the third will also be parallel to the other two.

15. (*a*) The length, breadth and the height of a closed box are 12 in., 10 in. and 8 in. respectively and the total inner surface is 376 sq. in. If the walls of the box are uniformly thick, find the thickness. [Ans. 1 inch.]

(*b*) If S be the area of the curved surface, v be the volume, h the height and α the semi-vertical angle of a right circular cone, prove that $S=\dfrac{\pi h^2 \sin\alpha}{\cos^2\alpha}$ and $v=\frac{1}{3}\pi h^3 \tan^2\alpha$.

B. U. Entrance Examination—1964

Group A—Algebra

(Answer any *two* questions)

1. (*a*) Same as Q. 3 (*a*) of H. S., 1963 (Compl.).

(*b*) Show that a real value of x will satisfy the equation $\dfrac{1-ix}{1+ix}=a-ib$ if $a^2+b^2=1$, the constants a, b being real.

2. (*a*) Solve (*any two*) :

(i) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=18,\ x+y=12.$ [Ans. $x=4,\ y=8$ or $x=8,\ y=4$]

(ii) $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}+6.\dfrac{2x-\sqrt{x}}{2x+\sqrt{x}}=5.$ [Ans. $x=1$, or $\frac{9}{4}$]

(iii) $8^x.4^y=128,$
$9^{x+y}=27^{xy}.$ $\left[\text{Ans. } \left.\begin{array}{l} x=1 \\ y=2\end{array}\right\} \text{ or } \begin{array}{l} x=\frac{14}{9} \\ y=\frac{7}{6}\end{array}\right]$

(*b*) If $x+y \propto z$ when y is constant and $x+z \propto y$ when z is constant, then show that $x+y+z \propto yz$ when both y and z vary.

3. (*a*) If the roots of the equation $ax^2+bx+c=0$ be the reciprocals of the roots of the equation $a'x^2+b'x+c'=0$, then show that $\dfrac{a}{c'}=\dfrac{b}{b'}=\dfrac{c}{a'}$.

(*b*) Prove that the condition that the two equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ should have one root common is $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$.

4. (*a*) Find the number of permutations of n letters in which there are x number of p's, y number of q's, z number of r's and the rest are all different.

(*b*) How many odd numbers of 4 stgnificant digits can be formed with the digits 3, 1, 5, 2, 0 ? [Ans. 54]

5. (*a*) Prove that the value of e lies between 2 and 3 where e represents the exponential series.

Show that

$$\left\{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\cdots\cdots\text{to}\infty\right\}^2-\left\{1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\cdots\cdots\text{to}\infty\right\}^2=1.$$

(*b*) Show that the middle term in the expansion of $(1+x)^{2n}$ is $\dfrac{1.3.5\ldots\ldots(2n-1)}{\lfloor n}.2^n.x^n$.

Group B—Trigonometry

(Answer any *two questions*)

6. (*a*) Same as Q. 8. (*a*) of H. S. 1965.

(*b*) Prove that $4(\cos^3 10°+\sin^3 20°)=3(\cos 10°+\sin 20°)$.

7. (*a*) Prove the following relations

(i) $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}$

(ii) $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}$.

(*b*) Solve the equation

$\sqrt{3}\cos x+\sin x=1$ for $-2\pi<x<2\pi$.

[Ans. $x=-\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}$]

8. (*a*) Same as Q. 10 (*a*) of H. S. 1964.

(*b*) The sides of a triangle are proportional to 3, 5 and 7 respectively. Show that the triangle is obtuse-angled and determine the obtuse angle. [Ans. 120°]

9. (*a*) Draw the graph of $y=\cos x$ in the range between $x=-\pi$ and $x=\pi$.

(*b*) A vertical post 15 ft. high is broken at a certain height and its upper part, not completely separated, meets the ground at an angle of 30°. Find the height at which the post is broken.

[Ans. 5 ft.]

Group C—Co-ordinate Geometry

(Answer any *two* questions)

10. (*a*) A point P divides the st. line AB internally in the ratio $\alpha:\beta$. The co-ordinates of A and B are (a_1, a_2) and (b_1, b_2) respectively. Find the co-ordinates of P.

[Ans. $\left(\frac{a_1\beta+b_1\alpha}{\alpha+\beta}, \frac{a_2\beta+b_2\alpha}{\alpha+\beta}\right)$]

(*b*) Find the equation of the locus of a point $P(x, y)$ if the ratio of its distance from the point $(-10, 0)$ to its distance from the point $(1, 0)$ be $2:1$. [Ans. $3x^2+3y^2-28x-96=0$.]

11. (*a*) Find an expression for the angle between the two st. lines given by $Ax+By=0$ and $A'x+B'y=0$ respectively.

(*b*) Find the distance of the point of intersection of the two st. lines $2x-3y+17=0$ and $3x+4y=0$ from the st. line $4x-3y=0$.

[Ans. -5 (numerically 5 units)]

12. (*a*) Same as Q. 9 (*a*) of B. U. E. 1961.

(*b*) Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and which has its centre on the straight line $3x+4y=5$. [Ans. $x^2+y^2-6x+2y+5=0$]

13. (*a*) Find the equation of the tangent to the parabola $y^2=8x$ which is perpendicular to the st. line $3x-y+7=0$.

[Ans. $x+3y+18=0$]

(*b*) Find the distance between the foci of the ellipse whose equation in standard form is $3x^2+4y^2=24$. [Ans. $2\sqrt{2}$ units]

Group D—Solid Geometry & Mensuration

(Answer any *two* questions)

14. (*a*) What do you mean by (i) the angle between a plane and a straight line ; (ii) the angle between two planes ?

If a right angle revolves about one of its arms, then prove that the other arm describes a plane.

(*b*) The base of a right prism is a triangle whose perimeter is 15 cm. and the radius of the in-circle of the triangle is 3 cm. If the volume of the prism be 270 c.c., find its height.

[Ans. 12 cm.]

15. (*a*) Same as Q. 7 (*b*) second part of H. S., 1960.

(*b*) Same as Q. 13 (*b*) of B. U. E., 1962.

CORRIGENDUM

Algebra :

Page 69, in line 12, read 'the last digit' for 'the digit'.
" 86, in line 5, read q for p.
" 98, in line 1, read 'team of eleven' for 'team'.

Trigonometry :

Page 14, in line 8, read cot $\theta=$cot α for cot$=\theta$ cot α.
" 55, in line 1 of Ex. 12, read 'cosines of two of the angles'.
" 69, in line 1 of Ex. 12, read $\left(1-\frac{r_1}{r_3}\right)$ for $\left(1-\frac{r_2}{r_3}\right)$.
" 74, in line 5, read 46°43′.
" 78, in last line, read 4·3955817 for 4·3955217.
" 85, in line 3 of Ex. 1, read 29° for 20°.
" 98, in line 4 of sum 6, read 56° for 55°.
" 105, in line 9 of Ex. 4, read 'Now, sin B' for 'Now, sin C'.

Co-ordinate Geometry :

Page 8, in line 7, read AP for AB.
" 34, in line 10 of Ex. 11, read 0 for 3.
" 35, in line 8, read $4f^2-4c$ for $4f^2=4c$.
" 42, line 7 should be 'fixed st. line on the same plane'.
" 58, in sum 13(d) read (1, 2) for (1, $-$2).
" 132, in line 8 of Ex. 7, read $m'=\frac{8}{9}$ for $m'=8$.
" 138, in line 15, read $\frac{SA}{AZ}$ for $\frac{SA}{SZ}$.
" 139, in line 19, read $(CS-CN)^2$ for $(CS-CN)$;
and in line 21, read e^2x^2 for ex^2.
" 149, in last line, read $\frac{xx_1}{a^2}$ for $\frac{xy_1}{a^2}$.
" 179, in Answer No. 17, read x^r for x.

Appendix :

Page 5, in line 9, read $\frac{x^2}{a^2}-1$ for $\frac{x^2}{a^2}=1$.

" 7, in line 11, read $+(y-\beta)^2$ for $+y(-\beta)^2$.

" 8, in line 3, read $+2\beta k$ for $-2\beta k$.

Question Papers :

Page 7, in Q. 4(*b*) read $3x$ for $2x$.

" 36, in Q. 2(*b*) read 8 in. for 6 in.

LOG TABLES

TABLE I

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	248	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	227	265	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	35	70	105	140	175	209	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301	32	65	97	129	162	193	225	258	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	180	210	240	270
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789	26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97	116	185	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89	107	124	142	160
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148
27	43136	[illegible]	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142
28	44716	[illegible]	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137
29	[illegible]	[illegible]	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132

LOGARITHMS OF NUMBERS

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30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	42	55	69	83	97	110	125
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	86	98	110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	52	63	73	84	94
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50	60	70	80	90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	75	84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	37	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45	53	62	71	80
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44	53	61	70	79
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42	51	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	62	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	53	60	68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	45	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	57	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35	42	49	56	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34	41	48	55	62
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	39	45	52	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31	37	44	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	18	25	31	37	43	49	55
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30	36	42	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	18	23	29	35	41	47	52

LOGARITHMS OF NUMBERS

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75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6	12	17	23	29	35	40	46	52
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6	11	17	23	29	34	40	46	51
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6	11	17	22	28	34	39	45	50
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6	11	17	22	28	33	39	44	50
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	5	11	16	22	27	33	38	44	49
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5	11	16	22	27	32	38	43	49
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	5	11	16	21	27	32	37	43	48
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5	11	16	21	26	32	37	42	47
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	5	10	16	21	26	31	36	42	47
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	5	10	15	21	26	31	36	41	46
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5	10	15	20	25	30	36	41	46
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5	10	15	20	25	30	35	40	45
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5	10	15	20	25	30	35	40	45
[illegible]	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5	10	15	20	25	29	34	39	44
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5	10	15	19	24	29	34	39	44
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5	10	14	19	24	29	34	38	43
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5	9	14	19	24	29	33	38	43
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5	9	14	19	24	28	33	38	42
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	5	9	14	19	23	28	33	37	42
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5	9	14	18	23	28	32	37	41
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5	9	14	18	23	27	32	36	41
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	5	9	14	18	23	27	32	36	41
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4	9	13	18	22	27	31	36	40
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4	9	13	18	22	26	31	35	40
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4	9	13	17	22	26	30	35	39

Table II

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2743	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

ANTILOGARITHMS

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE III

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01454	0·01745	89°	29	58	87	116	145	175	204	233	262
1°	·01745	·02036	·02327	·02618	·02908	·03199	·03490	88°	29	58	87	116	145	175	204	233	262
2°	·03490	·03781	·04071	·04362	·04653	·04943	·05234	87°	29	58	87	116	145	175	204	233	262
3°	·05234	·05524	·05814	·06105	·06395	·06685	·06976	86°	29	58	87	116	145	174	203	232	261
4°	·06976	·07266	·07556	·07846	·08136	·08426	·08716	85°	29	58	87	116	145	174	203	232	261
5°	0·08716	0·09005	0·09295	0·09585	0·09874	0·10164	0·10453	84°	29	58	87	116	145	174	203	232	261
6°	·10453	·10742	·11031	·11320	·11609	·11898	·12187	83°	29	58	87	116	145	174	203	232	261
7°	·12187	·12476	·12764	·13053	·13341	·13629	·13917	82°	29	58	87	116	145	173	202	231	260
8°	·13917	·14205	·14493	·14781	·15069	·15356	·15643	81°	29	58	86	115	144	173	202	230	259
9°	·15643	·15931	·16218	·16505	·16792	·17078	·17365	80°	29	57	86	115	144	172	201	230	258
10°	0·17365	0·17651	0·17937	0·18224	0·18509	0·18795	0·19081	79°	29	57	86	115	144	172	201	229	258
11°	·19081	·19366	·19652	·19937	·20222	·20507	·20791	78°	29	57	86	114	143	171	200	228	257
12°	·20791	·21076	·21360	·21644	·21928	·22212	·22495	77°	28	57	85	114	142	170	199	227	256
13°	·22495	·22778	·23062	·23345	·23627	·23910	·24192	76°	28	57	85	113	141	170	198	226	255
14°	·24192	·24474	·24756	·25038	·25320	·25601	·25882	75°	28	56	85	113	141	169	197	226	254
15°	0·25882	0·26163	0·26443	0·26724	0·27004	0·27284	0·27564	74°	28	56	84	112	140	168	196	224	252
16°	·27564	·27843	·28123	·28402	·28680	·28959	·29237	73°	28	56	84	112	140	167	195	223	251
17°	·29237	·29515	·29793	·30071	·30348	·30625	·30902	72°	28	56	83	111	139	166	194	222	250
18°	·30902	·31178	·31454	·31730	·32006	·32282	·32557	71°	28	55	83	110	138	166	193	221	248
19°	·32557	·32832	·33106	·33381	·33655	·33929	·34202	70°	27	55	82	110	137	164	192	219	247

20°	0·34202	0·34475	0·34748	0·35021	0·35293	0·35565	0·35837	69°	27	55	82	109	137	164	191	218	246
21°	·35837	·36108	·36379	·36650	·36921	·37191	·37461	68°	27	54	81	108	136	163	190	217	244
22°	·37461	·37730	·37999	·38268	·38537	·38805	·39073	67°	27	54	81	108	135	161	188	215	242
23°	·39073	·39341	·39608	·39875	·40142	·40408	·40674	66°	27	53	80	107	134	160	187	214	240
24°	·40674	·40939	·41204	·41469	·41734	·41998	·42262	65°	27	53	80	106	133	159	186	212	238
25°	0·42262	0·42525	0·42788	0·43051	0·43313	0·43575	0·43837	64°	26	52	79	105	131	157	184	210	236
26°	·43837	·44098	·44359	·44620	·44880	·45140	·45399	63°	26	52	78	104	130	156	182	208	234
27°	·45399	·45658	·45917	·46175	·46433	·46690	·46947	62°	26	52	77	103	129	155	181	206	232
28°	·46947	·47204	·47460	·47716	·47971	·48226	·48481	61°	26	51	77	102	128	154	179	204	230
29°	·48481	·48735	·48989	·49242	·49495	·49748	·50000	60°	25	51	76	101	127	152	177	202	228
30°	0·50000	0·50252	0·50503	0·50754	0·51004	0·51254	0·51504	59°	25	50	75	100	125	150	175	200	225
31°	·51504	·51753	·52002	·52250	·52498	·52745	·52992	58°	25	50	74	99	124	149	174	198	223
32°	·52992	·53238	·53484	·53730	·53975	·54220	·54464	57°	25	49	74	98	123	147	172	196	221
33°	·54464	·54708	·54951	·55194	·55436	·55678	·55919	56°	24	49	73	97	122	146	170	194	219
34°	·55919	·56160	·56401	·56641	·56880	·57119	·57358	55°	24	48	72	96	120	144	168	192	216
35°	0·57358	0·57596	0·57833	0·58070	0·58307	0·58543	0·58779	54°	24	47	71	95	119	142	166	190	213
36°	·58779	·59014	·59248	·59482	·59716	·59949	·60182	53°	23	47	70	94	117	140	164	187	211
37°	·60182	·60414	·60645	·60876	·61107	·61337	·61566	52°	23	46	70	92	116	139	162	185	208
38°	·61566	·61795	·62024	·62251	·62479	·62706	·62932	51°	23	46	68	91	114	137	159	182	205
39°	·62932	·63158	·63383	·63608	·63832	·64056	·64279	50°	22	45	67	90	112	135	157	179	202
40°	0·64279	0·64501	0·64723	0·64945	0·65166	0·65386	0·65606	49°	22	44	66	88	111	133	155	177	199
41°	·65606	·65825	·66044	·66262	·66480	·66697	·66913	48°	22	44	65	87	109	131	153	174	196
42°	·66913	·67129	·67344	·67559	·67773	·67987	·68200	47°	21	43	64	86	107	129	150	172	193
43°	·68200	·68412	·68624	·68835	·69046	·69256	·69466	46°	21	42	63	84	106	127	148	169	190
44°	·69466	·69675	·69883	·70091	·70298	·70505	·70711	45°	21	42	62	83	104	124	145	166	187
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	0·70711	0·70916	0·71121	0·71325	0·71529	0·71732	0·71934	44°	20	41	61	82	102	122	143	163	184
46°	·71934	·72136	·72337	·72537	·72737	·72937	·73135	43°	20	40	60	80	100	120	140	160	180
47°	·73135	·73333	·73531	·73728	·73924	·74120	·74314	42°	20	39	59	78	98	118	138	157	177
48°	·74314	·74509	·74703	·74896	·75088	·75280	·75471	41°	19	39	58	77	96	116	135	154	173
49°	·75471	·75661	·75851	·76041	·76229	·76417	·76604	40°	19	38	57	76	95	113	132	151	170
50°	0·76604	0·76791	0·76977	0·77162	0·77347	0·77531	0·77715	39°	19	37	56	74	93	111	130	148	167
51°	·77715	·77897	·78079	·78261	·78442	·78622	·78801	38°	18	36	54	72	91	109	127	145	163
52°	·78801	·78980	·79158	·79335	·79512	·79688	·79864	37°	18	35	53	71	89	106	124	142	159
53°	·79864	·80038	·80212	·80386	·80558	·80730	·80902	36°	17	35	52	69	87	104	121	138	156
54°	·80902	·81072	·81242	·81412	·81580	·81748	·81915	35°	17	34	51	68	85	101	118	135	152
55°	0·81915	0·82082	0·82248	0·82413	0·82577	0·82741	0·82904	34°	16	33	49	66	82	99	115	132	148
56°	·82904	·83066	·83228	·83389	·83549	·83708	·83867	33°	16	32	48	64	80	96	112	128	144
57°	·83867	·84025	·84182	·84339	·84495	·84650	·84805	32°	16	31	47	63	78	94	110	125	141
58°	·84805	·84959	·85112	·85264	·85416	·85567	·85717	31°	15	30	46	61	76	91	106	122	137
59°	·85717	·85866	·86015	·86163	·86310	·86457	·86603	30°	15	30	44	59	74	89	103	118	133
60°	0·86603	0·86748	0·86892	0·87036	0·87178	0·87321	0·87462	29°	14	29	43	57	72	86	100	114	129
61°	·87462	·87603	·87743	·87882	·88020	·88158	·88295	28°	14	28	42	55	69	83	97	111	125
62°	·88295	·88431	·88566	·88701	·88835	·88968	·89101	27°	13	27	40	54	67	81	94	103	121
63°	·89101	·89232	·89363	·89493	·89623	·89752	·89879	26°	13	26	39	52	65	78	91	104	117
64°	·89879	·90007	·90133	·90259	·90383	·90507	·90631	25°	13	25	38	50	63	75	88	100	113

TABLE III]

NATURAL SINES & COSINES

65°	0·90631	0·90753	0·90875	0·90996	0·91116	0·91236	0·91355	24°	12	24	36	48	60	72	84	96	108
66°	·91355	·91472	·91590	·91706	·91822	·91936	·92050	23°	12	23	35	46	58	70	81	93	104
67°	·92050	·92164	·92276	·92388	·92499	·92609	·92718	22°	11	22	33	45	56	67	78	89	100
68°	·92718	·92827	·92935	93042	·93148	·93253	·93358	21°	11	21	32	43	53	64	75	85	96
69°	·93358	·93462	·93565	·93667	·93769	·93869	·93969	20°	10	20	31	41	51	61	71	81	92
70°	0·93969	0·94068	0·94167	0·94264	0·94361	0·94457	0·94552	19°	10	19	29	39	49	58	68	78	87
71°	·94552	·94646	·94740	·94832	·94924	·95015	·95106	18°	9	18	28	37	46	55	64	74	83
72°	·95106	·95195	·95284	·95372	·95459	·95545	·95630	17°	9	18	26	35	44	52	61	70	79
73°	·95630	·95715	·95799	·95882	·95964	·96046	·96126	16°	8	17	25	33	41	50	58	66	74
74°	·96126	·96206	·96285	·96363	·96440	·96517	·96593	15°	8	16	23	31	39	47	54	62	70
75°	0·96593	0·96667	0·96742	0·96815	0·96887	0·96959	0·97030	14°	7	15	22	29	36	44	51	58	65
76°	·97030	·97100	·97169	·97237	·97304	·97371	·97437	13°	7	14	20	27	34	41	47	54	61
77°	·97437	·97502	·97566	·97630	·97692	·97754	·97815	12°	6	13	19	25	32	38	44	50	57
78°	·97815	·97875	·97934	·97992	·98050	·98107	·98163	11°	6	12	17	23	29	35	41	46	52
79°	·98163	·98218	·98272	·98325	·98378	·98430	·98481	10°	5	11	16	21	27	32	37	42	48
80°	0·98481	0·98531	0·98580	0·98629	0·98676	0·98723	0·98769	9°	5	10	14	19	24	29	34	38	43
81°	·98769	·98814	·98858	·98902	·98944	·98986	·99027	8°	4	9	13	17	22	26	30	34	39
82°	·99027	·99067	·99106	·99144	·99182	·99219	·99255	7°	4	8	11	15	19	23	27	30	34
83°	·99255	·99290	·99324	·99357	·99390	·99421	·99452	6°	3	7	10	13	17	20	23	26	30
84°	·99452	·99482	·99511	·99540	·99567	·99594	·99619	5°	3	6	8	11	14	17	20	22	25
85°	0·99619	0·99644	0·99668	0·99692	0·99714	0·99736	0·99756	4°	2	5	7	9	12	14	16	18	21
86°	·99756	·99776	·99795	·99813	·99831	·99847	·99863	3°	2	4	5	7	9	11	13	14	16
87°	·99863	·99878	·99892	·99905	·99917	·99929	·99939	2°	1	3	4	5	7	8	9	10	12
88°	·99939	·99949	·99958	·99966	·99973	·99979	·99985	1°									
89°	·99985	·99989	·99993	·99996	·99998	1·00000	1·00000	0°									
90°	1·00000																
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES

TABLE IV

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01455	0·01746	89°	29	58	87	116	146	175	204	233	262
1°	·01746	·02037	·02328	·02619	·02910	·03201	·03492	88°	29	58	87	116	146	175	204	233	262
2°	·03492	·03783	·04075	·04366	·04658	·04949	·05241	87°	29	58	87	116	146	175	204	233	262
3°	·05241	·05533	·05824	·06116	·06408	·06700	·06993	86°	29	58	88	117	146	175	204	234	263
4°	·06993	·07285	·07578	·07870	·08163	·08456	·08749	85°	29	58	88	117	146	175	204	234	263
5°	0·08749	0·09042	0·09335	0·09629	0·09923	0·10216	0·10510	84°	29	59	88	118	147	176	206	235	265
6°	·10510	·10805	·11099	·11394	·11688	·11983	·12278	83°	29	59	88	118	147	176	206	235	265
7°	·12278	·12574	·12869	·13165	·13461	·13758	·14054	82°	30	59	89	118	148	178	207	237	266
8°	·14054	·14351	·14648	·14945	·15243	·15540	·15838	81°	30	59	89	119	149	178	208	238	267
9°	·15838	·16137	·16435	·16734	·17033	·17333	·17633	80°	30	60	90	120	150	179	209	239	269
10°	0·17633	0·17933	0·18233	0·18534	0·18835	0·19136	0·19438	79°	30	60	90	120	151	181	211	241	271
11°	·19438	·19740	·20042	·20345	·20648	·20952	·21256	78°	30	61	91	121	152	182	212	242	273
12°	·21256	·21560	·21864	·22169	·22475	·22781	·23087	77°	31	61	92	122	153	183	214	244	275
13°	·23087	·23393	·23700	·24008	·24316	·24624	·24933	76°	31	62	92	123	154	185	216	246	277
14°	·24933	·25242	·25552	·25862	·26172	·26483	·26795	75°	31	62	93	124	155	186	217	248	279
15°	0·26795	0·27107	0·27419	0·27732	0·28046	0·28360	0·28675	74°	31	63	94	125	157	188	219	250	282
16°	·28675	·28990	·29305	·29621	·29938	·30255	·30573	73°	32	63	95	126	158	190	221	253	285
17°	·30573	·30891	·31210	·31530	·31850	·32171	·32492	72°	32	64	96	128	160	192	224	256	288
18°	·32492	·32814	·33136	·33460	·33783	·34108	·34433	71°	32	65	97	129	162	194	226	259	291
19°	·34433	·34758	·35085	·35412	·35740	·36068	·36397	70°	33	65	98	131	164	196	229	262	294

TABLE IV] NATURAL TANGENTS & COTANGENTS

20°	0·36397	0·36727	0·37057	0·37388	0·37720	0·38053	0·38386	69°	33	66	100	133	166	199	232	265	298
21°	·38386	·38721	·39055	·39391	·39727	·40065	·40403	68°	34	67	101	134	168	202	236	269	302
22°	·40403	·40741	·41081	·41421	·41763	·42105	·42447	67°	34	68	102	136	170	205	239	273	306
23°	·42447	·42791	·43136	·43481	·43828	·44175	·44523	66°	35	69	104	138	173	208	242	277	311
24°	·44523	·44872	·45222	·45573	·45924	·46277	·46631	65°	35	70	105	140	176	211	246	281	316
25°	0·46631	0·46985	0·47341	0·47698	0·48055	0·48414	0·48773	64°	36	71	107	143	179	214	250	286	321
26°	·48773	·49134	·49495	·49858	·50222	·50587	·50953	63°	36	73	109	145	182	218	254	291	327
27°	·50953	·51320	·51688	·52057	·52427	·52798	·53171	62°	37	74	111	148	185	222	259	296	333
28°	·53171	·53545	·53920	·54296	·54673	·55051	·55431	61°	38	75	113	151	189	226	264	302	339
29°	·55431	·55812	·56194	·56577	·56962	·57348	·57735	60°	38	77	115	154	192	230	269	307	346
30°	0·57735	0·58124	0·58513	0·58905	0·59297	0·59691	0·60086	59°	39	78	118	157	196	235	274	313	353
31°	·60086	·60483	·60881	·61280	·61681	·62083	·62487	58°	40	80	120	160	200	240	280	320	360
32°	·62487	·62892	·63299	·63707	·64117	·64528	·64941	57°	41	82	123	164	205	245	286	327	368
33°	·64941	·65355	·65771	·66189	·66608	·67028	·67451	56°	42	84	126	167	209	251	293	334	376
34°	·67451	·67875	·68301	·68728	·69157	·69588	·70021	55°	43	86	128	171	214	257	300	342	385
35°	0·70021	0·70455	0·70891	0·71329	0·71769	0·72211	0·72654	54°	44	88	132	176	220	263	307	351	395
36°	·72654	·73100	·73547	·73996	·74447	·74900	·75355	53°	45	90	135	180	225	270	315	360	405
37°	·75355	·75812	·76272	·76733	·77196	·77661	·78129	52°	46	92	139	185	231	277	324	370	416
38°	·78129	·78598	·79070	·79544	·80020	·80498	·80978	51°	48	95	143	190	238	285	333	380	428
39°	·80978	·81461	·81946	·82434	·82923	·83415	·83910	50°	49	98	147	196	245	293	342	391	440
40°	0·83910	0·84407	0·84906	0·85408	0·85912	0·86419	0·86929	49°	50	101	151	201	252	302	352	402	453
41°	·86929	·87441	·87955	·88473	·88992	·89515	·90040	48°	52	104	156	208	260	311	363	415	467
42°	·90040	·90569	·91099	·91633	·92170	·92709	·93252	47°	54	107	161	214	268	321	375	429	482
43°	·93252	·93797	·94345	·94896	·95451	·96008	·96569	46°	55	111	166	221	277	332	387	442	498
44°	·96569	·97133	·97700	·98270	·98843	·99420	1·00000	45°	57	114	172	229	286	343	400	457	515
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COTANGENTS

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	1·00000	1·00583	1·01170	1·01761	1·02355	1·02952	1·03553	44°	59	118	178	237	296	355	414	474	533
46°	·03553	·04158	·04766	·05378	·05994	·06613	·07237	43°	61	123	184	246	307	368	430	491	553
47°	·07237	·07864	·08496	·09131	·09770	·10414	·11061	42°	64	127	191	255	319	382	446	510	573
48°	·11061	·11713	·12369	·13029	·13694	·14363	·15037	41°	66	132	199	265	332	397	463	530	596
49°	·15037	·15715	·16398	·17085	·17777	·18474	·19175	40°	69	138	207	276	345	413	482	552	620
50°	1·19175	1·19882	1·20593	1·21310	1·22081	1·22758	1·23490	39°	72	144	216	288	360	431	503	575	647
51°	·23490	·24227	·24969	·25717	·26471	·27230	·27994	38°	75	150	225	300	376	451	526	601	676
52°	·27994	·28764	·29541	·30323	·31110	·31904	·32704	37°	78	157	235	314	392	471	549	628	707
53°	·32704	·33511	·34323	·35142	·35968	·36800	·37638	36°	82	164	247	329	411	493	576	658	740
54°	·37638	·38484	·39336	·40195	·41061	·41934	·42815	35°	86	172	259	345	431	517	603	690	776
55°	1·42815	1·43703	1·44598	1·45501	1·46411	1·47330	1·48256	34°	91	181	272	363	453	544	634	725	816
56°	·48256	·49190	·50133	·51084	·52043	·53010	·53987	33°	96	191	287	382	478	573	669	764	860
57°	·53987	·54972	·55966	·56969	·57981	·59002	·60033	32°	101	201	302	403	504	604	705	806	907
58°	·60033	·61074	·62125	·63185	·64256	·65337	·66428	31°	107	213	320	426	533	639	746	852	959
59°	·66428	·67530	·68643	·69766	·70901	·72047	·73205	30°	113	226	339	451	565	677	790	903	1016
60°	1·7321	1·7437	1·7556	1·7675	1·7796	1·7917	1·8040	29°	12	24	36	48	60	72	84	96	108
61°	1·8040	1·8165	1·8291	1·8418	1·8546	1·8676	1·8807	28°	13	25	38	51	64	77	89	102	115
62°	1·8807	1·8940	1·9074	1·9210	1·9347	1·9486	1·9626	27°	14	27	41	54	68	82	95	109	122
63°	1·9626	1·9768	1·9912	2·0057	2·0204	2·0353	2·0503	26°	15	29	44	58	73	88	102	117	131
64°	2·0503	2·0655	2·0809	2·0965	2·1123	2·1283	2·1445	25°	16	31	47	63	79	94	110	126	141

TABLE IV NATURAL TANGENTS & COTANGENTS

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	2·1445	2·1609	2·1775	2·1943	2·2113	2·2286	2·2460	24°	17	34	51	68	85	101	118	135	152
66°	2·2460	2·2637	2·2817	2·2998	2·3183	2·3369	2·3559	23°	18	37	55	73	92	110	128	146	165
67°	2·3559	2·3750	2·3945	2·4142	2·4342	2·4545	2·4751	22°	20	40	60	80	100	119	139	159	179
68°	2·4751	2·4960	2·5172	2·5386	2·5605	2·5826	2·6051	21°	22	43	65	87	109	130	152	174	195
69°	2·6051	2·6279	2·6511	2·6746	2·6985	2·7228	2·7475	20°	24	47	76	95	119	142	166	190	213
70°	2·7475	2·7725	2·7980	2·8239	2·8502	2·8770	2·9042	19°	26	52	78	104	131	157	183	209	235
71°	2·9042	2·9319	2·9600	2·9887	3·0178	3·0475	3·0777	18°	29	58	87	116	145	174	202	231	260
72°	3·0777	3·1084	3·1307	3·1716	3·2041	3·2371	3·2709	17°	32	64	97	129	161	193	225	258	290
73°	3·2709	3·3052	3·3402	3·3759	3·4124	3·4495	3·4874	16°	36	72	108	144	181	216	253	289	325
74°	3·4874	3·5261	3·5656	3·6059	3·6470	3·6891	3·7321	15°	41	81	122	163	204	244	285	326	366
75°	3·7321	3·7760	3·8208	3·8667	3·9136	3·9617	4·0108	14°	46	93	139	185	232	278	325	371	418
76°	4·0108	4·0611	4·1126	4·1653	4·2193	4·2747	4·3315	13°	53	107	160	214	267	320	374	427	481
77°	4·3315	4·3897	4·4494	4·5107	4·5736	4·6382	4·7046	12°	62	124	186	248	311	373	435	497	559
78°	4·7046	4·7729	4·8430	4·9152	4·9894	5·0658	5·1446	11°	73	146	220	293	366	439	512	586	659
79°	5·1446	5·2257	5·3093	5·3955	5·4845	5·5764	5·6713	10°	88	175	263	350	438	526	613	701	788
80°	5·6713	5·7694	5·8708	5·9758	6·0844	6·1970	6·3138	9°									
81°	6·3138	6·4348	6·5606	6·6912	6·8269	6·9682	7·1154	8°									
82°	7·1154	7·2687	7·4287	7·5958	7·7704	7·9530	8·1443	7°									
83°	8·1443	8·3450	8·5555	8·7769	9·0098	9·2553	9·5144	6°									
84°	9·5144	9·7882	10·078	10·385	10·712	11·059	11·4301	5°									
85°	11·430	11·826	12·251	12·706	13·197	13·727	14·301	4°									
86°	14·301	14·924	15·605	16·350	17·169	18·075	19·081	3°									
87°	19·081	20·206	21·470	22·904	24·542	26·432	28·636	2°									
88°	28·636	31·242	34·368	38·188	42·964	49·104	57·290	1°									
89°	57·290	68·750	85·940	114·59	171·89	343·77	+∞	0°									
90°	+∞																

Here the differences change very rapidly, so they can not be tabulated.

Angle x' being very small cot x or tan $(90° - x')$ is nearly equal to $3437{\cdot}7 \div x$

NATURAL COTANGENTS

TABLE V

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	$-\infty$	7·46373	7·76475	7·94084	8·06578	8·16268	8·24186	89°	Differences vary so rapidly here that tabulation is impossible. For small angles of x', log sin x' or log cos $(90°-x')=\log x+\bar{4}{\cdot}46373$.								
1°	8·24186	8·30879	8·36678	8·41792	8·46366	8·50504	8·54282	88°									
2°	8·54282	8·57757	8·60973	8·63968	8·66769	8·69400	8·71880	87°									
3°	8·71880	8·74226	8·76451	8·78568	8·80585	8·82513	8·84358	86°									
4°	8·84358	8·86128	8·87829	8·89464	8·91040	8·92561	8·94030	85°									
5°	8·94030	8·95450	8·96825	8·98157	8·99450	9·00704	9·01923	84°									
6°	9·01923	9·03109	9·04262	9·05386	9·06481	9·07548	9·08589	83°									
7°	9·08589	9·09606	9·10599	9·11570	9·12519	9·13447	9·14356	82°	96	192	288	384	480	576	672	768	864
8°	9·14356	9·15245	9·16116	9·16970	9·17807	9·18628	9·19433	81°	85	169	254	338	423	507	592	676	761
9°	9·19433	9·20223	9·20999	9·21761	9·22509	9·23244	9·23967	80°	76	151	227	302	378	453	529	604	680
10°	9·23967	9·24677	9·25376	9·26063	9·26739	9·27405	9·28060	79°	68	136	204	272	341	409	477	545	613
11°	·28060	·28705	·29340	·29966	·30582	·31189	·31788	78°	62	124	186	248	310	373	435	497	559
12°	·31788	·32378	·32960	·33534	·34100	·34658	·35209	77°	57	114	171	228	285	342	399	456	513
13°	·35209	·35752	·36289	·36819	·37341	·37858	·38368	76°	53	105	158	210	263	316	368	421	473
14°	·38368	·38871	·39369	·39860	·40346	·40825	·41300	75°	49	98	147	195	244	293	342	391	440
15°	9·41300	9·41768	9·42232	9·42690	9·43143	9·43591	9·44034	74°	46	91	137	182	228	273	319	364	410
16°	·44034	·44472	·44905	·45334	·45758	·46178	·46594	73°	43	85	128	171	213	256	299	341	384
17°	·46594	·47005	·47411	·47814	·48213	·48607	·48998	72°	40	80	120	160	201	241	281	321	361
18°	·48998	·49385	·49768	·50148	·50523	·50896	·51264	71°	38	76	113	151	189	227	264	302	340
19°	·51264	·51269	·51991	·52350	·52705	·53056	·53405	70°	36	71	107	143	179	214	250	285	321

TABLE V] LOGARITHMIC SINES & COSINES

20°	9·53405	9·53751	9·54093	9·54433	9·54769	9·55102	9·55433	69°	34	68	101	135	169	203	237	270	304
21°	·55433	·55761	·56085	·56408	·56727	·57044	·57358	68°	32	64	96	128	161	193	225	257	289
22°	·57358	·57669	·57978	·58284	·58588	·58889	·59188	67°	31	61	92	122	150	183	214	244	275
23°	·59188	·59484	·59778	·60070	·60359	·60646	·60931	66°	29	58	87	116	146	174	204	233	262
24°	·60931	·61214	·61494	·61773	·62049	·62323	·62595	65°	28	56	83	111	139	166	195	222	250
25°	9·62595	9·62865	9·63133	9·63398	9·63662	9·63924	9·64184	64°	27	53	80	106	133	159	186	212	239
26°	·64184	·64442	·64698	·64953	·65205	·65456	·65705	63°	25	51	76	102	127	152	178	203	229
27°	·65705	·65952	·66197	·66441	·66682	·66922	·67161	62°	24	49	73	97	122	146	170	194	219
28°	·67161	·67398	·67633	·67866	·68098	·68328	·68557	61°	23	47	70	93	117	140	163	186	210
29°	·68557	·68784	·69010	·69234	·69456	·69677	·69897	60°	22	45	67	89	112	134	156	179	201
30°	9·69897	9·70115	9·70332	9·70547	9·70761	9·70973	9·71184	59°	22	43	65	86	107	129	150	172	193
31°	·71184	·71393	·71602	·71809	·72014	·72218	·72421	58°	21	41	62	82	103	124	144	165	185
32°	·72421	·72622	·72823	·73022	·73219	·73416	·73611	57°	20	40	59	79	99	119	139	159	178
33°	·73611	·73805	·73997	·74189	·74379	·74568	·74756	56°	19	38	57	76	96	115	134	153	172
34°	·74756	·74943	·75128	·75313	·75496	·75678	·75859	55°	18	37	55	74	92	110	129	147	165
35°	9·75859	9·76039	9·76218	9·76395	9·76572	9·76747	9·76922	54°	18	35	53	71	89	106	124	142	159
36°	·76922	·77095	·77268	·77439	·77609	·77778	·77946	53°	17	34	51	68	86	103	120	137	154
37°	·77946	·78113	·78280	·78445	·78609	·78772	·78934	52°	17	33	50	66	83	99	116	132	149
38°	·78934	·79095	·79256	·79415	·79573	·79731	·70887	51°	16	32	48	64	80	95	112	127	143
39°	·79887	·80043	·80197	·80351	·80504	·80656	·80807	50°	15	31	46	62	77	92	108	123	138
40°	9·80807	9·80957	9·81106	9·81254	9·81402	9·81549	9·81694	49°	15	30	44	59	74	89	104	118	133
41°	·81694	·81839	·81983	·82126	·82269	·82410	·82551	48°	14	29	43	57	72	86	100	114	129
42°	·82551	·82691	·82830	·82968	·83106	·83242	·83378	47°	14	28	41	55	69	83	97	110	124
43°	·83378	·83513	·83648	·83781	·83914	·84046	·84177	46°	13	27	40	53	67	80	93	106	120
44°	·84177	·84308	·84437	·84566	·84694	·84822	·84949	45°	13	26	38	51	64	77	90	102	115
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COSINES

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	9·84949	9·85074	9·85200	9·85324	9·85448	9·85571	9·85693	44°	12	25	37	50	62	74	87	99	112
46°	·85693	·85815	·85936	·86056	·86176	·86294	·86413	43°	12	24	36	48	60	72	84	96	108
47°	·86413	·86530	·86647	·86763	·86879	·86993	·87107	42°	12	23	35	46	58	70	81	93	104
48°	·87107	·87221	·87334	·87446	·87557	·87668	·87778	41°	11	22	34	45	56	67	78	89	100
49°	·87778	·87887	·87996	·88105	·88212	·88319	·88425	40°	11	22	32	43	54	65	76	86	97
50°	9·88425	9·88531	9·88636	9·88741	9·88844	9·88948	9·89050	39°	10	21	31	42	52	62	73	83	94
51°	·89050	·89152	·89254	·89354	·89455	·89554	·89653	38°	10	20	30	40	50	60	70	80	90
52°	·89653	·89752	·89849	·89947	·90043	·90139	·90235	37°	10	19	29	39	49	58	68	78	87
53°	·90235	·90330	·90424	·90518	·90611	·90704	·90796	36°	9	19	28	37	47	56	65	84	84
54°	·90796	·90887	·90978	·91069	·91158	·91241	·91336	35°	9	18	27	36	45	54	63	72	81
55°	9·91336	9·91425	9·91512	9·91599	9·91686	9·91772	9·91857	34°	9	17	26	35	44	52	61	70	78
56°	·91857	·91942	·92027	·92111	·92194	·92277	·92359	33°	8	17	25	34	42	50	59	67	76
57°	·92359	·92441	·92522	·92603	·92683	·92763	·92842	32°	8	16	24	32	41	49	57	65	73
58°	·92842	·92921	·92999	·93077	·93154	·93230	·93307	31°	8	16	23	31	39	47	55	62	70
59°	·93307	·93382	·93457	·93532	·93606	·93680	·93753	30°	8	15	23	30	37	45	52	60	67
60°	9·93753	9·93826	9·93898	9·93970	9·94041	9·94112	9·94182	29°	7	14	22	29	36	43	50	57	64
61°	·94182	·94252	·94321	·94390	·94458	·94526	·94593	28°	7	14	21	27	34	41	48	55	62
62°	·94593	·94660	·94727	·94793	·94858	·94923	·94988	27°	7	13	20	26	33	40	46	53	59
63°	·94988	·95052	·95116	·95179	·95242	·95304	·95366	26°	6	13	19	25	32	38	44	50	57
64°	·95366	·95427	·95488	·95549	·95609	·95668	·95728	25°	6	12	18	24	30	36	42	48	54

TABLE V] LOGARITHMIC SINES & COSINES

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	9·95728	9·95786	9·95844	9·95902	9·95960	9·96017	9·96073	24°	6	12	17	23	29	35	40	46	52
66°	·96073	·96129	·96185	·96240	·96294	·96349	·96403	23°	6	11	17	22	28	33	38	44	50
67°	·96403	·96456	·96509	·96562	·96614	·96665	·96717	22°	5	10	16	21	26	31	36	42	47
68°	·96717	·96767	·96818	·96868	·96917	·96966	·97015	21°	5	10	15	20	25	29	34	40	44
69°	·97015	·97063	·97111	·97159	·97206	·97252	·97299	20°	5	9	14	19	24	28	33	38	42
70°	9·97299	9·97344	9·97390	9·97435	9·97479	9·97523	9·97567	19°	4	9	13	18	22	27	31	36	40
71°	·97567	·97610	·97653	·97696	·97738	·97779	·97821	18°	4	9	13	17	21	26	30	34	38
72°	·97821	·97861	·97902	·97942	·97982	·98021	·98060	17°	4	[illegible]	12	16	20	24	28	32	36
73	·98060	·98098	·98136	·98174	·98211	·98248	·98284	16°	4	8	11	15	19	22	26	30	34
74°	·98284	·98320	·98356	·98391	·98426	·98460	·98494	15°	4	7	11	14	18	21	25	28	32
75°	9·98494	9·98528	9·98561	9·98594	9·98627	9·98659	9·98690	14°	3	7	10	13	17	20	23	26	30
76°	·98690	·98722	·98753	·98783	·98813	·98843	·98872	13°	3	6	9	12	15	18	21	24	27
77°	·98872	·98901	·98930	·98958	·98986	·99013	·99040	12°	3	6	8	11	14	17	20	22	25
78°	·99040	·99067	·99093	·99119	·99145	·99170	·99195	11°	3	5	8	10	13	16	18	21	23
79°	·99195	·99219	·99243	·99267	·99290	·99313	·99335	10°	2	5	7	9	12	14	16	19	21
80°	9·99335	9·99357	9·99379	9·99400	9·99421	9·99442	9·99462	9°	2	4	6	8	11	13	15	17	19
81°	·99462	·99482	·99501	·99520	·99539	·99557	·99575	8°	2	4	6	8	10	11	13	15	17
82°	·99575	·99593	·99610	·99627	·99643	·99659	·99675	7°	2	[illegible]	5	7	8	10	12	13	15
83°	·99675	·99690	·99705	·99720	·99734	·99748	·99761	6°	1	3	4	6	7	9	10	12	13
84°	·99761	·99775	·99787	·99800	·99812	·99823	·99834	5°	1	3	4	5	6	8	9	10	11
85°	9·99834	9·99845	9·99856	9·99866	9·99876	9·99885	9·99894	4°	1	2	3	4	5	6	7	8	9
86°	·99894	·99903	·99911	·99919	·99926	·99934	·99940	3°	1	2	2	3	4	5	5	6	7
87°	·99940	·99947	·99953	·99959	·99964	·99969	·99974	2°	1	1	2	2	3	3	4	4	5
88°	·99974	·99978	·99982	·99985	·99988	·99991	·99993	1°	0	1	1	1	2	2	2	2	3
89°	·99993	·99995	·99997	·99998	·99999	10·00000	10·00000	0°									
90°	10·00000																

LOGARITHMIC COSINES

TABLE VI
LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
0°	−∞	7·46373	7·76476	7·94086	8·06581	8·16273	8·24192	89°	Here differences vary so rapidly that tabulation is impossible. For small angles of x', log tan x' or log cot $(90° - x') = \log x + \bar{4}{\cdot}46373$.								
1°	8·24192	8·30888	8·36689	8·41807	8·46385	8·50527	8·54308	88°									
2°	8·54308	8·57788	8·61009	8·64009	8·66816	8·69453	8·71940	87°									
3°	8·71940	8·74292	8·76525	8·78649	8·80674	8·82610	8·84464	86°									
4°	8·84464	8·86243	8·87953	8·89598	8·91185	8·92716	8·94195	85°									
5°	8·94195	8·95627	8·97013	8·98358	8·99662	9·00930	9·02162	84°									
6°	9·02162	9·03361	9·04528	9·05666	9·06775	9·07858	9·08914	83°									
7°	9·08914	9·09947	9·10956	9·11943	9·12909	9·13854	9·14780	82°	98	195	293	391	488	586	684	782	879
8°	9·14780	9·15688	9·16577	9·17450	9·18306	9·19146	9·19971	81°	87	178	260	346	433	519	606	692	779
9°	9·19971	9·20782	9·21578	9·22361	9·23130	9·23887	9·24632	80°	78	155	233	310	388	466	543	621	698
10°	9·24632	9·25365	9·26086	9·26797	9·27496	9·28186	9·28865	79°	71	141	212	282	354	420	494	564	635
11°	·28865	·29535	·30195	·30846	·31489	·32122	·32747	78°	65	129	194	259	323	388	453	518	582
12°	·32747	·33365	·33974	·34576	·35170	·35757	·36336	77°	60	120	179	239	299	359	419	478	538
13°	·36336	·36909	·37476	·38035	·38589	·39136	·39677	76°	56	111	167	222	278	334	389	445	500
14°	·39677	·40212	·40742	·41266	·41784	·42297	·42805	75°	52	104	156	208	261	313	365	417	469
15°	9·42805	9·43308	9·43806	9·44299	9·44787	9·45271	9·45750	74°	49	98	147	196	245	294	343	392	442
16°	·45750	·46224	·46694	·47160	·47622	·48080	·48534	73°	46	93	139	186	232	278	325	371	418
17°	·48534	·48984	·49430	·49872	·50311	·50746	·51178	72°	44	88	132	176	220	264	308	352	396
18°	·51178	·51606	·52031	·52452	·52870	·53285	·53697	71°	42	84	126	168	210	252	294	336	378
19°	·53697	·54106	·54512	·54915	·55315	·55712	·56107	70°	40	80	121	160	201	241	281	321	362

LOGARITHMIC TANGENTS & COTANGENTS

20°	9·56107	9·56498	9·56887	9·57274	9·57658	9·58039	9·58418	69°	39	77	116	154	193	231	270	308	347
21°	·58418	·58794	·59168	·59540	·59909	·60276	·60641	68°	37	74	111	148	185	222	259	296	333
22°	·60641	·61004	·61364	·61722	·62079	·62433	·62785	67°	36	72	107	143	179	214	250	286	322
23°	·62785	·63135	·63484	·63830	·64175	·64517	·64858	66°	35	69	104	138	173	208	242	277	311
24°	·64858	·65197	·65535	·65870	·66204	·66537	·66867	65°	34	67	101	134	168	201	235	268	302
25°	9·66867	9·67196	9·67524	9·67850	9·68174	9·68497	9·68818	64°	33	65	98	130	163	195	228	260	293
26°	·68818	·69138	·69457	·69774	·70089	·70404	·70717	63°	32	63	95	126	158	190	221	253	284
27°	·70717	·71028	·71339	·71648	·71955	·72262	·72567	62°	31	62	92	123	154	185	216	246	277
28°	·72567	·72872	·73175	·73476	·73777	·74077	·74375	61°	30	60	90	120	151	181	211	241	271
29°	·74375	·74673	·74969	·75264	·75558	·75852	·76144	60°	29	59	88	118	147	177	206	236	265
30°	9·76144	9·76435	9·76725	9·77015	9·77303	9·77591	9·77877	59°	29	58	87	116	144	173	202	231	260
31°	·77877	·78163	·78448	·78732	·79015	·79297	·79579	58°	28	57	85	113	142	170	198	227	255
32°	·79579	·79860	·80140	·80419	·80697	·80975	·81252	57°	28	56	84	112	139	167	195	223	251
33°	·81252	·81528	·81803	·82078	·82352	·82626	·82899	56°	28	55	83	110	137	165	192	220	247
34°	·82899	·83171	·83442	·83713	·83984	·84254	·84523	55°	27	54	81	108	136	162	190	217	244
35°	9·84523	9·84791	9·85059	9·85327	9·85594	9·85860	9·86126	54°	27	54	80	107	134	160	188	214	241
36°	·86126	·86392	·86656	·86921	·87185	·87448	·87711	53°	26	53	79	106	132	158	185	212	238
37°	·87711	·87974	·88236	·88498	·88759	·89020	·89281	52°	26	52	78	105	131	157	183	209	236
38°	·89281	·89541	·89801	·90061	·90320	·90578	·90837	51°	26	52	78	104	130	156	182	208	234
39°	·90837	·91095	·91353	·91610	·91868	·92125	·92381	50°	26	52	77	103	129	155	180	206	232
40°	9·92381	9·92638	9·92894	9·93150	9·93406	9·93661	9·93916	49°	26	51	77	102	128	154	179	205	230
41°	·93916	·94171	·94426	·94681	·94935	·95190	·95444	48°	25	51	76	102	127	153	178	204	229
42°	·95444	·95698	·95952	·96205	·96459	·96712	·96966	47°	25	51	76	101	127	152	177	203	228
43°	·96966	·97219	·97472	·97725	·97978	·98231	·98484	46°	25	51	76	101	127	152	177	202	228
44°	·98484	·98737	·98989	·99242	·99495	·99747	10·00000	45°	25	51	76	101	127	152	177	202	228
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COTANGENTS

LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	10·00000	10·00253	10·00505	10·00758	10·01011	10·01263	10·01516	44°	25	51	76	101	127	152	177	202	228
46°	·01516	·01769	·02022	·02275	·02528	·02781	·03034	43°	25	51	76	101	127	152	177	202	228
47°	·03034	·03288	·03541	·03795	·04048	·04302	·04556	42°	25	51	76	101	127	152	177	203	228
48°	·04556	·04810	·05065	·05319	·05574	·05829	·06084	41°	25	51	76	102	127	153	178	204	229
49°	·06084	·06339	·06594	·06850	·07106	·07362	·07619	40°	26	51	77	102	128	154	179	205	230
50°	10·07619	10·07875	10·08132	10·08390	10·08647	10·08905	10·09163	39°	26	52	77	103	129	155	180	206	232
51°	·09163	·09422	·09680	·09939	·10199	·10459	·10719	38°	26	52	78	104	130	156	182	208	234
52°	·10719	·10980	·11241	·11502	·11764	·12026	·12289	37°	26	52	78	105	131	157	183	209	236
53°	·12289	·12552	·12815	·13079	·13344	·13608	·13874	36°	26	53	79	106	132	158	185	212	238
54°	·13874	·14140	·14406	·14673	·14941	·15209	·15477	35°	27	54	80	107	134	160	188	214	241
55°	10·15477	10·15746	10·16016	10·16287	10·16558	10·16829	10·17101	34°	27	54	81	108	136	162	190	217	244
56°	·17101	·17374	·17648	·17922	·18197	·18472	·18748	33°	28	55	83	110	137	165	192	220	247
57°	·18748	·19025	·19303	·19581	·19860	·20140	·20421	32°	28	56	84	112	139	167	195	223	251
58°	·20421	·20703	·20985	·21268	·21552	·21837	·22123	31°	28	57	85	113	142	170	198	227	255
59°	·22123	·22409	·22697	·22985	·23275	·23565	·23856	30°	29	58	87	116	144	173	202	231	260
60°	10·23856	10·24148	10·24442	10·24736	10·25031	10·25327	10·25625	29°	29	59	88	118	147	177	206	236	265
61°	·25625	·25923	·26223	·26524	·26825	·27128	·27433	28°	30	60	90	120	151	181	211	241	271
62°	·27433	·27738	·28045	·28352	·28661	·28972	·29283	27°	31	62	92	123	154	185	216	246	277
63°	·29283	·29596	·29911	·30226	·30543	·30862	·31182	26°	32	63	95	126	158	190	221	253	284
64°	·31182	·31503	·31826	·32150	·32476	·32804	·33133	25°	33	65	98	130	163	195	228	260	293

LOGARITHMIC TANGENTS & COTANGENTS

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	10·33133	10 33468	10·33796	10·34130	10·34465	10·34803	10·35142	24°	34	67	101	134	168	201	235	268	302
66°	·35142	·35483	·35825	·36170	·36516	·36865	·37215	23°	35	69	104	138	173	208	242	277	311
67°	·37215	·37567	·37921	·38278	·38636	·38996	·39359	22°	36	72	107	143	179	214	251	286	322
68°	·39359	·39724	·40091	·40460	·40832	·41206	·41582	21°	37	74	111	148	185	222	259	296	333
69°	·41582	·41961	·42342	·42726	·43113	·43502	·43898	20°	39	77	116	154	193	231	270	308	347
70°	10·43893	10·44288	10·44685	10·45085	10·45488	10·45894	10·46303	19°	40	80	121	160	201	241	281	321	362
71°	·46303	·46715	·47130	·47548	·47969	·48394	·48822	18°	42	84	126	168	210	252	294	336	378
72°	·48822	·49254	·49689	·50128	·50570	·51016	·51466	17°	44	88	132	176	220	264	308	352	396
73°	·51466	·51920	·52378	·52840	·53306	·53776	·54250	16°	46	93	139	186	232	278	325	371	418
74°	·54250	·54729	·55213	·55701	·56194	·56692	·57195	15°	49	98	147	196	245	294	343	392	442
75°	10·57195	10·57703	10·58216	10·58734	10·59258	10·59788	10·60323	14°	52	104	156	208	261	313	365	417	469
76°	·60323	·60864	·61411	·61965	·62524	·63091	·63664	13°	56	111	167	222	278	334	389	445	500
77°	·63664	·64243	·64830	·65424	·66026	·66635	·67253	12°	60	120	179	239	299	359	419	478	538
78°	·67253	·67878	·68511	·69154	·69805	·70465	·71135	11°	65	129	194	259	323	388	453	518	582
79°	·71135	·71814	·72504	·73203	·73914	·74635	·75368	10°	71	141	212	282	354	420	494	564	635
80°	10·75368	10·76113	10·76870	10·77639	10·78422	10·79218	10·80029	9°	78	155	233	310	388	466	543	621	698
81°	·80029	·80854	·81694	·82550	·83423	·84312	·85220	8°	87	173	260	346	433	519	606	692	779
82°	·85220	·86146	·87091	·88057	·89044	·90053	·91086	7°	98	195	293	391	488	586	684	782	879
83°	·91086	·92142	·93225	·94334	·95472	·96639	·97838	6°									
84°	·97838	·99070	11·00338	11·01642	11·02987	11·04373	11·05805	5°									
85°	11·05805	11·07284	11·08815	11·10402	11·12047	11·13757	11·15536	4°									
86°	11·15536	11·17390	11·19326	11·21351	11·23475	11·25708	11·28060	3°									
87°	11·28060	11·30547	11·33184	11·35991	11·38991	11·42212	11·45692	2°	Here differences vary so rapidly that tabulation is impossible.								
88°	11·45692	11·49473	11·53615	11·58193	11·63311	11·69112	11·75808	1°									
89°	11·75808	11·83727	11·93419	12·05914	12·23524	12·53627	+∞	0°									
90°	+∞																

LOGARITHMIC COTANGENTS